RHEOLOGICAL-DYNAMICAL CONTINUUM DAMAGE MODEL FOR CONCRETE UNDER UNIAXIAL COMPRESSION AND ITS EXPERIMENTAL VERIFICATION

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ABSTRACT. A new analytical model for the prediction of concrete response under uniaxial compression and its experimental verification is presented in this paper. The proposed approach, referred to as the rheological-dynamical continuum damage model, combines rheological-dynamical analogy and damage mechanics. Within the framework of this approach the key continuum parameters such as the creep coefficient, Poisson's ratio and damage variable are functionally related. The critical values of the creep coefficient and damage variable under peak stress are used to describe the failure mode of the concrete cylinder. The ultimate strain is determined in the post-peak regime only, using the secant stress-strain relation from damage mechanics. The post-peak branch is used for the energy analysis. Experimental data for five concrete compositions were obtained during the examination presented herein. The principal difference between compressive failure and tensile fracture is that there is a residual stress in the specimens, which is a consequence of uniformly accelerated motion of load during the examination of compressive strength. The critical interpenetration displacements and crushing energy are obtained theoretically based on the concept of global failure analysis.

Nomenclature

A_0	damaged cross section area
\overline{A}	effective resisting cross section area
a	acceleration
$a_{\mathrm{cr}F}$	critical crack depth under peak stress
C	linear creep law constant
$c_{\rm cr}$	critical viscous damping
$c_{\rm load}$	loading rate
D	damage variable
$D_{\rm cr}$	critical damage variable
$D_{\mathrm{cr}F}$	critical damage variable under peak stress

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Key words and phrases: concrete under compression; RDA stress-strain curve; critical creep coefficient; critical damage variable; global failure analysis; crushing energy.

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S	frequency ratio
0 F	alastia modulus
L_H	
E_K	viscoelastic modulus
E_K^D	dynamic viscoelastic modulus
E_R	RDA modulus, RDA modulus function
E_{RF}	critical RDA modulus
E_T	tangent modulus
E(D)	variation in Young's modulus
$E(D_{\mathrm{cr}F})$	critical variation in Young's modulus
$\varepsilon, \varepsilon_j$	total strain
$\dot{\varepsilon}$	total strain rate
$\varepsilon_{S,j}$	static total strain
$\varepsilon_{D,j}$	dynamic total strain
$\varepsilon_{\rm el}$	elastic strain
$\varepsilon_{\rm ve}$	viscoelastic strain
ε _{vp}	viscoplastic strain
$\varepsilon_{\rm cr}$	peak compression strain
$\varepsilon_{\mathrm{cr}F}$	ultimate strain
f_c	measured strength
f_{γ}	acceleration coefficient
ϕ	diameter of concrete cylinder
φ	viscoelastic creep coefficient
φ^*	structural creep coefficient
$\varphi_{ m cr}$	critical creep coefficient
$\varphi_{\mathrm{cr}F}$	critical creep coefficient under the peak stress
G_C	crushing energy
g	gravity acceleration
γ	specific gravity
H	symbol for Hookean spring
H'	viscoplastic modulus
H'^D	dynamic viscoplastic modulus
Ι	minimum moment of inertia of cross-section about centroidal axis
i	minimum gyration radius
K	symbol for Kelvin's body
K_0, K_E	structural material constant
k	stiffness
$l_{0}, l_{E}, l_{D}, l_{j}$	cylinder height
$\lambda_0, \lambda_E, \lambda_D, \lambda_j$	slenderness ratio
λ_K	viscoelastic normal viscosity
λ_N	viscoplastic normal viscosity
m	mass
N	symbol for Newtonian dashpot
μ	Poisson's ratio
\overline{r}	effective radius
Q, Q_A	instantaneously applied loading

ω	natural angular frequency
ω_{σ}	load or stress angular frequency
ρ	mass density
StV	symbol for Saint-Venant's body
σ	time-dependent variation of stress
$\dot{\sigma}$	stress rate
σ_A	cyclic stress amplitude
σ_E	Euler's critical stress or stress at limit of elasticity
σ_Y	uniaxial yield stress of material
$\sigma_{\mathrm{cr}}, \sigma_{\mathrm{cr},j}$	inelastic critical stress
$\sigma_{{ m cr}F}$	critical failure stress
σ_D	dynamic stress
$\sigma_{D{ m cr}F}$	dynamic critical failure stress
σ_S	static stress
$\sigma_{S \mathrm{cr}F}$	static critical failure stress
$\sigma_{\rm fictitious}$	fictitious stress
$\sigma_{\rm residual}$	residual stress
T_K^D	dynamic time of retardation
t, Δ_t	time, very short time of loading
t_E	time when stress reaches limit of elasticity
t_F	time when failure occurs
$w_{ m cr}$	critical interpenetration displacement
Y	stress level for viscoplastic yielding
	symbol for parallel connection
	symbol for serial connection

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1. Introduction

Reinforced concrete materials have been studied and employed in diverse fields of science and engineering disciplines due to their wide application in infrastructure in many countries. From a practical standpoint, the ultimate strength design of reinforced concrete elements brought the stress-strain relationship into focus. The compression response of concrete, and in particular the compressive strength, ultimate strain and post-peak branch, have an important role in the design of concrete and concrete-based structures. In the last three decades, there has been a keen interest in compressive failure. In the early 1990's, an extensive Round/Robin test on compressive softening was carried out by the RILEM Technical Committee 148-SSC [1]. Compression failure can have a great variety of forms both from theoretical and experimental viewpoints, while the mode of failure is complex. The principal difference from tensile fracture is that there is a residual stress in the specimens. The correct evaluation of the constitutive parameters is also complicated by many other testing aspects. To interpret the test results, it is important to know the influence of the boundary conditions on the slope of the load-displacement curve and on the value of the crushing energy [2]. An in-depth review of this matter as well evidence that slenderness of the specimen has influence on the collapse mechanism is given in [3]. Considering the size-scale and slenderness effects in uniaxial compression tests, Carpinteri et al. [4] proposed an analytical model based on the concept of strain localization. The discussion of instantaneous deformations of concrete under load is timed from a theoretical viewpoint because deformations provide indirect information concerning the internal structure [5] as well as the microscopic fracture mechanism. Analytical models of time-dependent stress-strain response of concrete under compression are required. For global failure analysis, the failure mechanism must be treated in a smeared manner, as a continuum with damage. Since about 1998 (Milašinović, [6, 7]), a mathematical-physical analogy named rheological-dynamic analogy (RDA) has been proposed in explicit form to predict a range of inelastic and time-dependent problems related to 1D prismatic rods, such as buckling, fatigue etc. This theory defines the critical mechanical properties of viscoelastoplastic (VEP) materials. RDA is based on the propagation of elastic waves under instantaneously applied impact loading. A successful theoretical approach based on RDA and their practical applications have also been given in connection with the VEP behavior of metallic bars in tension [8]. In fact, cracking is accompanied by an emission of elastic waves which propagate within the bulk of the material. More information on the modalities of energy release and the development of cracking patterns can be obtained on the basis of the acoustic emission monitoring technique, which proves that it is possible to detect the occurrence and evolution of stress-induced cracks [9].

The aim of this paper is to experimentally verify an RDA model of prediction of stress-strain response of concrete in the pre-peak regime only, given the loadcontrolled compression test results used. On the other hand, the analytical model presented covers the post-peak regime combining the RDA and damage mechanics.

Many researchers have tried to represent the stress-strain relationship with standard mathematical curves, e.g., a parabola, hyperbola, ellipse, cubic parabola, or combinations like a parabola with a straight line and so on, see also Model Code 90 [10]. These curves may have the advantage of simplifying the computation of the ultimate moment of reinforced concrete sections. However, they can be classified only as empirical methods since the assumed stress distribution does not represent an observed physical phenomenon, such as the failure mechanism. Moreover, to identify global failure as a continuum with damage, relations between RDA parameters and damage mechanics must be formulated first. Based on these relations, the mode of failure is quantified by the characterizing parameters under the peak stress as the critical creep coefficient and critical damage variable. The ultimate strain is determined in the post-peak regime only, using the secant stress-strain relation from damage mechanics. Consequently, the proposed approach is referred to as the rheological-dynamical continuum damage model. The proposed model is shown to provide very good predictions for the stress versus strain response in the pre-peak regime. Within this global model the three main failure characteristics, the residual stress, the critical value of interpenetration displacement and the crushing energy, are theoretically evaluated. Finally, on the basis of four non-dimensional constants the crushing energy is calculated for five concrete compositions.

2. RDA of a specimen subjected to compression load

2.1. RDA-a short overview. Material micro cracking is accompanied by the loading of a specimen (concrete cylinder), leading to its damage and failure. Consider the case of the VEP strain of a cylindrical specimen presented in Figure 1a. In material investigations, both stress $\sigma(t)$ and inelastic strain $\varepsilon^*(t) = \varepsilon_{ve}(t) + \varepsilon_{vp}(t)$ are functions of time. If the total VEP strain $\varepsilon(t) = \varepsilon_{el} + \varepsilon^*(t)$ is presented as a sum of elastic (instantaneous), viscoelastic (VE) and viscoplastic (VP) components, each isochronous stress-strain diagram of a prismatic specimen (e.g., with a square or circular cross section A_0) can accurately be approximated by the rheological body H - K - (StV | N), consisting of five elements. The rheological body is shown in Figure 1b using the following symbols: N for the Newtonian dashpot, StV for Saint-Venant's body, H for the Hookean spring, "|" for a parallel connection and "—" for a connection in a series. Since the Hookean spring, Kelvin's body (K = H|N) and VP body (StV |N) are connected in a series, stresses $\sigma(t)$ in all the bodies are equal. Based on this rheological body, Milašinović [6] derived a governing differential equation, i.e.,

(2.1)
$$\ddot{\varepsilon}(t) + \dot{\varepsilon}(t) \left(\frac{E_K}{\lambda_K} + \frac{H'}{\lambda_N}\right) + \varepsilon(t) \frac{E_K H'}{\lambda_K \lambda_N}$$
$$= \frac{\ddot{\sigma}(t)}{E_H} + \dot{\sigma}(t) \left(\frac{E_K}{\lambda_K E_H} + \frac{H'}{\lambda_N E_H} + \frac{1}{\lambda_K} + \frac{1}{\lambda_N}\right)$$
$$+ \sigma(t) \left(\frac{E_K}{\lambda_K \lambda_N} + \frac{H'}{\lambda_K \lambda_N} + \frac{E_K H'}{\lambda_K \lambda_N E_H}\right) - \sigma_Y \frac{E_K}{\lambda_K \lambda_N}$$

in which E_H is the elastic modulus and σ_Y is the uniaxial yield stress. The yield condition is $Y = \sigma_Y + H' \varepsilon_{vp}(t)$. The four properties at fixed step times are:

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extensional VE viscosity λ_K , extensional VP viscosity λ_N , VE modulus E_K and VP modulus H'. However, these constants cannot easily be determined in physical experiments, especially Trouton's viscosities λ_K and λ_N . The corresponding homogeneous equation of the total VEP strain is as follows

(2.2)
$$\ddot{\varepsilon}(t)\lambda_K\lambda_N + \dot{\varepsilon}(t)(E_K\lambda_N + H'\lambda_K) + \varepsilon(t)E_KH' = 0$$



FIGURE 1. Rheological-dynamical continuum damage model for concrete cylinder $(l_0/\phi = 2)$ under compression: a) damage state of concrete cylinder; b) rheological model; c) dynamical model.

On the other hand, a mechanical longitudinal disturbance (strain) propagates in an elastic medium at the finite initial phase velocity $v_0 = (E_H/\rho)^{1/2}$, where ρ is the density of the medium. The vibration at an arbitrary point M of the specimen lags in the phase behind that at the source of the wave. If l_0 is the initial distance between the two ends of the specimen, the time required for a wave to travel from one to the other end of it is $t - t_0 = T_K^D = l_0/v_0$. The natural angular frequency ω of the discrete dynamical model, which represents the undamped free longitudinal vibration of a specimen, is given by

(2.3)
$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{E_H A_0}{l_0} \frac{1}{\rho A_0 l_0}} = \frac{v_0}{l_0} = \frac{1}{T_K^D} \Rightarrow T_K^D = \frac{l_0}{v_0} = \frac{1}{\omega}$$

m is the mass of the specimen and k its axial stiffness, as shown in Figure 1c. Bearing in mind (2.2), an expression similar to (2.3) can be formulated, setting the rheological model of the specimen into the state of critical viscous damping ($c = c_{\rm cr}$)

(2.4)
$$\omega = \sqrt{\frac{E_K H'}{\lambda_K \lambda_N}} = \sqrt{\frac{1}{T_K T^*}} = \frac{1}{T_K^D}$$

where $E_K/\lambda_K = H'/\lambda_N$, $\lambda_K = E_K T_K$, $\lambda_N = H'T^*$ and $T_K = T^* = T_K^D$. Therefore,

(2.5)
$$\sqrt{\frac{E_H}{\rho}}\frac{1}{l_0} = \sqrt{\frac{E_K H'}{\lambda_K \lambda_N}} \Rightarrow \lambda_K \lambda_N = \frac{E_K H' \gamma l_0^2}{E_H g} \Rightarrow \frac{\lambda_K \lambda_N}{\gamma} = \frac{E_K H' A_0 l_0^2 \rho}{E_H \gamma A_0}$$

where γ is the specific gravity. Thus,

(2.6)
$$m = \frac{\lambda_K \lambda_N}{\gamma} = k (T_K^D)^2, \ k = \frac{E_K H'}{\gamma}, \ c_{\rm cr} = 2\sqrt{km} = 2kT_K^D$$

Consequently, the propagation of longitudinal elastic waves under instantaneously applied impact loading Q represents a physical basis for the analogy between two different physical phenomena, the rheological and dynamical. Then, (2.2) may be expressed as follows

(2.7)
$$\ddot{\varepsilon}(t)m + \dot{\varepsilon}(t)c_{\rm cr} + \varepsilon(t)k = 0$$

Therefore, a very complicated nonlinear problem in the VEP range of strains may be solved as a simple linear dynamic one. Generally speaking, the RDA is derived in order to solve dynamic problems [7], but it can be used in the analysis of quasi-static loading ($\delta \rightarrow 0$) considering the corresponding limit values of derived analytical expressions. For instance, each quasi-static stress-strain curve of a specimen (bar, column, concrete cylinder, etc.) can be obtained using the RDA modulus function, including the peak stress or compressive strength, which is the main parameter for energy analysis. On the other hand, the *J* integral cannot be applied to solve the problem of loading in compression of concrete cylinder although the other fracture mechanics approaches can be applied.

2.2. RDA modulus and the structural-material constant of a specimen. The inelastic strain has been solved analytically using the sinusoidal excitation $\sigma(t) = \sigma_0 + \sigma_A \sin(\omega_{\sigma} t)$ in a uniaxial loaded specimen [6]. The RDA modulus in the state of critical viscous damping was obtained as follows

(2.8)
$$E_R(t) = \frac{\frac{1+\delta^2}{E_H} + \frac{1}{E_K^D(t)} + \frac{1}{H'^D(t)}}{\frac{1+\delta^2}{E_H^2} + \left(\frac{1}{E_K^D(t)} + \frac{1}{H'^D(t)}\right)\left(\frac{2}{E_H} + \frac{1}{E_K^D(t)} + \frac{1}{H'^D(t)}\right)}$$

where δ is the angular frequency ratio,

(2.9)
$$\delta = \frac{\omega_{\sigma}}{\omega} = T_K^D \omega_{\sigma}$$

The previous properties are replaced by the corresponding dynamic (marked D), $E_K^D(t)$ and $H'^D(t)$. ω_{σ} is the frequency of excitation. When a specimen is loaded cyclically, the rheological behavior is characterized by the dynamic time of retardation $T_K^D = 1/\omega$, with the following dynamic VE modulus

(2.10)
$$E_K^D(t) = \frac{E_H}{\varphi(t)} \left(1 - e^{-t/T_K^D}\right) = \frac{E_H}{\varphi(t)}$$

 $\varphi(t)$ is the VE creep coefficient $(e^{-t/T_K^D} \approx 0)$. According to the second formula of (2.6), the dynamic VP modulus may be expressed by

(2.11)
$$H'^{D}(t) = \frac{k\gamma}{E_{K}^{D}(t)} = \frac{A_{0}\gamma\varphi(t)}{l_{0}}$$

In the case of quasi-static loading $(\delta \to 0)$, the RDA modulus is given by

(2.12)
$$E_R(t) = \frac{1}{\frac{1}{E_H} + \frac{1}{E_K^D(t)} + \frac{1}{H'^D(t)}}$$

If the behavior of the material cannot be VE $(E_K^D \to \infty)$, its only response is elastoviscoplastic. Therefore,

(2.13)
$$E_R(t) = \frac{{H'}^D(t)}{1 + \frac{{H'}^D(t)}{E_H}}$$

Obviously, E_R is equal to the tangent modulus E_T at the selected moment in time. In order to obtain the structural-material constant of a specimen in the inelastic range, the elastic strain may be ignored or its value considered negligible $(1/E_H \sim 0)$. Therefore,

(2.14)
$$E_R(t) = {H'}^D(t) = \frac{A_0 \gamma \varphi(t)}{l_0} = \frac{\varphi(t)}{K_0 E_H}$$

where

(2.15)
$$K_0 = \frac{1}{E_K H'} = \frac{1}{k\gamma} = \frac{l_0}{A_0} \frac{1}{E_H \gamma}$$

 K_0 is the structural-material constant of a specimen (bar, column, concrete cylinder, etc.).

2.3. RDA modulus function. Consider now the linear law $\sigma(\varphi)$ in concrete in the interval $t_E < t < t_F$, as shown in Figure 2

(2.16)
$$\sigma(t) = C \cdot \varphi(t)$$



FIGURE 2. Linear change of stress with creep coefficient in concrete.

Time t_E , when stress reaches the limit of elasticity, and t_F , when failure occurs, are experimentally measured values. The linear law gives

(2.17)
$$\sigma(t) = \frac{\sigma_E}{\varphi^*} \varphi(t)$$

The Euler critical stress

(2.18)
$$\sigma_E = \sigma_{\rm cr} = \frac{E_H \pi^2}{\lambda_E^2}$$

can be used as a failure criterion for slender two-hinged specimens. The structural creep coefficient φ^* for the specimen length l_E (slender elastic specimens) is determined based on the Euler and RDA curve intersection at the limit of elasticity [6]

(2.19)
$$\varphi^* = \pi^2 \frac{i^3}{I} \frac{1}{\gamma \lambda_E}$$

in which $i = \sqrt{I/A_0}$ is the minimum radius of gyration, $\lambda_E = l_E/i$ is the slenderness ratio at the limit of elasticity, and I is the minimum moment of inertia of the cross section about the centroidal axis. Hence, beyond the limit of elasticity, the linear creep law takes the form of

(2.20)
$$\sigma_{\rm cr} = \frac{1}{K_E} \varphi_{\rm cr}$$

where K_E is the structural-material constant at the limit of elasticity.

(2.21)
$$K_E = \lambda_E \frac{i^3}{I} \frac{1}{E_H} \frac{1}{\gamma}$$

Thus, the RDA modulus function follows from (2.12)

(2.22)
$$E_R = \frac{1}{\frac{1}{E_H} + \frac{\varphi^*}{E_H} + \frac{1}{H'}} = \frac{E_H}{1 + \varphi_{\rm cr}} = \frac{E_H}{1 + \sigma_{\rm cr} K_E}$$

where the critical creep coefficient includes inelastic strain, as found in concrete.

(2.23)
$$\varphi_{\rm cr} = \varphi^* + \frac{E_H}{H'} = \sigma_{\rm cr} K_E$$

2.4. Quasi-static inelastic buckling curve. The inelastic critical stress of the two-hinged specimen may be obtained as explained in [6]

(2.24)
$$\sigma_{\rm cr} = H' E_H = \frac{A_0 \gamma \varphi^*}{l_0} E_H = \frac{E_H}{\frac{l_0}{A_0} \frac{1}{\gamma \varphi^*}} = \frac{E_H}{\frac{l_0}{i} \frac{1^3}{I} \frac{1}{\gamma \varphi^*}} = \frac{E_H}{\lambda_0 \frac{i^3}{I} \frac{1}{\gamma \varphi^*}}$$

The above critical stress is only an upper bound of critical stress in the inelastic range of strains, because the elastic modulus and structural creep coefficient are used. However, the elastic modulus must be replaced by the RDA modulus given by (2.22), if the failure compressive strength is explored. This viewpoint will be considered in Section 4.1. The minimal slenderness for which the Euler critical stress can still be applied is slenderness λ_E at the limit of elasticity. Hence, the quasi-static elastic and inelastic buckling curves are as follows

(2.25)
$$\sigma_{\mathrm{cr},j} = \frac{\pi^2}{\lambda_j^2} E_H, \quad \text{for } \lambda_j \ge \lambda_E,$$

(2.26)
$$\sigma_{\mathrm{cr},j} = \frac{E_H}{\lambda_j \frac{i^3}{I} \frac{1}{\gamma \varphi^*}}, \quad \text{for } \lambda_j \leqslant \lambda_E$$

where the effect of the specimen shape and size is included through the slenderness

(2.27)
$$\lambda_j = l_j/i, \quad (j = 1, 2, 3, \ldots)$$

2.5. Average RDA stress-strain curve. The RDA modulus function can be used further to obtain the quasi-static stress-strain curve, as follows

(2.28)
$$\varepsilon = \frac{\sigma_{\rm cr}}{E_R(0)} = \frac{\sigma_{\rm cr}}{E(0)} (1 + \varphi_{\rm cr}) = \frac{\sigma_{\rm cr}}{E(0)} (1 + \sigma_{\rm cr} K_E)$$

Thus, one quadratic equation takes the form of

(2.29)
$$\sigma_{\rm cr}^2 K_E + \sigma_{\rm cr} - E(0)\varepsilon = 0$$

Slope E(0) is the elastic modulus of the material in its initial state. The root of (2.29) under the initial conditions $\varepsilon(0) = 0$ and $\sigma_{\rm cr}(0) = 0$ is the limit value of critical stress for the selected strain or the average stress-strain curve. Then

(2.30)
$$\sigma_{\rm cr} = \frac{1}{2K_E} \left(\sqrt{1 + 4K_E E(0)\varepsilon} - 1 \right)$$

At the limit of elasticity, the slope is equal to the elastic modulus E_H (known value). Therefore,

$$(2.31) E_R(0) = E_H$$

Thus,

(2.32)
$$E(0) = E_H (1 + \varphi^*)$$

As detailed by Van Mier et al. [1], the stress-strain curves of concrete are dependent on two major parameters, testing conditions and concrete characteristics. The key experimental parameters cited in [1] included the frictional restraint between the loading platen and the specimen, the rotation of the loading platen during the experiment, the gauge length of the control LVDT, the stiffness of the testing machine, the type of the feed-back signal, the loading rate, the shape and size of the test specimen and the concrete composition. It is therefore important when using experimental data for verification and comparison that the experimental parameters are fully listed. Concrete characteristics depend on many interrelated variables such as the water-cement ratio, the mechanical and physical properties of the cement and aggregate, and the age of the specimen when tested [11].

In this study, cylinders of slenderness $l_0/\phi = 2$ loaded between the steel plates with low-fiction are tested only. The analytical stress-strain curve of concrete in the ascending branch is obtained as a critical curve and can be computed using (2.30) if the compressive strength, elastic modulus, concrete density and Poisson's ratio are experimentally evaluated. It is valid for various prismatic concrete samples (e.g., with a square or circular cross section A_0) and different concrete compositions. To identify the basic mechanism that leads to concrete damage growth, some primary features of concrete behavior experimentally observed are presented first.

3. Relations between RDA parameters and damage mechanics

3.1. Creep coefficient-Poisson's ratio dependence according to RDA. The RDA approach to the analysis of the problem of effect of Poisson's ratio on the creep coefficient has already been described in [7]. In this approach, based on the Bernoulli energy theorem and assuming that $\varepsilon_E = \sigma_E/E_H = 0.001$, the creep coefficient can be expressed by the formula

(3.1)
$$\varphi(\mu) = \left[\left(\frac{1}{1 - 0.001\mu} \right)^4 - 1 \right] \frac{1}{2 \cdot 0.001} / \left\{ 1 - \left[\left(\frac{1}{1 - 0.001\mu} \right)^4 - 1 \right] \frac{1}{2 \cdot 0.001} \right\}$$

Figure 3 presents the function given by (3.1), whose results are in excellent agreement with the experimentally obtained values.



FIGURE 3. Variation of creep coefficient with Poisson's ratio.

Concrete is a semi-brittle material determined by the linear dependence of $\log \varphi$ on $\log \mu$. As it is shown, according to the criterion of the linear logarithm $\varphi - \mu$ dependence, the upper boundary value of Poisson's ratio is 0.25. Given the observed variation in the concrete composition and experimental data, in the absence of experimental data, a value of Poisson's ratio between 0.15 and 0.20 is appropriate for characterizing the elastic material response. In the inelastic regime it increases to a certain critical value between 0.20–0.35. The above equation can be simplified by neglecting the products of second-order exponents

$$\left[\left(\frac{1}{1 - 0.001\mu} \right)^4 - 1 \right] \frac{1}{2 \cdot 0.001}$$

$$= \left\{ \frac{1}{\left[1 - 0.002\mu + (0.001\mu)^2 \right] \left[1 - 0.002\mu + (0.001\mu)^2 \right]} - 1 \right\} \frac{1}{0.002}$$

$$= \left[\frac{1}{\left[(1 - 0.002\mu)(1 - 0.002\mu)} - 1 \right] \frac{1}{0.002}$$

$$= \left[\frac{1}{1 - 0.004\mu + (0.002\mu)^2} - 1 \right] \frac{1}{0.002} = \left(\frac{1}{1 - 0.004\mu} - 1 \right) \frac{1}{0.002}$$

$$= \left(\frac{0.004\mu}{1 - 0.004\mu} \right) \frac{1}{0.002} = \frac{2\mu}{1 - 0.004\mu} \approx 2\mu$$

Thus,

(3.3)
$$\varphi = \frac{2\mu}{1 - 2\mu}$$

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3.2. Dependence of damage variable on creep coefficient and Poisson's ratio. Since the development of micro cracks induces a reduction in the stiffness of materials, the damage state can also be characterized by variation in the elastic modulus (Lemaitre and Chaboche [12]). Thus, the damage variable D is characterized by variation in Young's modulus E(D), as follows

(3.4)
$$E(D) = (1-D)E_H$$

If we suppose that variation E(D) is equal to the RDA modulus, (3.4) can be expressed by

(3.5)
$$(1 - D_{\rm cr})E_H = E_R = E_H \frac{1 + \varphi_{\rm cr} + \delta^2}{\left(1 + \varphi_{\rm cr}\right)^2 + \delta^2} \Rightarrow D_{\rm cr} = \frac{\left(1 + \varphi_{\rm cr}\right)\varphi_{\rm cr}}{\left(1 + \varphi_{\rm cr}\right)^2 + \delta^2}$$

In the case of quasi-static loading ($\delta \rightarrow 0$), the critical damage variable is given by

(3.6)
$$D_{\rm cr} = \frac{\varphi_{\rm cr}}{1 + \varphi_{\rm cr}} \Rightarrow \varphi_{\rm cr} = \frac{D_{\rm cr}}{1 - D_{\rm cr}}$$

Figure 4 shows one important relation between the critical damage variable and the critical creep coefficient, where $D_{\rm cr} \rightarrow 1$ when $\varphi_{\rm cr} \rightarrow \infty$. Hence, damage is described by a scalar $D_{\rm cr}$ taking on a value between 0 and 1.



FIGURE 4. Variation of critical damage variable with critical creep coefficient.

Also, using (3.3), linear variation of the critical damage variable $D_{\rm cr}$ with Poisson's ratio is obtained. It is consistent with the known phenomenon of Poisson's ratio increasing under compressive load due to micro cracking.

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(3.7)
$$D_{\rm cr} = 2 \mu$$

Eq. (3.4) provides a basis for the hypothesis of strain equivalence [12]. However, the physical meaning of D_{cr} is usually interpreted as a decrease in the effective load-carrying area (or effective resisting cross section area) due to void development [13]

(3.8)
$$A = (1 - D_{\rm cr})A_0$$

4. RDA study of uniaxial compression tests

4.1. RDA stress-strain curve. A compressive strength test was performed on a concrete cylinder with a diameter of $\phi = 150 \text{ mm}$ and a height of $l_0 = 300 \text{ mm}$, see Figure 5. The cylinder was made of SikaGrout®212 concrete, a product of the Swiss company Sika AG, which is primarily intended for structural reinforcement (principle 4, method 4.2 of EN 1504-9).



FIGURE 5. Tested cylinder SG1 before (left) and after (right) encumbering.

A concentric compressive load was applied using an automated hydraulic testing machine Controls Automax 5 with a 3000 kN capacity. Low-friction loading was applied through steel plates, one of them pivoting. The load on the tested specimen was measured with an oil pressure gauge with a 1 kN margin error. The local strains were measured by means of three strain gauges parallel to the direction of the applied load and centered at mid-height of the cylinder. The unstable descending part of the stress-strain curve was not measurable with the experimental setup used (load-controlled compression test). Loading Q was applied at rate c_{load} . However, the loading rate could not be established instantaneously, but the current increased from zero to a final constant value in a very short time Δt , see Figure 6.

Due to the change of the loading rate, the uniformly accelerated motion of load $(a = c_{\text{load}}/\Delta t)$, which acts on the concrete cylinder, must be analyzed as shown in Figure 7.

Let us determine the stresses σ_D in an arbitrary cross section at a distance x of the cylinder shown in Figure 7. The observed part moves with acceleration a, which means that a force equal to its mass times acceleration a must be included in the equilibrium equation

(4.1)
$$Q_D = \frac{Q + \gamma A_0 x}{g} a$$

Thus, the dynamic stresses σ_D acting in the sectional plane of the cylinder will balance not only the static load $Q + \gamma A_0 x$, but also the additional force Q_D

(4.2)
$$\sigma_D = \frac{Q + \gamma A_0 x}{A_0} + \frac{Q + \gamma A_0 x}{g A_0} a = \frac{Q + \gamma A_0 x}{A_0} \left(1 + \frac{a}{g}\right)$$

The ratio $(Q + \gamma A_0 x)/A_0$ is the static stress σ_S in the observed section of the cylinder. Therefore

(4.3)
$$\sigma_D = \sigma_S \left(1 + \frac{a}{g} \right)$$

i.e., the dynamic stress is equal to the static stress multiplied by the coefficient (1 + a/g).

The uniformly accelerated motion of load remains the same until the specimen fails, because the loading rate is constant. After the failure, the acceleration is lost and particles of the material are returned to the gravitational acceleration. Therefore, the acceleration coefficient f_{γ} , which must be used to adjust the specific weight of the material $\gamma = f_{\gamma} \cdot \gamma_g$ ($\gamma_g = \rho \cdot g$) in order to obtain its dynamic strength, is given by

(4.4)
$$f_{\gamma} = \frac{g}{g-a}$$

crete cylinder SG1.

It should be noted that this coefficient corresponds to the recommended increase of the measured elastic modulus [10]

(4.5)
$$E_H = 2.2 \cdot 10^4 [f_c/10]^{0.3}$$

 f_c is the measured strength of the standard concrete cylinder. The experimentally evaluated mechanical properties of cylinder SG1 at the time of the test (28-day strength) are listed in Table 1.

TABLE 1. Experimentally evaluated mechanical properties of con-

Concrete property	Test data
Density $[kg/m^3]$	2265.5
Elastic modulus [MPa]	30994
Poisson's ratio	0.185
Measured strength [MPa]	65.91



FIGURE 6. Increases and decreases of loading rate in very short time Δt .

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FIGURE 7. Stresses in cylinder under uniformly accelerated motion of load.

The structural-material characteristics of concrete cylinder SG1 are listed in Table 2.

The properties obtained at the limit of elasticity are presented in Table 3.

Depending on the acceleration coefficient f_{γ} it is possible to construct any number of curves. Two stress-strain curves are computed here using the properties from Table 3 and the four experimentally evaluated characteristics of the material given in Table 1 (see Figure 8). The first SG RDA-g curve is computed using the gravitational acceleration g ($f_{\gamma} = 1$) and can be treated as static

TABLE 2. Structural-material characteristics of concrete cylinder SG1.



TABLE 3. Structural-material properties of concrete cylinder SG1 at the limit of elasticity.

	RDA	Gravitational accel-	Uniformly accelerated motion of
	prediction	eration $\gamma_g = \rho \cdot g$	load $\gamma = f_{\gamma} \cdot \gamma_g, f_{\gamma} = 1.37$
λ_E	$\pi^2 \frac{i^3}{I} \frac{1}{\gamma \varphi^*}$	157.41	114.89
σ_E	$\frac{\pi^2 E_H}{\lambda_E^2} = \frac{\varphi^* E_H \gamma}{\lambda_E(\frac{i^3}{I})}$	12.35	23.17
\overline{K}_E	$\lambda_E \frac{i^3}{I} \frac{1}{E_H \gamma}$	0.047569	0.025344

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$$\sigma_S = \frac{1}{2 \cdot 0.047569} (\sqrt{1 + 4 \cdot 0.047569 \cdot 49196.83 \cdot \varepsilon} - 1),$$

for $\varepsilon_{\rm cr} = 0.003575 \Rightarrow \sigma_{{\rm cr}F} = 51.20$ MPa.

The dynamic SG RDA curve is computed using the acceleration coefficient $f_{\gamma} = 1.37$, which fits the percentage increase in measured elastic modulus of 37%, as recommended in [10]

$$\sigma_D = \frac{1}{2 \cdot 0.025344} (\sqrt{1 + 4 \cdot 0.025344 \cdot 49196.83 \cdot \varepsilon} - 1),$$

for $\varepsilon_{\rm cr} = 0.003575 \Rightarrow \sigma_{{\rm cr}F} = 65.88$ MPa.

As a consequence of the hypothesis of strain equivalence, the same values of peak strains follow from the static and dynamic stress-strain curves for peak stresses.

As it is shown in Figure 8, the results of dynamic SG RDA curve are in agreement with the experimentally evaluated values, especially beyond the limit of elasticity. It should be noted that limit of elasticity is the border from which RDA is developed.

Figure 8 also presents the static and dynamic RDA strain-stress curves calculated by (2.28) for different slenderness λ_j as explained in the Appendix.

$$\varepsilon_{S,j} = \frac{\sigma_{S\,\mathrm{cr}F,j}}{49196.83} (1 + \sigma_{S\,\mathrm{cr}F,j} \cdot 0.047569),$$

$$\varepsilon_{D,j} = \frac{\sigma_{D\,\mathrm{cr}F,j}}{49196.83} (1 + \sigma_{D\,\mathrm{cr}F,j} \cdot 0.025344),$$

$$\lambda_j = l_j/i, \quad (j = 1, 2, 3, \ldots)$$

Although the strain-stress curves are derived from the RDA failure curve using iterations, they are close to the stress-strain curves. It means that the theoretical stress-strain curves are unique. The RDA curve (inelastic buckling curve) shown in Figure 9 gives only an upper bound of inelastic critical stresses in the inelastic range, because the elastic modulus E_H and structural creep coefficient φ^* are used in (2.26). However, when stress exceeds the limit of elasticity the elastic modulus must be replaced by the RDA modulus given by (2.22) to obtain the failure stress. The RDA failure curve is obtained using the iterative procedure for all slenderness $\lambda_i (j = 1, 2, 3, \ldots)$ as explained in the Appendix. The analyzed concrete cylinder SG1 has slenderness $\lambda_0 = 8(l_0/\phi = 2)$, see Table 1. The lower slenderness limit $\lambda_D = 15(l_D = 56.25 \text{ cm}, l_D/\phi = 3.75)$ is found based on the intersection of the RDA failure curve and the horizontal line corresponding to the measured strength. The higher cylinders have smaller compressive strengths because of the size effect, see Figure 9. Consequently, the RDA results are in full accordance with the conclusions reported in [1], where for slenderness l_i/ϕ greater than 2 (and up to 4) a constant strength was measured.

The stress-strain curves are unique because they depend only on the four experimentally evaluated properties of the concrete. The specimen shape and size effect are included through the slenderness.

4.2. Critical parameters under peak stress and ultimate strain. Van Mier showed that localization of deformations occur in the post-peak regime in



FIGURE 8. Test data for stress-strain pairs and RDA results for stress-strain curves.

compression [14, 15]. Also, it has been clearly demonstrated by Hillerborg and co-workers in 1976 [16] that in tensile fracture of concrete strain localization occurs after stress has passed the peak point. Therefore, Hillerborg [17] suggested that in compression failure a description of the stress-deformations properties by means of a stress-strain diagram before the peak and a stress-deformation diagram after the peak may be expected to be a more realistic material model than just a simple stress-strain curve. Also, based on the Van Mier observations, Bažant [18] analyzed the post-peak stress-strain relation on the basis of a series coupling hypotheses. However, from the rheological point of view, the strain localization hypothesis means that strain at the peak point is permanently restrained. Hence, the relaxation of stress takes place as explained in [19]

(4.6)
$$\sigma_{\operatorname{Re} l.} = \frac{\sigma_{\operatorname{cr}}}{1 + \varphi_{\operatorname{cr} F}}$$



FIGURE 9. Quasi-static inelastic buckling curves for concrete cylinder SG1.

Let us consider the concrete cylinder at the lower slenderness limit $\lambda_D = 15$ $(l_D/\phi = 3.75)$. (2.26) gives the upper bound of the inelastic critical stress

$$\sigma_{\rm cr} = \frac{E_H \cdot f_\gamma \cdot \gamma_g \cdot \varphi^*}{\lambda_D (i^3/I)} =$$

$$= \frac{30994 \cdot 1.37 \cdot 2.2655 \cdot 10^{-3} \cdot 0.5873}{15 \cdot 0.02122} =$$

$$= \frac{30994 \cdot 0.0031 \cdot 0.5873}{15 \cdot 0.02122} = 177 \,\text{MPa}$$

The critical creep coefficient for the measured strength $f_c = 65.91 \,\mathrm{MPa}$ is as follows

$$\varphi_{\mathrm{cr}F} = \sigma_{\mathrm{cr}F} \cdot K_E = 65.91 \cdot 0.025344 = 1.67.$$

Obviously, the relaxation of stress is equal to the measured strength

$$\sigma_{\text{Re}\,l.} = \frac{177}{1+1.67} = 66 \,\text{MPa}$$

It means that the rheological-dynamical theory is in accordance with the hypothesis of strain localization, and the measured strength can only be confirmed by another test, i.e. a displacement controlled experiment. This paper is concerned with the development of a new model of inelastic material behavior of concrete in compression, alternative to other methods of nonlinear fracture mechanics or plasticity previously published in the literature. The model has already been developed for metals [8]. The model allows that a very complicated nonlinear problem may be solved as a simple linear dynamic one. It is important to note that, from the mathematical point of view, the creep coefficient is a global quantity, and therefore it permits to characterize structural behavior without the need to model the details of the actual failure mode of the specimen, which may vary from pure crushing to diagonal shear failure, or even to splitting, depending on its size-scale and slenderness [1]. The dynamic stress-strain curve is well suited for taking into account

the nonlinear response of concrete cylinder in the ascending branch and gives the measured strength. The corresponding peak strain for the above analyzed cylinder is $\varepsilon_{\rm cr} = 0.003575$. Consequently, the critical creep coefficient for the peak dynamic stress can be determined according to (2.27)

$$\varphi_{\mathrm{cr}F} = \sigma_{\mathrm{cr}F} K_E = 65.88 \cdot 0.025344 = 1.67$$

The critical damage variable for the peak dynamic stress can be determined according to (3.6)

$$D_{\rm cr}F = \frac{\varphi_{\rm cr}F}{1 + \varphi_{\rm cr}F} = \frac{1.67}{2.67} = 0.6255$$

In the end, from the secant stress-strain relation given by (3.4), the ultimate strain is given as follows

$$\sigma_{\rm cr}F = (1 - D_{\rm cr}F)E_H\varepsilon_{\rm cr}F \Rightarrow \varepsilon_{\rm cr}F$$
$$\varepsilon_{\rm cr}F = \frac{\sigma_{\rm cr}F}{(1 - D_{\rm cr}F)E_H} = \frac{65.88}{(1 - 0.6255)30994} = 0.005679$$

As a consequence of the hypothesis of strain equivalence, the same ultimate strain follows from the static stress-strain curve

$$\begin{aligned} \varphi_{\rm cr} F &= 51.2 \cdot 0.047569 = 2.44, \\ D_{\rm cr} F &= \frac{2.44}{3.44} = 0.7089, \\ \varepsilon_{\rm cr} F &= \frac{51.2}{(1 - 0.7089)30994} = 0.005679 \end{aligned}$$

4.3. Mode of failure. Concrete is a semi-brittle material which collapse under the peak stress σ_{crF} (measured strength). The effective resisting cross section area and the effective radius of the cylinder after the failure are shown in Figure 10 left. These failure properties can be determined according to (3.8)

$$\bar{A} = (1 - D_{\mathrm{cr}F})A_0 \Rightarrow \bar{r} = 7.5\sqrt{1 - 0.6255} = 4.59\,\mathrm{cm}$$

Consequently, the critical crack depth is equal to the difference between the radius of the damaged cross section area A_0 and the radius of the effective resisting cross section area \overline{A} .

$$a_{\rm cr}F = 7.5 - 4.59 = 2.91 \,\rm cm$$

Figure 10 right shows cylinder SG1 after the experiment, with the mode of failure clearly visible. The calculated value of the critical crack depth is in good agreement with the measured value (2.43 cm).

It is well known that the compression strength after an experiment does not vanish, but it tends to a residual value. According to the nature of the action explained in Section 4.1, the specimen is loaded dynamically. After the failure, the acceleration is lost in a very short time Δt , see Figure 6. The result of this process is a decrease in the dynamic compressive strength to the static value (SG RDA-g curve). However, because of the negative acceleration after the failure, a fictitious stress exists for a very short time Δt

(4.7)
$$\sigma_{\text{fictitious}} = \sigma_S \left(1 - \frac{a}{g} \right)$$



FIGURE 10. Mode of failure left, and measured value of critical crack depth right.

Therefore, the stress-strain curve ends as shown in Figure 11. It means that the softening branch exists. Consequently, the established difference between the dynamic and the static strength is the residual stress level remaining in the material, and it is a consequence of the uniformly accelerated motion of load during the examination of compressive strength

(4.8)
$$\sigma_{\text{residual}} = \sigma_{\text{cr}F} - \sigma_{\text{cr}}$$

The residual stress is transmitted through the effective resisting area of the cylinder that remains after its failure, see Figure 10. For the above analyzed concrete cylinder $\sigma_{\text{residual}}/f_c = 0.212\%$ is obtained. Equations of the uniformly accelerated motion of load and acceleration coefficient give a = 0.27g, $f\gamma = 1.37$.

In this paper damage mechanics is referred to for the post-peak regime only. The damage parameter D_{crF} given in (3.8) is defined as a scalar, with the effective resisting area \bar{A} corresponding to the damaged area A_0 , as presented in Figure 10. This concept pertains because the total inelastic strain, which is free after the failure for a very short time Δt (Figure 6), is really the ultimate strain caused by the critical failure stress. From the secant stress-strain relation given by (3.4), the ultimate strain is given as presented in the diagram shown in Figure 11

(4.9)
$$\varepsilon_{\mathrm{cr}F} = \frac{\sigma_{\mathrm{cr}F}}{(1 - D_{\mathrm{cr}F})E_H} = \frac{\sigma_{\mathrm{cr}F}}{E_H}(1 + \varphi_{\mathrm{cr}F}) = \frac{\sigma_{\mathrm{cr}F}}{E_{RF}}$$

Also, from the strain-stress curve given by (2.28), the ultimate strain is obtained as follows

(4.10)
$$\varepsilon_{\rm cr} = \frac{\sigma_{\rm cr}F}{E_H(1+\varphi^*)} (1+\varphi_{\rm cr}F) \Rightarrow \varepsilon_{\rm cr}F = \varepsilon_{\rm cr}(1+\varphi^*)$$

It means that the total inelastic strain is really the ultimate strain and the postpeak behavior depends on the pre-peak behavior. The proposed RDA model relies on global parameters to connect pre-peak behaviour with post-peak behaviour, and should only be considered holistically. That is precisely what distinguishes it from the Hillerborg model [17], where two independent curves are used for analysis, a stress-strain curve for the ascending part (before the peak stress), and a stressdeformation curve for the descending part. The damaged material of the cylinder is assumed to be able to transfer compressive stress between the overlapping surfaces, see Figure 10 left. Concerning such stresses, they are assumed to be a decreasing function of the interpenetration w [4]. The critical value of the interpenetration displacement is determined so that areas F_{II} and F_{III} under the softening branch (including the elastic unloading portion F_I), as shown in Figure 11, are equal to the crushing energy G_C . When the residual stress is reached in load-controlled experiments, the tested specimen is no longer under loading, and the ultimate strain cannot increase. Accordingly, two key failure properties can be calculated. The critical value of the interpenetration displacement shown in Figure 11 is as follows

(4.11)
$$w_{\rm cr} = [\varepsilon_{\rm cr}F - \varepsilon_{\rm cr} + (\sigma_{\rm cr}F - \sigma_E)/E_H]l_0$$

The crushing energy can be calculated as presented in the diagram shown in Figure $11\,$

(4.12)
$$G_C = (F_I + F_{II} + F_{III})l_0$$
$$= \left[\frac{1}{2}\frac{\sigma_{\rm cr}^2}{E_H} + \frac{1}{2}\sigma_{\rm residual}(\varepsilon_{\rm cr}F - \varepsilon_{\rm cr}) + (\sigma_{\rm cr} - \sigma_{\rm residual})(\varepsilon_{\rm cr}F - \varepsilon_{\rm cr})\right]l_0$$

For the above analyzed concrete cylinder is obtained

$$\begin{split} w_{\rm cr} &= [0.005679 - 0.003575 + (65.91 - 23.17)/30994] \cdot 300 = 1.04\,{\rm mm} \\ G_C &= (0.042289 + 0.015475 + 0.076775) \cdot 300 = 40.36\,{\rm N/mm} \end{split}$$



FIGURE 11. Stress-strain curve, critical interpenetration displacement and crushing energy.

It is well known that the sharp bend at the peak stress in the uniaxial stressstrain diagram is not realistic. Because of that a decrease in the dynamic compressive strength to the static value, which leads to the linear post-peak softening, is a logical explanation based on load-controlled experiments (Figure 6). Hence, the calculated values of the two key parameters are in agreement with the observed test values for similar concrete samples that can be found in literature, although they were obtained using displacement controlled experiments. Also, it should be noted that the critical interpenetration displacement corresponds to $\sigma_{\text{residual}} = 0.2\sigma_{\text{crF}}$ according to the experimental evidence [4]. However, it is to be noted that these MILAŠINOVIĆ

two key failure properties essentially describe the response of the central zone of the cylinders only, which is generally accepted to be subjected to a near-uniform uniaxial compressive stress, in contrast to the complex and indefinable compressive state of stress imposed on the end zones by frictional restraints due to the interaction between the specimen and loading device [3]. From the findings which have been presented the static loading may be simulated if uniformly acceleration motion of load is zero (a = 0). Hence, $\sigma_D = \sigma_S$ and $\sigma_{\text{fictitious}} = \sigma_S$. Also, the acceleration coefficient $f_{\gamma} = 1$ and RDA-g curve valid as shown in Figure 12. The residual stress level is zero. It means that presented stress-strain diagram for concrete in compression is the same type as proposed in many current codes only for static loading. The expression given in (3.8) implies that $D_{\rm cr}F = 0.6255$ for the peak dynamic stress corresponds to the damage state with the effective resisting area of the cylinder that remains after its failure. However, $D_{crF} = 0.7089$ for the peak static stress is a critical value which corresponds to the failure of the cylinder in two parts, because the residual stress level is zero. It is important to note that according to Lemaitre [20], the critical value of the damage variable lies in the range $0.2 \leq D_{\rm crF} \leq 0.8$ for metals. Hence, two key failure properties under static loading can be calculated as follows

$$w_{\rm cr} = [\varepsilon_{\rm cr}F - \varepsilon_{\rm cr} + (\sigma_{\rm cr} - \sigma_E)/E_H]l_0$$

= [0.005679 - 0.003575 + (51.2 - 12.35)/30994] \cdot 300 = 1.01 mm

$$G_C = (F_I + F_{II})l_0 = \left[\frac{1}{2}\frac{\sigma_{\rm cr}^2}{E_H} + \sigma_{\rm cr}(\varepsilon_{\rm cr}F - \varepsilon_{\rm cr})\right]l_0$$

= (0.042289 + 0.107725) \cdot 300 = 45 N/mm



FIGURE 12. Stress-strain curve, critical interpenetration displacement and crushing energy under static loading.

5. Comparative analysis for five concrete compositions

5.1. Experimental tests and model verifications. An experimental investigation was carried out to explain the compression behavior of standard concrete cylinders with a strength range of 20 - 80 MPa. The presented experimental re-

search was conducted at the Materials and Structures Testing Laboratory of the Faculty of Civil Engineering in Subotica, Serbia. A series of tests were performed on five concrete compositions. The measured values of concrete density, elastic modulus, Poisson's ratio and compression strength are listed in Table 4.

Type	Concrete	Elastic	Poisson's	Compression
of	density	modulus	ratio	strength
concrete	$[kg/m^3]$	[MPa]		[MPa]
HCS-04	2289	49211	0.175	76.82
SG	2265.5	30994	0.185	65.91
NSC	2325	28700	0.217	57.27
CC-1	2228.6	27404	0.180	38.90
C20	2125	23421	0.165	25.00

TABLE 4. Experimentally evaluated mechanical properties for five concrete compositions.

HCS-04: Repair mortar cement, polymers, minerals, and chemicals and fillersbased; SG: SikaGrout®212 repair mortar; NSC: Normal-strength concrete; CC-1: Normal-strength concrete; C20: Concrete strength class.

TABLE 5. Numerically defined dynamic and static RDA curves for five concrete compositions.

Dynamic RDA curve				
HCS-04	$\frac{1}{2 \cdot 0.017054} (\sqrt{1 + 4 \cdot 0.017054 \cdot 75709.23 \cdot \varepsilon} - 1)$			
SG	$\frac{1}{2 \cdot 0.025344} (\sqrt{1 + 4 \cdot 0.025344 \cdot 49196.83 \cdot \varepsilon} - 1)$			
NSC	$\frac{1}{2 \cdot 0.019906} (\sqrt{1 + 4 \cdot 0.019906 \cdot 50707.16 \cdot \varepsilon} - 1)$			
CC-1	$\frac{1}{2 \cdot 0.030928} (\sqrt{1 + 4 \cdot 0.030928 \cdot 42818.75 \cdot \varepsilon} - 1)$			
C20	$\frac{1}{2 \cdot 0.045456} (\sqrt{1 + 4 \cdot 0.045456 \cdot 34956.72 \cdot \varepsilon} - 1)$			
	Static RDA curve			
HCS-04	$\frac{1}{2 \cdot 0.032009} (\sqrt{1 + 4 \cdot 0.032009 \cdot 75709.23 \cdot \varepsilon} - 1)$			
SG	$\frac{1}{2 \cdot 0.047569} (\sqrt{1 + 4 \cdot 0.047569 \cdot 49196.83 \cdot \varepsilon} - 1)$			
NSC	$\frac{1}{2 \cdot 0.037360} (\sqrt{1 + 4 \cdot 0.037360 \cdot 50707.16 \cdot \varepsilon} - 1)$			
CC-1	$\frac{1}{2 \cdot 0.058048} (\sqrt{1 + 4 \cdot 0.058048 \cdot 42818.75 \cdot \varepsilon} - 1)$			
C20	$\frac{1}{2\cdot 0.085316}(\sqrt{1+4\cdot 0.085316\cdot 34956.72\cdot \varepsilon}-1)$			

The HCS-04 cylinder was made of a high-quality, two-component concrete mix, commercially available by the name Polimag®HK-04. The liquid component contains water, cement polymer and plasticizer, while the powder component contains cement, crushed carbonaceous stone aggregates, powdered filler and minerals. These components were mixed in a concrete mixer and no additional materials were added. The SG type concrete was a high strength, low shrinkage, expanding material commercially known as SikaGrout®212. It is a powdered concrete mix

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which contains cement, crushed stone aggregate and powdered cement additives. In accordance to the manufacturer recommendations, fresh concrete was prepared by adding 3.7 liters of drinking water to one 28 kg bag of SikaGrout. The mix proportions for other concrete compositions were: NSC (portland cement (PC) CEM II/B-M: 500 kg/m³; water: 200 kg/m³; fine aggregate: 991 kg/m³; coarse aggregate: 633 kg/m³), CC-1 (PC CEM II/A-M: 445 kg/m³; water: 250 kg/m³; fine aggregate: 465 kg/m³; coarse aggregate: 1100 kg/m³) and C20 (PC CEM II/B-M: 395 kg/m³; water: 280 kg/m³; sand: 420 kg/m³; coarse aggregate: 1165 kg/m³). The concrete mix was designed for compressive cylinder strength f_c at 28 days of approximately 20-80 MPa. The concrete properties at the time of the test are presented in Table 4. The numerically defined dynamic and static RDA curves are presented in Table 5. The acceleration coefficient $f_{\gamma} = 1.37$ was used in the computation for all concrete compositions.

The experimentally evaluated and numerically computed dynamic stress-strain curves shown in Figure 13 are in excellent agreement beyond the limit of elasticity, because the limit of elasticity is the border from which the rheological-dynamical theory is developed.

It was observed that the mechanical response of concrete in uniaxial compression can be well explained through the measured strengths. Figure 14(1) shows that all recommended elastic modules are between 30 and 40 percent greater than the measured values, except for concrete with the maximum strength, where the measured modulus is higher than recommended. Consequently, it is proved in this study that the presented procedure for the computation of the stress-strain curves is more objective than standard computation involving elastic modulus adjustment procedures. Figure 14(2) shows that different strengths can be obtained for materials of different densities, with strength obviously growing with the growth of concrete density. This opens up possibilities for new studies on the basis of which two important questions may be answered. First, what is the relationship between density and strength if the porosity (density) of the same concrete mixtures is varied and second, what is this relationship if the concrete density is changed by varying the aggregate density? An adequate answer to these questions is possible to obtain with the application of the proposed RDA model because the model uses four measured mechanical properties, density, elastic modulus, Poisson's ratio and compressive strength, which are functionally linked in the proposed model. Figure 15(1) presents the variation of the measured Poisson's ratio with strength, which coincides with the variation of the computed VE creep coefficient shown in Figure 15(2).

Figure 16(1) illustrates the variation of the elastic stress with the function at its minimum for the above analyzed concrete cylinder, while the one (2) illustrates the variation of the structural-material constant, with the function at its maximum for the same specimen.

Figure 17(1) shows the logical linear relationship between the dynamic and static strengths. However, the peak strain shown in Figure 17(2) is a function at its maximum value for the above analyzed concrete cylinder.



FIGURE 13. Comparison of test data for stress-strain pairs and dynamic RDA curves for five concrete compositions.

The next important aspect of this study is the computation of the critical parameters under peak stress and ultimate strains (see Section 4.2). Figures 18 and 19 illustrate the same type of function with maximum values for the above analyzed concrete cylinder.

The failure mode of the standard concrete cylinder is discussed in Section 4.3. The relevant global properties are: effective resisting cross section area, critical crack depth, critical interpenetration displacement and residual stress level. These properties are shown in Figures 20 and 21. The effective resisting cross section area for the above analyzed concrete cylinder is at its minimum in the presented function, while the other properties are at their maximum. The calculated values of critical crack depths are in excellent agreement with the measured value (2.43 cm), see Figure 10(2). It should be noted that critical crack depths are measurable after the failure of tested cylinders in the load-controlled compression test and because of that they can be used for experimental verification of RDA approach. When the interpenetration displacement reaches the critical value, the concrete cylinder can



FIGURE 14. Comparison of measured elastic modulus with recommended values (1), and variation of concrete density with varying strength (2).

transfer only constant residual stress through the effective resisting cross section area. According to the experimental and theoretical investigation in this study, the critical interpenetration displacement corresponds to $\sigma_{\text{residual}} = 0.2124\sigma_{\text{cr}F}$, see Figure 21(2).

The common key properties that characterize the global failure mode of the concrete cylinder are the crushing energy and brittleness ratio, see Figure 22. The growth of the crushing energy with the growth of strength shown in Figure 22(1) is logical and expected. However, the presented function is at its maximum in the case of the concrete cylinder analyzed in Section 4.3, although one of the specimens has greater measured strength. Obviously, the reason lies in the fact that the analyzed cylinder has greater ultimate strain, as shown in Figure 19(2).



FIGURE 15. Variation of Poisson's ratio with strength (1), and VE creep coefficient with strength (2).

The function of the brittleness ratio is shown in Figure 22(2), with the function minimum occurring for the above analyzed concrete cylinder. This material characteristic is closely related to the ultimate strain because ductile materials have a lower brittleness ratio.

It is important to emphasize that the measured concrete properties are characterized by much higher dispersion than the relevant parameters which characterize the failure mode of concrete cylinders, see Table 6.

Although the concrete properties measured in the case of the SG cylinder are not the best, the most favorable failure characteristics are obtained if the criterion of the brittleness ratio is adopted as authoritative. On the other hand, the MB20 concrete cylinder test data are least favorable, and this is where the brittleness ratio



FIGURE 16. Variation of elastic stress with strength (1), and structural-material constant with strength (2).

has a maximum value. This means that this type of concrete is the worst. Hence, it is obvious that an optimization approach to find the best concrete composition may be useful.

5.2. Model parameters identification. The presented theory describes the critical stress-strain response of material in compression before the peak for five different concrete compositions, which are experimentally verified. This approach covers also the degradation of stiffness and strength with limited ductility in the post-peak regime, because it combines damage mechanics. Consider the SG concrete cylinder, whose measured properties are given in Table 1 (concrete density:



FIGURE 17. Variation of dynamic and static strengths (1), and peak strain and strength (2).

 $\rho = 2265.5 \text{ kg/m}^3$; elastic modulus: 49211 MPa; Poisson's ratio: 0.185; compression strength: $f_c = 65.91 \text{ MPa}$). The acceleration coefficient is $f_{\gamma} = 1.37$. The model parameters identification where the structural creep coefficient is the main parameter gives

$$\varphi^* = \frac{2 \cdot \mu}{1 - 2 \cdot \mu} = \frac{2 \cdot 0.185}{1 - 2 \cdot 0.185} = 0.5873$$

 $E(0) = (1 + \varphi^*) \cdot E_H = (1 + 0.5873) \cdot 30994 = 49196.83 \text{ MPa}$

$$\lambda_E = \pi^2 \frac{i^3}{I} \frac{1}{f_\gamma \cdot \gamma_g \cdot \varphi^*} = \pi^2 \cdot 0.02122 \frac{1}{1.37 \cdot 2.2655 \cdot 10^{-3} \cdot 0.5873} = 114.89$$



FIGURE 18. Variation of critical creep coefficient and strength (1), and critical damage variable and strength (2).

$$K_E = \lambda_E \frac{i^3}{I} \frac{1}{E_H \cdot f_\gamma \cdot \gamma_g} = 114.89 \cdot 0.02122 \frac{1}{30994 \cdot 1.37 \cdot 2.2655 \cdot 10^{-3}} = 0.025344$$

The peak compression strain may be obtained from the measured strength using (2.28)

$$\varepsilon_{\rm cr} = \frac{f_c}{E(0)} (1 + f_c K_E) = \frac{65.91}{49196.83} (1 + 65.91 \cdot 0.025344) = 0.003577$$

The global mode of failure is determined by the critical parameters under the peak stress as well as ultimate strain

$$\varphi_{\rm cr}F = f_c K_E = 65.91 \cdot 0.025344 = 1.67$$



FIGURE 19. Variation of critical Poisson's ratio and strength (1), and ultimate strain and strength (2).

$$D_{\rm cr}F = \frac{\varphi_{\rm cr}F}{1 + \varphi_{\rm cr}F} = \frac{1.67}{2.67} = 0.6255$$
$$\varepsilon_{\rm cr}F = \frac{f_c}{(1 - D_{\rm cr}F)E_H} = \frac{65.91}{(1 - 0.6255)30994} = 0.005679$$

The area below the stress-strain curve in Figure 11 represents the crushing energy per unit area for standard concrete cylinders, which can be calculated from

(5.1)
$$G_C = 93 \cdot \frac{f_c^2}{E_H} + 204 \cdot f_c \cdot (\varepsilon_{\rm cr}F - \varepsilon_{\rm cr})$$

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FIGURE 20. Variation of effective resisting area and strength (1), and critical crack depth and strength (2).

with four non-dimensional constants valid for all concrete compositions

(5.2)

$$\sigma_{\text{residual}}/f_c = 0.2124$$

 $\sigma_S/f_c = 0.7876$
 $a/g = f_c/\sigma_S - 1 = 0.27$
 $f_{\gamma} = \frac{1}{1 - a/g} = 1.37$

Owing to the present, fundamentally new investigations, our understanding of concrete failure in compression may be achieved from both theoretical and experimental viewpoints. The residual stress level is what principally distinguishes it



FIGURE 21. Variation of critical interpenetration displacement and strength (1), and $\sigma_{\text{residual}}/\sigma_{\text{cr}F}$ ratio and strength (2).

from tensile fracture, but it is a consequence of the uniformly accelerated motion of load during the examination of compressive strength. The main concrete properties which must be used for the numerical calculation of the crushing energy by (5.1) are given in Table 7.

6. Conclusions

The proposed approach for global failure analysis of concrete in compression combines the RDA and damage mechanics. The RDA modulus is used to obtain one simple continuous modulus function and a stress-strain curve. The key global quantities, such as the creep coefficient, Poisson's ratio and damage variable, are functionally related. The critical values of the creep coefficient and damage variable under the peak stress (measured strength) are used to describe the global



FIGURE 22. Variation of crushing energy and strength (1), and brittleness ratio and strength (2).

failure mode of concrete cylinders. The ultimate strain is formulated in the postpeak regime only, using the secant stress-strain relation from damage mechanics. The softening branch exists as explained in Section 4.3. The residual stress is a consequence of uniformly accelerated motion of load during the examination of compressive strength. The present study analyzes experimentally five different concrete compositions before the peak stress. The dynamic and static stress-strain curves are computed taking into account four measured properties of concrete. The same acceleration coefficient for all concrete compositions is used to obtain the dynamic curves, which are in excellent agreement with the values measured beyond the limit of elasticity. The results are also in full compliance with the conclusions reported by RILEM TC 148-SSC [1]. The measured concrete properties have much higher dispersion than relevant parameters which characterize the failure mode. Hence,

Identified model parameter	HCS-04	SG	NSC	CC-1	C20
Density			max.		min.
Elastic modulus	max.				min.
Poisson's ratio			max.		min.
Compression strength	max.				min.
Structural creep coefficient			max.		min.
Critical creep coefficient		max.			min.
Critical Poisson's ratio		max.			min.
Critical damage variable		max.			min.
Critical crack depth		max.			min.
Peak compression strain		max.			min.
Ultimate compression strain		max.			min.
Interpenetration displacement		max.			min.
Crushing energy		max.			min.

TABLE 6. Influence of identified model parameters.

TABLE 7. Main concrete properties and crushing energies calculated with four non-dimensional constants valid for all concrete compositions.

Type of	f_c	E_H	$\varepsilon_{ m cr}$	$\varepsilon_{\mathrm{cr}F}$	G_C
concrete	$[N/mm^2]$	$[N/mm^2]$			[N/mm]
HCS-04	76.82	49211	0.00235	0.00361	30.84
SG	65.91	30994	0.003577	0.00568	41.32
NSC	57.27	28700	0.0024	0.00427	32.48
CC-1	38.90	27404	0.002	0.00313	14.08
C20	25.00	23421	0.001525	0.00228	6.33

it is obvious that an optimization approach to find the best concrete properties may be useful. Finally, the crushing energy per unit area for standard concrete cylinders is calculated, with four non-dimensional constants valid for all concrete compositions. The calculated crushing energies are in agreement with the observed test values of similar concrete specimens that can be found in literature, which were obtained in displacement controlled experiments.

7. Appendix

RDA strain-stress curve Following Milašinović [6], the RDA modulus function $E_R(\sigma_{\rm cr})$ may be used to obtain compressive strength $\sigma_{{\rm cr}F}$, which occurs when the current critical stress does not change the RDA modulus in the next iteration. For the computed value of $\sigma_{\rm cr}^{(0)}$, according to (2.24)

(7.1)
$$\sigma_{\rm cr}^{(0)} = \frac{E_H}{\lambda_0 \frac{i^3}{I} \frac{1}{\gamma \varphi^*}}$$

an appropriate $E_R^{(1)}$ may be calculated according to (2.22)

(7.2)
$$E_R^{(1)} = \frac{E_H}{1 + \sigma_{\rm cr}^{(0)} K_{\varphi}}$$

and then a new $\sigma_{\rm cr}^{(1)}$ recalculated

(7.3)
$$\sigma_{\rm cr}^{(1)} = \frac{E_R^{(1)}}{\lambda_0 \frac{i^3}{I} \frac{1}{\gamma \varphi^*}}$$

This iterative procedure must be performed until there is convergence to stress σ_{crF}

(7.4)
$$\sigma_{\rm cr}^{(n)} = \sigma_{\rm cr}F = \frac{E_R^{(n)}}{\lambda_0 \frac{i^3}{I} \frac{1}{\gamma \varphi^*}}$$

The above stress represents the compressive strength of the concrete cylinder. The RDA modulus is as follows

(7.5)
$$E_{R}^{(n)} = \frac{E_{H}}{1 + \sigma_{\rm cr}^{(n-1)} K_{\varphi}} = \frac{E_{H}}{1 + \frac{E_{R}^{(n-1)}}{E_{H}} \frac{\lambda_{E}}{\lambda_{0}} \varphi^{*}}$$

This iterative procedure may be repeated for different slenderness ratios $\lambda_j = l_j/i(j = 1, 2, 3, ...)$ and the RDA failure curve can be drawn (see the RDA failure and RDA-g failure curves shown in Figure 9. In the end, compressive strengths $\sigma_{\rm cr} F_j$ may be used to obtain the RDA strain-stress curve using (2.28)

(7.6)
$$\varepsilon_j = \frac{\sigma_{\operatorname{cr} Fj}}{E_{R,j}(0)} = \frac{\sigma_{\operatorname{cr} Fj}}{E(0)} (1 + \varphi_{\operatorname{cr} Fj}) = \frac{\sigma_{\operatorname{cr} Fj}}{E_H(1 + \varphi^*)} (1 + \sigma_{\operatorname{cr} Fj} K_{\varphi})$$

Figure 8 presents strain-stress curves named as the RDA strain-stress curve and the RDA strain-stress curve-g, which are obtained using (7.6).

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References

- J.G.M. Van Mier, S.P. Shah, M. Arnaud, et al., Strain-softening of concrete in uniaxial compression, Mater. Struct. 30(4) (1997), 195–209.
- M.D. Kotsovos Effect of testing techniques on the post-ultimate behavior of concrete in compression, Materiaux et Constructions 16(91) (1983), 3–12.
- F. Indelicato, M. Paggi, Specimen shape and the problem of contact in the assessment of concrete compressive strength, Mater. Struct. 41(2) (2008), 431–441.
- A. Carpinteri, M. Corrado, M. Paggi, An analytical model based on strain localisation for the study of size-scale and slenderness effects in uniaxial compression tests, Strain 47 (2011), 351–362.
- M. A. Youssef, M. Moftah, General stress-strain relationship for concrete at elevated temperatures, Eng. Struct. 29 (2007), 2618–2634.

- D. D. Milašinović, Rheological-dynamical analogy: prediction of buckling curves of columns, Int. J. Solids Struct. 37(29) (2000), 3965–4004.
- <u>,</u> Rheological-dynamical analogy: modeling of fatigue bahavior, Int. J. Solids Struct. 40(1) (2003), 181–217.
- <u>,</u> Rheological-dynamical analogy: visco-elasto-plastic behaviour of metallic bars, Int. J. Solids Struct. 41(16-17) (2004), 4599–4634.
- A. Carpinteri, G. Lacidogna, N. Pugno, Structural damage diagnosis and life-time assessment by acoustic emission monitoring, Eng. Fract. Mech. 74(1-2) (2007), 273–289.
- Comité Euro-International du Béton (1993) CEB-FIB Model Code 1990, Thomas Teford Ltd, Lausanne, CEB Bulletin No. 213/214.
- X. Chen, S. Wu, J. Zhou, Experimental study and analytical formulation of mechanical behavior of concrete, Construction and Buildings Materials 47 (2013), 662–670.
- J. Lemaitre, J. L. Chaboche, Aspect phénoménologique de la rupture par endommagement, Journal de Mécanique Appliqué 2 (1978), 317–365.
- L. M. Kachanov, Introduction to Continuum Damage Mechanics, Martinus Nijhoff, Dordrecht, 1986.
- J. G. M. Van Mier, Multiaxial strain-softening of concrete part I: Fracture, Mater. Struct. 19(3) (1986), 179–190.
- Multiaxial strain-softening of concrete part II: Load-Histories, Mater. Struct. 19(3) (1986), 190–200.
- A. Hillerborg, M. Modeer, P. E. Petersson, Analysis of crack formation and crack growth in concrete by means of fracture mechanics and finite elements, Cement and Concrete Research 6(6) (1976), 773–781.
- A. Hillerborg, Fracture mechanics concepts applied to moment capacity and rotational capacity of reinforced concrete beams, Eng. Fract. Mech. 35(1/2/3) (1990), 233–240.
- Z. P. Bažant, Identification of strain-softening constitutive relation from uniaxial tests by series coupling model for localization, Cement and Concrete Research 19(6) (1989), 973–977.
- D. D. Milašinović, Fatigue crack growth and failure of inelastic rods based on rheologicaldynamical analogy, Int. J. Fatigue 33(3) (2011), 372–381.
- 20. J. Lemaitre, How to use damage mechanics, Nucl. Eng. Des. 80 (1984), 233-245.

РЕОЛОШКО-ДИНАМИЧКИ КОНТИНУУМСКИ МОДЕЛ ОШТЕЋЕЊА БЕТОНА ИЗЛОЖЕНОГ ЈЕДНОАКСИЈАЛНОМ ПРИТИСКУ И ЊЕГОВА ЕКСПЕРИМЕНТАЛНА ПОТВРДА

РЕЗИМЕ. У раду је представљен нови модел за предвиђање одзива бетона изложеног једноаксијалном притиску. Предложени приступ који је назван реолошко-динамички континуумски модел оштећења материјала комбинује реолошко-динамичку аналогију и механику оштећења. У оквиру овог приступа функционално су повезани књучни параметри континуума, као што су Поасонов кеофицијент и скаларна варијабла оштећења. За одређивање облика и врсте лома узорка изложеног притиску кориштене су критичне вриједности поасоновог коефицијента и скаларне варијабле оштећења при напону који одговара чврстоћи на притисак. Гранична деформација је одређена у пост-критичном подручју кориштењем секантне везе напон-деформација из механике лома. Приказани су подаци за пет бетонских мјешавина добивених експерименталним истраживањима. Основна разлика између лома узорка изложеног притиску и лома узорка изложеног затезању је заостали напон који је посљедица једнолико убрзаног кретања оптерећења током испитивања чврстоће на притисак. Критично помјерање односно међусобно продирање у материјалу и енергија лома при притиску су добивени теоријски на бази концепта глобалне анализе лома.

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