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A NOTE ON THE PROOF OF BERTRAND'S THEOREM

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Abstract. In this paper we fill a common gap in the proof of Bertrand' theorem present both the in Bertrand's original paper *Théorème relatif au movement d'un point attiré vers un centre fixe* and in the Arnold's book *Mathematical methods of classical mechanics*, by providing missing details which pertain to the problem of how to single out elastic and gravitational potentials among the power law ones.

1. Introduction

Motion of a particle with a unit mass in a central field is described by the Newton's equations

$$\ddot{\vec{r}} = -\frac{\partial U}{\partial \vec{r}}, \tag{1.1}$$

where $\vec{r} = \vec{r}(t) \in \mathbb{R}^3$ is the radius vector of the particle at time t, and $U = U(r)$ is the potential energy or simply the potential; here $r = |\vec{r}|$ is the Euclidean norm of the vector $\vec{r}$. It is a common fact that every motion in a central field actually occurs in a fixed plane. If we denote by π the plane in which a given motion occurs, then we can write $\vec{r} = r(\cos\varphi\,\vec{e_1} + \sin\varphi\,\vec{e_2})$, where $\vec{e_1}, \vec{e_2}$ is an orthonormal basis of π, and (r, φ) are polar coordinates in this plane. Let us denote $\vec{e_r} = \frac{\vec{r}}{r} = \cos\varphi\,\vec{e_1} + \sin\varphi\,\vec{e_2}$. Then (1.1) can be rewritten as

$$\ddot{\vec{r}} = -U'(r)\vec{e_r}. \tag{1.2}$$

If $\vec{e_\varphi} = -\sin\varphi\,\vec{e_1} + \cos\varphi\,\vec{e_2}$, then $\dot{\vec{e_r}} = \dot\varphi\,\vec{e_\varphi}$ and $\dot{\vec{e_\varphi}} = -\dot\varphi\,\vec{e_r}$. Therefore, $\vec{r} = r\vec{e_r}$ and (1.2) imply

$$\ddot r - r\dot\varphi^2 = -U'(r), \quad 2\dot r\dot\varphi + r\ddot\varphi = 0. \tag{1.3}$$

From the second equation in (1.3) follows

$$r^2\dot\varphi = M,$$

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for a certain constant M. The first equation in (1.3) we transform into

$$\ddot{r} = -V'(r), \tag{1.4}$$

where $V(r) = U(r) + \frac{M^2}{2r^2}$ is *the amended potential*. From (1.4) follows that the energy of the system (1.1) is a constant of the motion:

$$\frac{1}{2}\dot{r}^2 + V(r) = E.$$

Our considerations will have much to do with circular orbits: these are solutions of (1.1) with $r = \rho$ constant. From (1.4) we deduce that $V'(\rho) = 0$ is necessary and sufficient condition for $r = \rho$ to be a circular orbit. Assume that $V''(\rho) > 0$. In [**3**, subsections 12.6 and 12.7] is developed a theory which asserts that for a given energy level $E > V(\rho)$ close to $V(\rho)$, the solutions of (1.4) periodically oscillate in the interval $[r_{\min}, r_{\max}] \ni \rho$ (see Figure 1). The position of the particle, when $r = r_{\min}$ is called a *pericenter* and for $r = r_{\max}$ an *apocenter*. The angle Φ between two consecutive pericenters and apocenters, the so–called *apsidal angle*, is given by (see [**2**])

$$\Phi = \int_{r_{\min}}^{r_{\max}} \frac{M/r^2}{\sqrt{2(E - V(r))}}\, dr. \tag{1.5}$$

A simple but fundamental observation is that an orbit of (1.1) is closed if and only if $\Phi/\pi \in \mathbb{Q}$.

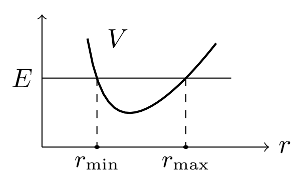

FIGURE 1

Already Newton knew that all bounded orbits for elastic and gravitational potentials are ellipses (see [**9**]). However, if there are other central potentials which give rise to closed orbits under arbitrary initial conditions (provided they are bounded) had been unknown for almost two centuries until Bertrand in 1873 published his paper [**4**] in which he proved that the answer was no. This assertion is today known as *Bertrand's theorem*.

A lot of effort has been spent to extend Bertrand's theorem to the geometries and spaces other than the Euclidean three–dimensional one: for example, in [**8**] it is proved for the Lobachevsky and spherical geometry, and in [**5**], [**10**] for surfaces of revolution. Let us stress that all these generalizations are based upon Bertrand's original proof. For further development in this direction see [**11**].

Some authors used quite different tools to handle the problem, in particular perturbation theory of integrable Hamiltonian systems. For example, Albouy [**1**] and Féjoz, Kaczmarek [**6**] applied Birkhoff normal forms calculated along circular orbits.

Bertrand's original proof can be divided into three steps: first, he calculated the limit of the apsidal angle when the orbit approaches the circular one, and then employing continuous dependence of the apsidal angle on orbits he established its independence on energy, which consequently implies that the angle is equal to the aforementioned limit. In the second step is used the fact that the apsidal angle is independent on momentum as well, and, as a consequence, one arrives to the power law for the potential: $U(r) = ar^\alpha$. The third step deals with the problem of how to single out the gravitational and elastic potentials among the power law ones. Bertrand resolved this step essentially by taking some special limits in the initial form of the apsidal angle. This step of the Bertrand's proof is also treated in Arnold's book [**2**]. However, in both cases details lack and it turns out that they are not so easy to prove. This is exactly the purpose of the paper: to provide some details missing in [**4**] and [**2**] in the proof of this step. It is the content of Lemma (2.1).

An original approach to the third step based on the fractional calculus can be found in [**7**].

2. Formulation and proof of Lemma

LEMMA 2.1. *If $U(r) = ar^\alpha$ and $a > 0$, then*

$$\lim_{E\to\infty} \Phi = \frac{\pi}{2},$$

for all $\alpha > 0$.

PROOF. Let us introduce a new variable x by

$$\frac{M}{r} = x.$$

Define the function W by $W(x) = V(r)$, that is

$$W(x) = \frac{1}{2}x^2 + U\Big(\frac{M}{x}\Big) = \frac{1}{2}x^2 + \frac{1}{2}kx^{-\alpha},$$

where $k = 2aM^\alpha$. Let $x_1 = M/r_{\max}$ and $x_2 = M/r_{\min}$. Following [**2**] we apply the substitution $x = yx_2$ in (1.5) and get

$$\Phi = \int_{y_1}^{1} \frac{dy}{\sqrt{H(y)}},$$

where $y_1 = x_1/x_2$ and

$$H(y) = 1 - y^2 - kx_2^{-\alpha-2}(y^{-\alpha} - 1).$$

Note that $H'(y) = 0$ only for $y = y_*$, where

$$y_* = lx_2^{-1}, \tag{2.1}$$

with $l = (\alpha k/2)^{1/(\alpha+2)}$ and

$$H''(y) = -2 - k\alpha(\alpha+1)y^{-(\alpha+2)} < 0, \quad y \in (y_1, 1).$$

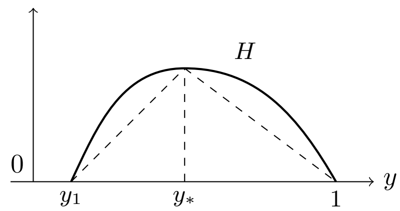

FIGURE 2

Consequently,

$$H(y) \geqslant \frac{H(y_*) - H(y_1)}{y_* - y_1}(y - y_1) + H(y_1), \quad y \in (y_1, y_*),$$
$$H(y) \geqslant \frac{H(y_*) - H(1)}{y_* - 1}(y - 1) + H(1), \quad y \in (y_*, 1),$$

(see Figure 2) and $H(y_1) = H(1) = 0$ imply

$$\frac{1}{\sqrt{H(y)}} < \sqrt{\frac{y_* - y_1}{H(y_*)}} \frac{1}{\sqrt{y - y_1}}, \quad y \in (y_1, y_*), \tag{2.2}$$

and

$$\frac{1}{\sqrt{H(y)}} < \sqrt{\frac{1 - y_*}{H(y_*)}} \frac{1}{\sqrt{1 - y}}, \quad y \in (y_*, 1). \tag{2.3}$$

It is also

$$\frac{1}{\sqrt{1 - y^2}} < \frac{1}{\sqrt{H(y)}}, \quad y \in (y_1, 1). \tag{2.4}$$

Due to $r_{\min} \to 0$, $r_{\max} \to \infty$ as $E \to \infty$, it is $x_1 \to 0+$, $x_2 \to \infty$, whence

$$\begin{aligned} &y_1 \to 0, \ y_* \to 0 \\ &H(y_*) = 1 - y_*^2 - kl^{-\alpha} x_2^{-2} + k x_2^{-\alpha-2} \to 1. \end{aligned} \tag{2.5}$$

The proof of the Lemma is based upon the following inequality

$$\begin{aligned} \Big| \int_{y_1}^1 \frac{dy}{\sqrt{H(y)}} - \int_{y_1}^1 \frac{dy}{\sqrt{1 - y^2}} \Big| \leqslant & \Big| \int_{y_*}^1 \frac{dy}{\sqrt{H(y)}} - \int_{y*}^1 \frac{dy}{\sqrt{1 - y^2}} \Big| \\ & + \int_{y_1}^{y_*} \frac{dy}{\sqrt{H(y)}} + \int_{y_1}^{y_*} \frac{dy}{\sqrt{1 - y^2}}. \end{aligned} \tag{2.6}$$

Evidently, $\int_{y_1}^1 \frac{dy}{\sqrt{1-y^2}} \to \frac{\pi}{2}$ and $\int_{y_1}^{y_*} \frac{dy}{\sqrt{1-y^2}} \to 0$. The inequalities (2.4) and (2.3), imply

$$\Big| \int_{y_*}^1 \frac{dy}{\sqrt{H(y)}} - \int_{y*}^1 \frac{dy}{\sqrt{1 - y^2}} \Big|$$

$$
\begin{aligned}
&= \int_{y_*}^{1} \frac{\sqrt{1-y^2} - \sqrt{H(y)}}{\sqrt{H(y)}\sqrt{1-y^2}}\, dy \\
&= \int_{y_*}^{1} \frac{1-y^2-H(y)}{\sqrt{H(y)}\sqrt{1-y^2}(\sqrt{H(y)}+\sqrt{1-y^2})}\, dy \\
&= kx_2^{-\alpha-2} \int_{y_*}^{1} \frac{y^{-\alpha}-1}{\sqrt{H(y)}\sqrt{1-y^2}(\sqrt{H(y)}+\sqrt{1-y^2})}\, dy \\
&\leqslant kx_2^{-\alpha-2} \int_{y_*}^{1} \frac{y^{-\alpha}-1}{H(y)^{3/2}}\, dy \\
&\leqslant \Big(\frac{1-y^*}{H(y_*)}\Big)^{3/2} kx_2^{-\alpha-2} \int_{y_*}^{1} \frac{y^{-\alpha}-1}{1-y}\, \frac{1}{\sqrt{1-y}}\, dy.
\end{aligned}
$$

The last term tends to 0: indeed, owing to (2.5), it is

$$
\Big(\frac{1-y^*}{H(y_*)}\Big)^{3/2} \to 1,
$$

and thanks to $y^{-\alpha} - 1 < y^{-\alpha} \leqslant y_*^{-\alpha}$ for $y \geqslant y_*$, and the equality $x_2\, y_* = l$ (see (2.1)), respectively, it follows

$$
\begin{aligned}
&kx_2^{-\alpha-2} \int_{y_*}^{1} \frac{y^{-\alpha}-1}{1-y}\, \frac{1}{\sqrt{1-y}}\, dy \\
&\quad = kx_2^{-\alpha-2} \int_{1/2}^{1} \frac{y^{-\alpha}-1}{1-y}\, \frac{1}{\sqrt{1-y}}\, dy + kx_2^{-\alpha-2} \int_{y_*}^{1/2} \frac{y^{-\alpha}-1}{1-y}\, \frac{1}{\sqrt{1-y}}\, dy \\
&\quad \leqslant kx_2^{-\alpha-2} \int_{1/2}^{1} \frac{y^{-\alpha}-1}{1-y}\, \frac{1}{\sqrt{1-y}}\, dy + kx_2^{-\alpha-2} \int_{y_*}^{1/2} \frac{y_*^{-\alpha}}{(1-y^2)^{3/2}}\, dy \\
&\quad = kx_2^{-\alpha-2} \int_{1/2}^{1} \frac{y^{-\alpha}-1}{1-y}\, \frac{1}{\sqrt{1-y}}\, dy + kl^{-\alpha} x_2^{-2} \int_{y_*}^{1/2} \frac{dy}{(1-y^2)^{3/2}} \to 0.
\end{aligned}
$$

Thus, the first expression on the r. h. s. in (2.6) tends to 0. It only remains to prove the same for the second term on r. h. s. of (2.6). It is, however, a simple consequence of (2.2):

$$
\int_{y_1}^{y_*} \frac{dy}{\sqrt{H(y)}} \leqslant \sqrt{\frac{y_*-y_1}{H(y_*)}} \int_{y_1}^{y_*} \frac{dy}{\sqrt{y-y_1}} \to 0. \qquad \square
$$

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О ДОКАЗУ БЕРТРАНОВЕ ТЕОРЕМЕ

Резиме. У овом раду допуњујемо доказе Бертранове теореме из Бертрановог оригиналног рада *Théorème relatif au movement d'un point attiré vers un centre fixe*, и Арнољдове књиге Математичке методе класичне механике, представљајући недостајуће кораке који омогућују да се еластични и гравитациони потенцијал издвоје међу потенцијалима облика степене функције.

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