Dual-series equations formulation for static deformation of plates with a partial internal line support

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Abstract

The paper deals with the application of dual-series equations to the problem of rectangular plates having at least two parallel simply supported edges and a partial internal line support located at the centre where the length of internal line support can be varied symmetrically, loaded with a uniformly distributed load. By choosing the proper finite Hankel transform, the dual-series equations can be reduced to the form of a Fredholm integral equation which can be solved conveniently by using standard techniques. The solutions of integral equation and the deformations for each case of the plates are given and discussed in details.

Keywords : rectangular plate, dual-series equations, internal line support, Hankel transform, Fredholm integral equation

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1   Introduction

Problems of rectangular plate [1-4] and circular plate [5-7] with the combination of clamped, simply supported, and free boundary conditions have previously been treated and founded in the scientific or technical literature on this subject. Much attention was received to investigate the vibration [5-7], buckling [4,5-7], and bending [1-3,5] of the plates. It is generally well-known that, for a plate with mixed edge conditions, singularities in the bending fields are to be expected at the transition points of discontinuous boundary. The nature of these singularities has been studied by using the Fadle eigenfunction expansion techniques [8,9]. From these methods, it can be pointed out that the proper singularity in the moments is of the inverse-square-root type.

Normally, mixed boundary value problems of the theory of elasticity and of mathematical physics for the regions with partly infinite boundaries [10-13] can often be reduced to studying the dual integral equations; while for the finite regions [14-17], the problems can often be reduced to studying the dual-series equations. However, the local behavior at the point of discontinuity of the boundary conditions must be the same for both finite and infinite regions. The analysis in these problems is usually led to the final form of integral equations of special type by utilizing the method of integral transform techniques [18].

Focusing on the treatment of dual-series equations that is concerned in the present investigation, there are developments for this purpose. Kelman and Simpson [19] and Kelman [20] proposed an algorithm to obtain the closed form analytic solutions to dual cosine series equations. In their works, the solutions were reduced to a finite number of precisely specified elementary arithmetic operations, which derived by the method of orthogonality relations. The convergence of solutions was also studied by Kelman [21]. A more general type of certain classic dual trigonometric (sine and cosine) series equations that occurred in solving mixed boundary value problems in rectangular domains in the $x - y$ plane was presented by Kelman et al.[22]. It is interesting to note that various mixed boundary value problems of physics and mechanics are mostly reduced to singular integral equations, but the mentioned works [19,20,22] are, however, avoided to solve them. Other applications of dual-series equations have been used by Mills and Dudukovic [23,24] in which the solution procedure reduces the determination of the series coefficients to the solution of a large system of algebraic equations. Recently, Malits [25] presented a new form of the solution of the dual and triple Fourier-Bessel series equations where the final form of solution can be transformed into the Fredholm integral equations of the second kind or into the

singular integral equations of a well-known type.

In problems concerning the bending of rectangular plates under various types of loading, they are of interest and have many applications in structural engineering. Closed-form solutions can be found for the case of regular boundary conditions [26], but not for the case of mixed boundary conditions. In the latter case, one analytical method used in the past allowed the problem to be reduced to finding the solution to a Fredholm integral equation of the first kind [5]. It can be noted that the singularities [8] were not taken into account. The necessity of considering the singularity in problem solutions was confirmed by Leissa et al.[27] and Chen and Pickett [28]. Both of these numerical works concluded that to get sufficient accuracy in the solutions, it requires the use of appropriate singularity functions to represent the singular behavior at that point of singularity. Leissa [29] suggested that, for the problems of plate having singularities due to the abrupt change in geometry or mixed boundary conditions; the singularities are not negligible and have to be incorporated in the analysis. If these singularities are not considered properly, then highly inaccurate, or even meaningless, results can arise. Therefore the infinite quantities may have strong effects upon the global behavior of the configuration such as static or dynamic deflections, free vibration frequencies and buckling loads. Another analytical method used was the application of dual-series equations [30-32] for the problem formulation in connection with a modification of the technique presented by Westmann and Yang [33] who analyzed the mixed boundary value problems arising from the longitudinal shear and Saint-Venant torsion and flexure of cracked rectangular beams. Following this method, the dual-series equations can be reduced to a Fredholm integral equation of the second kind for an unknown auxiliary function by means of certain integral representations. It is worth to notice that the moment singularities [8] are introduced at the points of discontinuous support as seen in [30-32], and the method requires that the singular part of the solution is isolated and treated analytically.

The present investigation concerns the bending of rectangular plates in which there are two opposite simply supported edges. A partial internal line support is symmetrically placed at the plate centre and it is perpendicular to the two simple supports. The remaining edges of the plate have the same type of support conditions that are either clamped or free supports as shown in Fig. 1. The loading is taken as uniform. An interesting feature of the specific problems considered is that there are few articles in the past on this current subject. It would appear that the static bending problem of plate having a partial internal line support would have numerous solutions in the literature. To the best knowledge

of the authors, the first investigation for a closely related problem of rectangular plate with a partial internal line support is that of Yang [34]. In his work, the problem formulation was reduced to the solution of a singular integral equation governing the pressure distribution along the line of internal support. It found that the shear singularities exist at the tips of internal support in the order of an inverse-square-root type by means of a finite Hilbert transform. However, this result does not agree with the work of Williams [8], who find that a moment singularity should exist outside of the support region and then, the pressure distribution along the support (Kirchhoff shear) is nonintegrable. In addition to static bending, Stahl and Keer [35] considered the free vibration and buckling of simply supported rectangular plate. The method used is similar to the work that presented by Westmann and Yang [33], and the correct moment singularities are taken into consideration.

Because the bending problems of rectangular plate having a partial internal line support subjected to a uniformly distributed load, however, have never been treated correctly, thus the objective of this paper is to formulate the problem of rectangular plates by use of the dual-series equations which differs from the preceding [34]. The singularities of the solution are made explicitly by means of a finite Hankel transform technique as in the same manner with Kiattikomol and Sriswasdi [31], Stahl and Keer [35], and Kiattikomol et al.[36]. The solution can be determined from an inhomogeneous Fredholm integral equation of the second kind. This equation, with the Simpson's rule of integration, is then reduced to a set of simultaneous equations suitable for numerical solution where the numerical analysis is also discussed in details. The advantages of the present method can be concluded as follows: the singular part of the solution is isolated and treated analytically, and the solutions of problem can be expressed in the closed-form expressions. The physical quantities for the deflection and slope of the plates along the line outside of an internal support are carried out in this paper.

2   Dual series equations formulation

Considering the geometry of rectangular plates that illustrated in Fig. 1, the coordinates and dimensions shown on the figures are scaled by the factor π/a, where a is the actual plate length. Since the problems involve a thin elastic plate with flexural rigidity (D) carrying a uniformly distributed load (q), thus the main objective is to determine the solution of the fourth-order partial differential

equation

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{qa^4}{\pi^4 D} \tag{1}$$

for the deflection $w(x, y)$. The flexural rigidity of the plate can be defined by the expression as $D = Eh^3/12(1 - \nu^2)$, while h is the plate thickness, and the material properties ν and E are the Poisson's ratio and the Young's modulus, respectively.

In this paper, the boundary conditions are prescribed according to the type of support along the plate edges and an additional partial internal support along part of the plate centre line. Therefore, the boundary conditions of a partial internal line support have to be accounted in addition to the constraints from boundary supports.

Due to the two-fold symmetry in deflection function, the boundary conditions need only to be written on one quadrant of the plate that bounded by the region $0 \le x \le \pi/2$ and $0 \le y \le b$, so that the boundary conditions satisfying the edge at $0 \le x \le \pi/2$ and $y = b$ are separately given in the following below [26]:

$$w = 0 \ , \ \frac{\partial w}{\partial y} = 0 \tag{2}$$

for the plate with two opposite clamped edges at $|y| = b$ (Fig. 1(a)), and

$$\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2} = 0 \ , \ \frac{\partial^3 w}{\partial y^3} + (2 - \nu)\frac{\partial^3 w}{\partial x^2 \partial y} = 0 \tag{3}$$

for which the plate has two opposite free edges at $|y| = b$, see Fig. 1(b).

At $y = 0$ along an internal line support, the internal constraint conditions [35] can be specified as follows:

$$\frac{\partial w}{\partial y} = 0 \ ; \ 0 \le x \le \pi/2, \tag{4}$$

$$w = 0 \ , \ \frac{\partial w}{\partial x} = 0 \ , \ \frac{\partial^2 w}{\partial x^2} = 0 \ ; \ e < x \le \pi/2, \tag{5}$$

$$\frac{\partial^3 w}{\partial y^3} + (2 - \nu)\frac{\partial^3 w}{\partial x^2 \partial y} = 0 \ ; \ 0 \le x < e. \tag{6}$$

It is remarkable that the condition given in equation (4) is caused by the symmetry of the deflection function, while the boundary conditions presented in equations (7) to (6) are the mixed boundary conditions which have important

to the problem formulation. The boundary condition shown in equation (4) is represented to the bending moment expression. Both of equations (5) and (6) are equivalent to the expression of supplemented, or Kirchhoff, shearing force.

Since the edges of the plate at $x = 0, \pi$ are of the simple support type, the deflection function that satisfied equation (1) can be sought in the form of Levy type solution approach [26] as the sum of the particular (w_p) and complementary (w_c) solutions

$$w = w_p + w_c, \tag{7}$$

where

$$w_p = \frac{4qa^4}{\pi^5 D} \sum_{m=1,3,5,\dots}^{\infty} m^{-5} \sin(mx), \tag{8}$$

$$w_c = \sum_{m=1,3,5,\dots}^{\infty} Y_m(y) \sin(mx), \tag{9}$$

and

$$Y_m(y) = \frac{qa^4}{D}[A_m \cosh(my) + B_m my \sinh(my)$$

$$+ C_m \sinh(my) + D_m my \cosh(my)], \tag{10}$$

in which A_m, B_m, C_m, and D_m are the unknown constants to be determined from the boundary conditions at $y = b$ for each case of the plates and at $y = 0$ along the internal line support.

First considering the plate shown in Fig. 1(a), the application of boundary conditions that provided by equations (2), (3), and (4) yields the following relations

$$A_m = -\frac{4 \sinh(mb)[1 + mb \coth(mb)]}{\pi^5 m^5 [mb + \sinh(mb) \cosh(mb)]} + D_m \frac{\Delta^{(2)}}{\Delta^{(1)}}, \tag{11}$$

$$\Delta^{(1)} B_m = \frac{4}{\pi^5 m^5 \cosh(mb)} - D_m \tanh(mb), \tag{12}$$

$$C_m = -D_m, \tag{13}$$

where

$$\Delta^{(1)} = 1 - mb[\tanh(mb) - \coth(mb)], \tag{14}$$

$$\Delta^{(2)} = \tanh(mb) + (mb)^2[\tanh(mb) - \coth(mb)]. \tag{15}$$

It can be noted that the problem is now reduced to the determination of the unknown constant D_m. For this purpose, the dual-series equations are needed

to be formulated, which can be obtained from the remaining mixed boundary conditions as given in equations (7) to (6).

In deriving the dual-series equations, it may be convenient to use the conditions of equations (9) and (6) due to the permission of dual-series equations to be cast into the proper form for solution [36]. Substitution of equation (7) into equations (9) and (6), leads to the dual-series equations as follows:

$$\sum_{m=1,3,5,...}^{\infty} m^2 P_m \sin(mx) = 0 \; ; \; e < x \le \pi/2, \tag{16}$$

$$\sum_{m=1,3,5,...}^{\infty} m^3 P_m (1 + F_m) \sin(mx) = \sum_{m=1,3,5,...}^{\infty} G_m \sin(mx) \; ; \; 0 \le x < e, \tag{17}$$

where

$$P_m = \frac{4\Delta^{(3)}}{\pi^5 m^5} + D_m \frac{\Delta^{(2)}}{\Delta^{(1)}}, \tag{18}$$

$$1 + F_m = \frac{\Delta^{(1)}}{\Delta^{(2)}} \;\; , \; G_m = \frac{4\Delta^{(1)}\Delta^{(3)}}{\pi^5 m^2 \Delta^{(2)}}, \tag{19}$$

and

$$\Delta^{(3)} = \frac{[1 - \cosh(mb)][mb - \sinh(mb)]}{mb + \sinh(mb)\cosh(mb)}. \tag{20}$$

It should be noted that the constant value of 1 in the left-hand side of equation (17) serves to isolate the singularity at $x = e$ since the weight function F_m approaches zero as $m \to \infty$.

The solution to the preceding dual-series equations (equations (16) and (17)) can be determined by choosing the unknown function P_m in the form of finite Hankel integral transforms. Based on this integral transform technique, there are many applications in various fields of problem which can be seen continually in Gradwell and Iyer [37] and Tsai [38] for contact problems, De and Patra [39], Fildis and Yahsi [40], and Wang et al.[41] for crack problems. Other applications are still to be found in the scattering scientific or technical literature. For the problems of plate with mixed boundary conditions, Keer and Sve [42], Keer and Stahl [43], Stahl and Keer [44], and Dundurs et al.[45] have been applied this type of integral transform to the vibration, buckling, and bending problems of the plates.

To solve the dual-series equations of equations (16) and (17), the method is made by representing the unknown function P_m in the form [36]

$$m^2 P_m = \bar{E} J_1(me) + \int_0^e t\phi(t) J_1(mt) dt \; ; \; m = 1, 3, 5, ..., \tag{21}$$

in which $\phi(\cdot)$ is the unknown auxiliary function and $J_1(\cdot)$ is the Bessel function of the first kind and order 1. The unknown constant $\bar{E}$ can be determined from the condition of equation (7); the details have been explained in [36]. Most importantly, this choice of P_m can automatically be satisfied the first dual-series equations that presented in equation (16) and the internal constraint condition that given by equation (8) with the assistance of the identities as listed below [35,36,42]

$$\sum_{m=1,3,5,...}^{\infty} m^{-2} J_1(mt) \sin(mx) = \frac{1}{4} \left[\frac{x}{t}(t^2 - x^2)^{1/2} + t \sin^{-1}\left(\frac{x}{t}\right) \right] \; : \; x < t \; ; \; x+t < \pi,$$
$$\tag{22}$$

$$= \frac{\pi}{8} t \; : \; x \geq t \; ; \; x + t < \pi, \tag{23}$$

$$\sum_{m=1,3,5,...}^{\infty} J_1(mt) \sin(mx) = \frac{1}{2} x t^{-1} (t^2 - x^2)^{-1/2} H(t - x) \; ; \; x + t < \pi, \tag{24}$$

$$\sum_{m=1,3,5,...}^{\infty} m^{-1} J_1(mt) \cos(mx) = \frac{1}{2} t^{-1} (t^2 - x^2)^{1/2} H(t - x) \; ; \; x + t < \pi, \tag{25}$$

where $H(\cdot)$ is the Heaviside unit step function. It can be observed that equation (25) is derived from a direct integration of equation (24) between the limits of 0 and x.

As mentioned above that referred to [36], integrating equation (16) twice with respect to x, the obtained result is equivalent to the condition as given in equation (7). After that substituting P_m from equation (21) and also setting $x = \pi/2$ with using the identity of equation (23), the unknown constant $\bar{E}$ is found to be

$$\bar{E} = - \int_0^e \frac{t^2}{e} \phi(t) dt. \tag{26}$$

Using equation (26), then, equation (21) becomes

$$m^2 P_m = \int_0^e t\phi(t) \left[J_1(mt) - \frac{t}{e} J_1(me) \right] dt \; ; \; m = 1, 3, 5, ... \tag{27}$$

To verify that the condition on the slope $\partial w/\partial x$ in equation (8) and the first dual-series equations shown in equation (16) are both satisfied when P_m is introduced by equation (27), the procedure can be exemplified as follows. In case of equation (8), integrating equation (16) once with respect to x and substituting P_m, yields

$$\int_0^e t\phi(t) \sum_{m=1,3,5,...}^{\infty} m^{-1} \left[J_1(mt) - \frac{t}{e} J_1(me) \right] \cos(mx) dt = 0 \; ; \; e < x \leq \pi/2. \tag{28}$$

With the help of identity given in equation (25), it is seen that equation (28) is automatically satisfied.

Similarly, it is easy to verify that P_m satisfies equation (16). By substituting equation (27) into equation (16) and changing the order of summation and integration with using equation (24) leads to

$$\int_0^e t\phi(t) \left[\frac{1}{2} xt^{-1}(t^2 - x^2)^{-1/2} H(t - x) - \frac{xt}{2e^2}(e^2 - x^2)^{-1/2} H(e - x) \right] dt$$

$$= 0 \; ; \; e < x \leq \pi/2. \tag{29}$$

Since x is always larger than t and e leading to $H(t - x) = 0$, thus, equation (29) is also satisfied.

According to [8], the correct singularities at the points of discontinuity are of $O(\varepsilon^{-1/2})$ in the moments and of $O(\varepsilon^{-3/2})$ in the shearing forces, where ε is the small distance measured from the singular point. Therefore, the object is to show that the function P_m given by equation (27) yields the correct singularities at $x = e$, $y = 0$. For the current purpose, the shear distribution at $y = 0$ may be expressed in the form

$$\frac{V_y(x, 0)}{qa\pi^3} = -2 \left[-\frac{d}{dx} \sum_{m=1,3,5,...}^{\infty} m^2 P_m \cos(mx) \right]$$

$$+ \sum_{m=1,3,5,\ldots}^{\infty} (m^3 F_m P_m - G_m) \sin(mx) \Bigg] \ ; \ e < x \leq \pi/2. \tag{30}$$

Substitution of P_m defined by equation (21) into equation (30) and considering only the singular part, yields

$$\frac{V_y(e+\varepsilon,0)}{qa\pi^3} \sim \left[\bar{E} \frac{d}{dx} \sum_{m=1,3,5,\ldots}^{\infty} J_1(me) \cos(mx) + \ldots \right]\Bigg|_{x=e+\varepsilon}, \tag{31}$$

and using the identity [36] that given below

$$\sum_{m=1,3,5,\ldots}^{\infty} J_1(mt) \cos(mx) = \frac{1}{2}t^{-1} - \frac{1}{2}xt^{-1}(x^2-t^2)^{-1/2}H(x-t)$$

$$+ \int_0^{\infty} [\exp(\pi s) + 1]^{-1} I_1(ts) \cosh(xs) ds \ ; \ x+t < \pi, \tag{32}$$

where $I_1(\cdot)$ is the modified Bessel function of the first kind and order 1.

Therefore, equation (31) becomes

$$\frac{V_y(e+\varepsilon,0)}{qa\pi^3} \sim \bar{E}\frac{e}{2}(2e\varepsilon)^{-3/2} + O(\varepsilon)^{-1/2} + \ldots \tag{33}$$

From the equation shown above, this indicates that there is an inverse square root singularity in moment when P_m is of such form that conditions of equations (7) to (9) are automatically satisfied. It remains to reduce the second dual-series equations that given in equation (17) to the form of an integral equation which can be solved numerically [30-32,35,36,42-45]. The details will be explained in the next section.

In the second case, see Fig. 1(b), the total deflection is chosen to be the same form of equation (7), but both unknown constants A_m and B_m that presented in equation (10) must be redefined. The internal constraint conditions given in equations (4) to (6) still govern, thus the relation of equation (13) for C_m and the form of equations (16), (17) for the dual-series equations can also be used with some modifications for this case.

Applying the boundary conditions of equations (4) and (5) leads to the following relations:

$$\bar{\Delta}^{(1)} A_m = -\frac{4\nu[mb\cosh(mb) - \eta''\sinh(mb)]}{\pi^5 m^5} + D_m \bar{\Delta}^{(2)}, \tag{34}$$

$$\bar{\Delta}^{(1)}B_m = \frac{4\nu \sinh(mb)}{\pi^5 m^5} - D_m[2 + (3+\nu)\sinh^2(mb)], \tag{35}$$

where

$$\bar{\Delta}^{(1)} = (3+\nu)\sinh(mb)\cosh(mb) - (1-\nu)mb, \tag{36}$$

$$\bar{\Delta}^{(2)} = (1-\nu)(mb)^2 + (1+\nu)\eta'' + (3+\nu)\cosh^2(mb), \tag{37}$$

$$\eta'' = \frac{1+\nu}{1-\nu}. \tag{38}$$

It should be noted in both equations (16) and (17) that the unknown function P_m, unknown constant G_m, and the weight function F_m have to be changed in the following forms

$$P_m = \frac{4\bar{\Delta}^{(3)}}{\pi^5 m^5 \bar{\Delta}^{(1)}} + D_m \frac{\bar{\Delta}^{(2)}}{\bar{\Delta}^{(1)}}, \tag{39}$$

$$1 + F_m = \frac{\bar{\Delta}^{(1)}}{\bar{\Delta}^{(2)}} \;,\; G_m = \frac{4\bar{\Delta}^{(3)}}{\pi^5 m^2 \bar{\Delta}^{(2)}}, \tag{40}$$

and

$$\bar{\Delta}^{(3)} = (3+\nu)\sinh(mb)\cosh(mb) - (1-\nu)mb$$
$$+\nu[\eta''\sinh(mb) - mb\cosh(mb)]. \tag{41}$$

By the same procedure as indicated in the previous case, the unknown function P_m is represented in the form of equation (27) which automatically satisfied the conditions on $w = 0$, $\partial w/\partial x$ in equations (7), (8), respectively, and the first dual-series equations provided by equation (16). Furthermore, it gives the square root moment singularity at the tip of internal support.

3   Integral equation

Integrating equation (17) once with respect to x and substituting P_m given by equation (27), after changing the order of integration and summation leads to

$$\int_0^e t\phi(t) \sum_{m=1,3,5,\ldots}^{\infty} (1 + F_m)\left[J_1(mt) - \frac{t}{e}J_1(me)\right]\cos(mx)dt$$

$$= \sum_{m=1,3,5,\ldots}^{\infty} m^{-1}G_m\cos(mx) \;;\; 0 \le x < e. \tag{42}$$

In view of equation (32), equation (42) can be written as

$$\int_0^x \frac{x\phi(t)}{\sqrt{x^2 - t^2}} dt = h(x) \; ; \; 0 \le x < e, \tag{43}$$

where

$$h(x) = e \int_0^1 \phi(er)$$

$$\times \left\{ 1 - r^2 + 2er \int_0^\infty [\exp(\pi s) + 1]^{-1} [I_1(ser) - rI_1(se)] \cosh(xs) ds \right\} dr$$

$$+ 2e^2 \int_0^1 r\phi(er) \sum_{m=1,3,5,\ldots}^\infty F_m [J_1(mer) - rJ_1(me)] \cos(mx) dr$$

$$- 2 \sum_{m=1,3,5,\ldots}^\infty m^{-1} G_m \cos(mx) \; ; \; 0 \le x < e. \tag{44}$$

Note that the change of variable $t = er$ is introduced in equation (44).

Equation (43) may be cast in the form of Abel's integral equation; therefore the solution to this equation (43) takes the form

$$\phi(t) = \frac{2}{\pi} \frac{d}{dt} \int_0^t \frac{h(x)}{\sqrt{t^2 - x^2}} dx \; ; \; 0 \le t < e. \tag{45}$$

It is remarkable that equation (42) should include an arbitrary constant of integration resulting from the integration of equation (17) with respect to x, however, it has no effect on the solution process of solving the Abel's integral equation, and thus the constant can be excluded.

Substituting equation (44) into equation (45) and with the help of certain identities that found in Gradshteyn and Ryzhic [46] and then, after some manipulations, the final result becomes

$$\Phi(\rho) + \int_0^1 K(\rho, r) \Phi(r) dr = f(\rho) \; ; \; 0 \le \rho \le 1, \tag{46}$$

where

$$\Phi(\rho) = \phi(e\rho) , \ \Phi(r) = \phi(er), \tag{47}$$

$$K(\rho, r) = 2e^2 r \left\{ \sum_{m=1,3,5,...}^{\infty} mF_m[J_1(mer) - rJ_1(me)]J_1(me\rho) \right.$$

$$\left. - \int_0^{\infty} s[\exp(\pi s) + 1]^{-1}[I_1(ser) - rI_1(se)]I_1(se\rho)ds \right\}, \tag{48}$$

$$f(\rho) = 2 \sum_{m=1,3,5,...}^{\infty} G_m J_1(me\rho). \tag{49}$$

Equation (46) is called the inhomogeneous Fredholm integral equation of the second kind and can be solved numerically to obtain an unknown auxiliary function $\Phi(\rho)$. The functions F_m and G_m are separately defined for each case of the plates as shown previously.

4 Numerical treatment for integral equation

It can be seen that the problems are reduced to determine the solution of an integral equation for the unknown function $\Phi(\rho)$ as presented in equation (46). This equation can be approximated by a sum over discrete values of r and ρ, leading to the system of linear inhomogeneous algebraic equations in the form

$$\Phi(r_i) + \sum_{j=1}^{N} W_j K(r_i, r_j)\Phi(r_j) = f(r_i) \ ; \ 0 \leq r_i, \ r_j \leq 1, \tag{50}$$

where $i = 1, 2, 3, ..., N$ and W_j is the weight function depending on the numerical integration procedure. Both the abscissas r_i and r_j are equally spaced from 0 to 1. Referring to equation (50), $K(r_i, r_j)$ is the discretized kernel of integral equation, while $f(r_i)$ and $\Phi(r_i)$ are the discretized values for the right-hand side function of integral equation and auxiliary function, respectively.

To solve equation (50), Simpson's rule is chosen for this purpose to set up the simultaneous equations; therefore the weight function W_j has the form

$$\left. \begin{array}{l} W_1 = W_N = \frac{1}{3(N-1)} \\ W_2 = W_4 = ... = \frac{4}{3(N-1)} \\ W_3 = W_5 = ... = \frac{2}{3(N-1)} \end{array} \right\}. \tag{51}$$

In view of equation (50), one expands to the linear simultaneous N equations and they can be tabulated in the following matrix form as

$$
\begin{bmatrix}
1 + W_1 K(r_1, r_1) & W_2 K(r_1, r_2) & \cdots & W_N K(r_1, r_N) \\
W_1 K(r_2, r_1) & 1 + W_2 K(r_2, r_2) & \cdots & W_N K(r_2, r_N) \\
\vdots & \vdots & \ddots & \vdots \\
W_1 K(r_N, r_1) & W_2 K(r_N, r_2) & \cdots & 1 + W_N K(r_N, r_N)
\end{bmatrix}
$$

$$
\times
\begin{bmatrix}
\Phi(r_1) \\
\Phi(r_2) \\
\vdots \\
\Phi(r_N)
\end{bmatrix}
=
\begin{bmatrix}
f(r_1) \\
f(r_2) \\
\vdots \\
f(r_N)
\end{bmatrix}.
\tag{52}
$$

It is noted that the improper infinite integral term in the kernel as appeared in equation (48) can be treated numerically using a 16-point Gauss-Legredre quadrature formula [47]. It can also be found that the characteristic of integrand of the improper infinite integral increases monotonically, up to some maximum value, and then decays exponentially. The function of an integrand would be converged to zero only if its argument $s \to \infty$, hence, it can be integrated numerically.

The infinite series in the kernel $K(\rho, r)$ and in the inhomogeneous part of integral equation $f(\rho)$ will be calculated to a relative error of 0.00001, i.e., the series are terminated when the ratio of the absolute value of the last term to the absolute value of the summation of all previous terms become less than 0.00001. Thus, equation (52) can be solved for the discretized value of the unknown auxiliary function $\Phi(\rho)$ using the Gaussian elimination with partial pivoting [48].

The numerical evaluation was carried out only for the case of a square plate of actual length a or the scaled length π, and the scaled half-length of internal line support c was varied from 0.005π to 0.5π. The Poisson's ratio was taken as 0.3. The results are shown in Figs. 2 and 3 for the cases of a square plate clamped and free edges at $|y| = \pi/2$, respectively. Therefore, the quantities in each case of the plates pertaining to the terms of unknown auxiliary function can be computed.

5   Plate deformation

In this section, the physical quantities of the plate are determined numerically. The quantities of interest are the deflected shape of the plate under loading and

the slope for both directions along the centre line of the plate.

The total deflection function for each case of the plates can be expressed in the closed-form by substitution of the unknown constants A_m, B_m, C_m, and D_m into equation (10). These unknown constants are related to the unknown auxiliary function $\Phi(\rho)$ that obtained from equation (46). Using equations (18) to (24) and equations (39) to (45), and the integral representation of P_m given in equation (27), then, the unknown constants A_m, C_m and D_m can be taken in the same form but they are different in the function $\Phi(\rho)$ while the functions F_m and G_m have been described previously. Therefore,

$$A_m = -\frac{4}{\pi^5 m^5} + \left(\frac{e}{m}\right)^2 \int_0^1 \rho\Phi(\rho)[J_1(me\rho) - \rho J_1(me)]d\rho, \tag{53}$$

$$C_m = -D_m = \frac{G_m}{m^3} + (1 + F_m)\left(\frac{e}{m}\right)^2 \int_0^1 \rho\Phi(\rho)[J_1(me\rho) - \rho J_1(me)]d\rho. \tag{54}$$

In case of the unknown constant B_m, it can separately be expressed as

$$B_m = \frac{4\operatorname{sech}(mb)}{\pi^5 m^5 \Delta^{(1)}}$$

$$+ \frac{\tanh(mb)}{\Delta^{(1)}}\left\{ \frac{G_m}{m^3} - (1 + F_m)\left(\frac{e}{m}\right)^2 \int_0^1 \rho\Phi(\rho)[J_1(me\rho) - \rho J_1(me)]d\rho \right\} \tag{55}$$

for the plate with two opposite clamped edges at $|y| = b$, and

$$B_m = \frac{4\nu \sinh(mb)}{\pi^5 m^5 \bar{\Delta}^{(1)}}$$

$$+ \frac{\bar{\Delta}^{(4)}}{\bar{\Delta}^{(1)}}\left\{ \frac{G_m}{m^3} - (1 + F_m)\left(\frac{e}{m}\right)^2 \int_0^1 \rho\Phi(\rho)[J_1(me\rho) - \rho J_1(me)]d\rho \right\} \tag{56}$$

for the plate with two opposite free edges at $|y| = b$, where $\bar{\Delta}^{(4)}$ is defined by

$$\bar{\Delta}^{(4)} = 2 + (3 + \nu)\sinh^2(mb). \tag{57}$$

Substituting equations (8) and (9) into equation (7) and utilizing equations (53) to (61) for the function $Y_m(y)$ that given in equation (10), then, numerical

results for the deflection function can be evaluated. Because of the two-fold symmetry of the deflection function, only the deflection surfaces in the lower left quadrant of the plate are given in Figs. 4 and 5 for two cases of plate presented in Figs. 1(a) and 1(b), respectively.

Another interested quantity that is the slope of the plate. In the present investigation, the slopes θ_x and θ_y are considered and they can be defined in the scaled coordinates by the derivatives as follows:

$$\theta_x = \frac{\pi}{a}\frac{\partial w}{\partial x} \ , \ \theta_y = \frac{\pi}{a}\frac{\partial w}{\partial y}. \tag{58}$$

To determine the slope along an internal line support in the direction of x, substituting equation (7) into equation (63) and letting $y = 0$ yields

$$\frac{\theta_x(x,0)}{(qa^3/D)} = \pi \sum_{m=1,3,5,\ldots}^{\infty} mP_m\cos(mx) \ ; \ 0 \le x \le e. \tag{59}$$

Using equation (27) for P_m and the help of identity given in equation (25), then equation (59) can be expressed in the integral representation form as

$$\frac{\theta_x(x,0)}{(qa^3/D)} = \frac{\pi}{2}\left[\int_{\xi}^{1}(\rho^2-\xi^2)^{-1/2}\Phi(\rho)d\rho - \int_{0}^{1}(1-\xi^2)^{-1/2}\rho^2\Phi(\rho)d\rho\right]$$

$$; \ \xi = x/e \text{ and } 0 \le \xi, \rho \le 1. \tag{60}$$

The slope along the centre line of the plate in the direction of y and normal to an internal line support is determined by substitution of equation (7) and setting $x = \pi/2$ into equation (64). The result is

$$\frac{\theta_y(\pi/2,y)}{(qa^3/D)} = \pi \sum_{m=1,3,5,\ldots}^{\infty}\left\{\begin{array}{l}A_m\sinh(my) - C_m my\sinh(my)\\ +B_m[my\cosh(my) + \sinh(my)]\end{array}\right\}m(-1)^{\frac{(m-1)}{2}}$$

$$; \ 0 \le y \le b, \tag{61}$$

where A_m, B_m, C_m, and D_m are provided by equations (53) to (61).

In Figs. 6 and 7 present the numerical results for the slopes of both plates that computed from equations (60) and (61), respectively.

6   Results and conclusions

The solution of integral equation in term of an unknown auxiliary function $\Phi(\rho)$ and the results for quantities of the plate, namely the deflection $w(x,y)$ and the slopes $\theta_x(x,0)$, $\theta_y(\pi/2,y)$ were carried out numerically. It is found that when e/π equals 0.5, the kernel of integral equation becomes infinity because the infinite integral in the kernel fails to converge. This case is one of the singular problems for the plate supported by a point column support at the plate centre. However, for the other cases when e/π approaches 0.5 such as $e/\pi = 0.495$, the kernel in this case can be computed but more equations are needed in order to achieve a better accuracy for the function $\Phi(\rho)$. The results of $\Phi(\rho)$ value are illustrated in Figs. 2 and 3 only for the case of square plate corresponding to the boundary conditions of plates shown in Figs. 1(a) and 1(b), respectively.

From the obtained results, the integral equation was approximated by using the different numbers of equation with various e/π-ratios. This can be summarized as follows: in the first case of the plate shown in Fig. 1(a), the numbers of equation used are 11 equations for $e/\pi \leq 0.2$, 21 equations for $0.2 < e/\pi \leq 0.3$, 31 equations for $0.3 < e/\pi \leq 0.45$, and 61, 91 equations for $e/\pi = 0.49$ and $e/\pi = 0.495$, respectively, and in the second case of the plate as shown in Fig. 1(b), 11 equations for $e/\pi \leq 0.3$, 21 equations for $0.3 < e/\pi \leq 0.45$, and 61, 81 equations for $e/\pi = 0.49$ and $e/\pi = 0.495$, respectively. These results indicated that the number of equations used increases as the ratio of e/π increased because it is difficult to calculate the integrand of infinite integral in the kernel. Although the present investigation is restricted to the Poisson's ratio taken as 0.3, however, it can be observed from equation (46) that the unknown auxiliary function is independent of the Poisson's ratio for the case of plate having the conditions shown in Fig. 1(a) because the functions F_m and G_m are not involved to the Poisson's ratio but not to be true for the plate having the conditions as shown in Fig. 1(b). Therefore, only the results given in Fig. 2 can be implied to use with the plate having other Poisson's ratios.

In Figs. 4 and 5 present the normalized deflection surface with various e/π-ratios for a quarter segment of square plates having the edge conditions similar to Figs. 1(a) and 1(b), respectively. It can be remarked in Fig. 4 that the results are independent of the Poisson's ratio due to the auxiliary function $\Phi(\rho)$ as described previously. The deflections $w(x,0)$ of the plate are quite close to zero when the ratios of e/π are less than 0.15 and 0.25 for the plates as seen in Figs. 4 and 5, respectively.

The slopes for both directions that expressed in equations (60) and (61), are,

respectively, presented in Figs. 6 and 7. Their characteristics can be discussed in the same manner of the normalized deflection surface for both cases of the plate boundary conditions in which the slopes of the plate having the boundary conditions shown in Fig. 1(a) are not depended on the Poisson's ratio. As seen in Fig. 6, the slopes $\theta_x(x, 0)$ are very small when e/π-ratios are less than 0.25, while the slopes $\theta_y(\pi/2, y)$ for the case of $e/\pi < 0.35$ are much closed to the case when $e/\pi = 0.05$. The latter is clearly seen in Fig. 7.

Finally, from the obtained results, the conclusions can be drawn that the paper proposes an analytical method for studying the bending problem of rectangular plates having a partial internal line support where the correct moment singularities are taken into the problem formulation. Two extreme cases of the plate with different boundary conditions are formulated through the dual-series equations. Taking advantage of finite Hankel integral transform technique, the dual-series equations can finally be reduced to an inhomogeneous Fredholm integral equation of the second kind which is solved numerically by using a simple numerical method named the Simpson's rule of integration. It can be revealed that there is no approximation before the numerical evaluation of the integral equation solution. Therefore, the interested quantities such as the deflected function and slope of the plate can be expressed in the analytical closed-form expressions. In addition, the present results can be served as benchmark for the other investigations of this current problem.

References

[1] H. Zorski, Plates with discontinuous supports, Archiwum Mechanika Stosowanej, 10, (1958), 271-313.

[2] G.M.L. Gladwell, Some mixed boundary value problems in isotropic thin plate theory, Q. Jl. Mech. Appl. Math., 11, (1958), 159-171.

[3] M. Kurata, Bending of simply supported rectangular plates with clamped portions along arbitrary sections of the edges, Ingenieur-Archiv, XXVII. Band, (1960), 385-416.

[4] M. Hamada, Y. Inoue and H. Hashimoto, Buckling of simply supported but partially clamped rectangular plates uniformly compressed in one direction, Bulletin of JSME, 10, (1967), 35-40.

[5] W. Nowacki and Z. Olesiak, Vibration, buckling and bending of a circular plate clamped along part of its periphery and simply supported on the remaining part, Bull. Acad. Pol. Sci., 4, (1956), 247-258.

[6] C.C. Bartlett, The vibration and buckling of a circular plate clamped on part of its boundary and simply supported on the remainder, Q. Jl. Mech. Appl. Math., 16, (1963), 431-440.

[7] B. Noble, The vibration and buckling of a circular plate clamped on part of its boundary and simply supported on the remainder, In Proc. Ninth Midwestern Mechanics Conference, Univ. of Wis., Madison, Wis., (1965), 141-146.

[8] M.L. Williams, Surface stress singularities resulting from various boundary conditions in angular corners of plates under bending, In Proc. First U.S. National Congress of Applied Mechanics (ASME), Ill. Inst. of Tech, Chicago, Ill., 1, (1952), 325-329.

[9] M.L. Williams, The bending stress distribution at the base of a stationary crack, J. Appl. Mech. (ASME), 28, (1961), 78-82.

[10] I. Zlatina, Dual integral equations applied to the steady-state vibration elastic contact problem for the half-space, Journal of Elasticity, 7, (1977), 437-447.

[11] S. Itou and Q. Rengen, Thermal stresses around two parallel cracks in two bonded dissimilar elastic half-planes, Arch. Appl. Mech., 63, (1993), 377-385.

[12] M. Rahman, A note on the polynomial solution of a class of dual integral equations arising in mixed boundary value problems of elasticity, Z. angew Math. Phys. (ZAMP), 46, (1995), 107-121.

[13] Z.-G. Zhou and B. Wang, Investigation of behavior of mode-I interface crack in piezoelectric materials by using Schmidt method, Appl. Math. Mech. (English Edition), 27, (2006), 871-882.

[14] S. Davidson, The linear steady thermoelastic problem for a strip with a collinear array of Griffith cracks parallel to its edges, J. Eng. Math., 27, (1993), 89-98.

[15] F. Ashida, Temperature and stress analysis of an elastic circular cylinder in contact with heated rigid stamps, Arch. Appl. Mech., 71, (2001), 221-232.

[16] W.-J. Feng, R.-J. Hao, J.-X. Liu and S.-M. Duan, Scattering of SH waves by arc-shaped interface cracks between a cylindrical magneto-electro-elastic inclusion and matrix: near fields, Arch. Appl. Mech., 74, (2005), 649-663.

[17] W.-J. Feng, R.K.L. Su, and Y.Q. Liu, Scattering of SH waves by an arc-shaped interface crack between a cylindrical magneto-electro-elastic inclusion and matrix with the symmetry of 6 mm, Acta Mechanica, 183, (2006), 81-102.

[18] I.N. Sneddon, The use of integral transforms, McGraw-Hill Book Co., New York, 1972.

[19] R.B. Kelman and J.T. Simpson, An algorithm for solving a dual cosine series, Comp. Math. Appl., 1, (1975), 195-199.

[20] R.B. Kelman, Algorithm for the classic dual cosine equation, Appl. Math. Comp., 6, (1980), 7-19.

[21] R.B. Kelman, Convergence of solutions to the classic dual cosine equation, Appl. Math. Comp., 9, (1981), 217-225.

[22] R.B. Kelman, J.P. Maheffy and J.T. Simpson, Algorithms for classic dual trigonometric equations, Comp. Math. Appl., 3, (1977), 203-215.

[23] P.L. Mills and M.P. Dudukovic, Application of the method of weighted residuals to mixed boundary value problems dual-series relations, Chem. Eng. Sci., 35, (1980), 1557-1570.

[24] P.L. Mills and M.P. Dudukovic, Application of the method of weighted residuals to mixed (split) boundary value problems triple series equations, Comp. Chem. Eng., 6, (1982), 141-154.

[25] P. Malits, Dual and triple Fourier-Bessel series equations, Comp. Math. Appl., 48, (2004), 823-831.

[26] S.P. Timoshenko and S. Woinowsky-Krieger, Theory of plates and shells, 2nd ed., McGraw-Hill, Singapore, 1959.

[27] A.W. Leissa, W.E. Clausen, L.E. Hulbert and A.T. Hopper, A comparison of approximate methods for the solution of plate bending problems, AIAA Journal, 7, (1969), 920-928.

[28] S.S-H. Chen and G. Pickett, Bending of plates of any shape and with any variation in boundary conditions, J. Appl. Mech. (ASME), 34, (1967), 217-218.

[29] A.W. Leissa, Singularity considerations in membrane, plate and shell behaviors, Int. J. Solids Struct., 38, (2001), 3341-3353.

[30] B. Stahl and L.M. Keer, Bending of a uniformly loaded circular plate with mixed boundary conditions, AIAA Journal, 10, (1972), 739-743.

[31] K. Kiattikomol and V. Sriswasdi, Bending of a uniformly loaded annular plate with mixed boundary conditions, AIAA Journal, 26, (1988), 487-492.

[32] K. Kiattikomol, S. Porn-anupapkul and N. Thonchangya, Application of dual series equations to a square plate with varying corner supported lengths, In Proc. Recent Advances in Structural Engineering, Japan-Thai Civil Engineering Conference, Bangkok, Thailand, (1985), 115-130.

[33] R.A. Westmann and W.H. Yang, Stress analysis of cracked rectangular beams, J. Appl. Mech. (ASME), 34, (1967), 693-701.

[34] W.H. Yang, On an integral equation solution for a plate with internal support, Q. Jl. Mech. Appl. Math., 21, (1968), 503-513.

[35] B. Stahl and L.M. Keer, Vibration and buckling of a rectangular plate with an internal support, Q. Jl. Mech. Appl. Math., 25, (1972), 467-478.

[36] K. Kiattikomol, L.M. Keer and J. Dundurs, Application of dual series to rectangular plates, J. Eng. Mech. Div. (ASCE), 100, (1974), 445-456.

[37] G.M.L. Gradwell and K.R.P. Iyer, Unbonded contact between a circular plate and an elastic half-space, Journal of Elasticity, 4, (1974), 115-130.

[38] Y.M. Tsai, Symmetrical contact problem of a thick transversely isotropic plate, Journal of Elasticity, 16, (1986), 179-188.

[39] J. De and B. Patra, Cruciform crack in an orthotropic elastic plane, Int. J. Fracture, 53, (1992), 387-397.

[40] H. Fildis and O.S. Yahsi, The axisymmetric crack problem in a non-homogeneous interfacial region between homogeneous half-spaces, Int. J. Fracture, 78, (1996), 139-164.

[41] B. Wang, J. Han and S. Du, Dynamic response for functionally graded materials with penny-shaped cracks, Acta Mechanica Solida Sinica (AMSS), 12, (1999), 106-113.

[42] L.M. Keer and C. Sve, On the bending of cracked plates, Int. J. Solids Struct., 6, (1970), 1545-1559.

[43] L.M. Keer and B. Stahl, Eigenvalue problems of rectangular plates with mixed edge conditions, J. Appl. Mech. (ASME), 39, (1972), 513-520.

[44] B. Stahl and L.M. Keer, 1972, Vibration and stability of cracked rectangular plates, Int. J. Solids Struct., 8, (1972), 69-91.

[45] J. Dundurs, K. Kiattikomol and L.M. Keer, 1974, Contact between plates and sagged supports, J. Eng. Mech. Div. (ASCE), 100, (1974), 445-456.

[46] I.N. Gradshteyn and I.M. Ryzhik, Table of integrals, series, and products, 4th ed., Academic Press, New York, 1956.

[47] M. Abramowitz and I.A. Stegun, Handbook of mathematical functions with formulas, graphs, and mathematical tables, Dover Publications, New York, 1964.

[48] G.H. Golub and C.F. Van Loan, Matrix computations, 2nd ed., The Johns Hopkins, University Press, Baltimore, 1990.
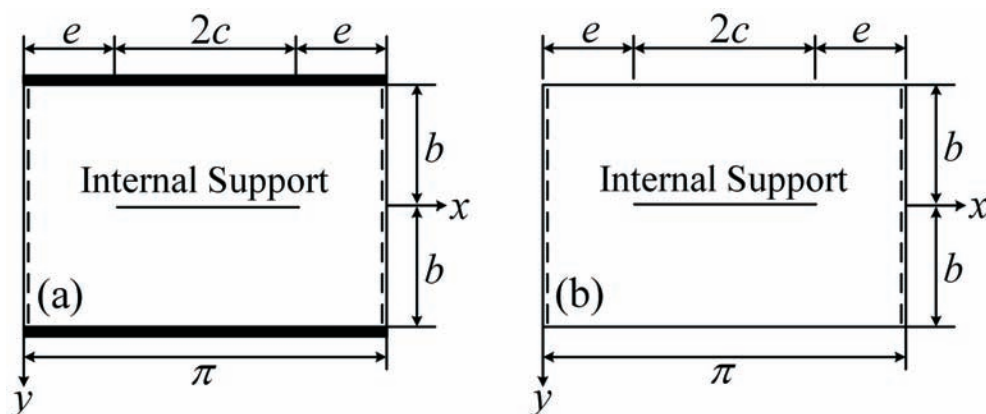
Figure 1: Plates with internal line support (a) clamped edges at $|y| = b$ and (b) free edges at $|y| = b$
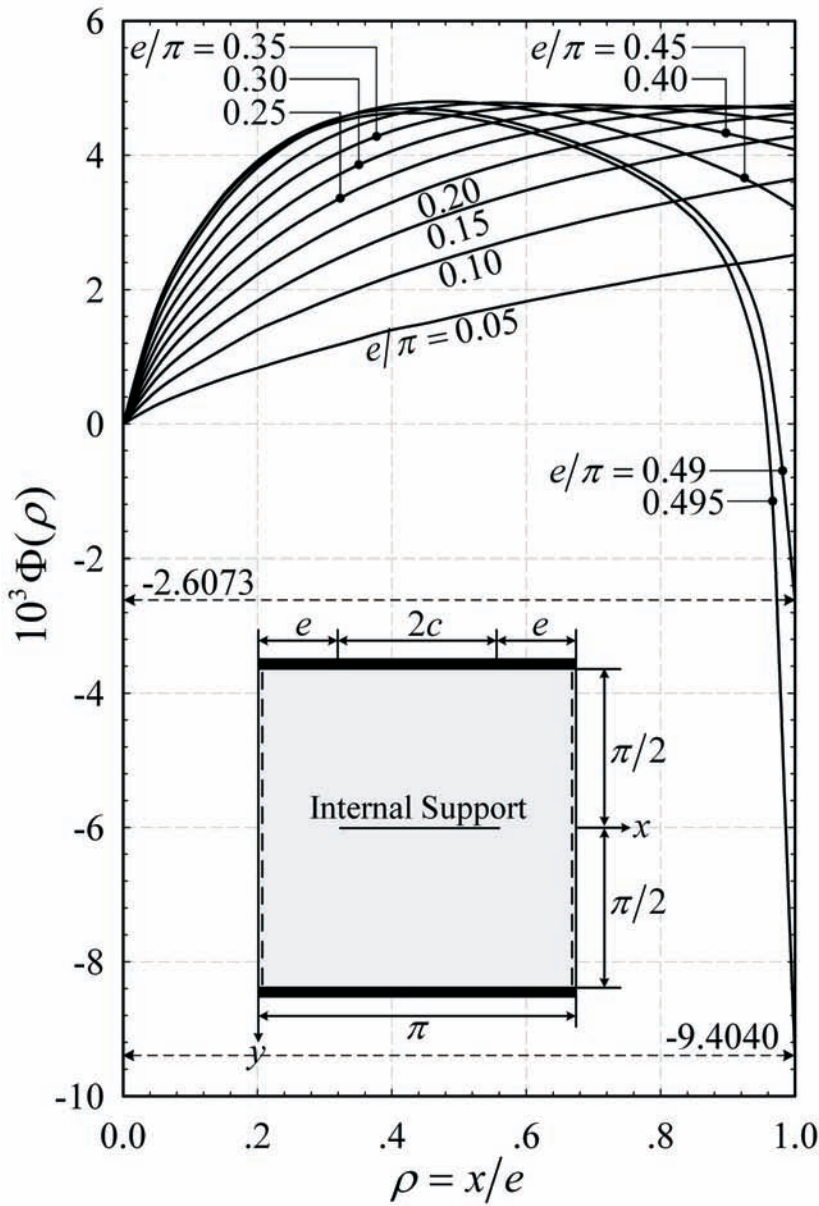
Figure 2: Auxiliary function $\Phi(\rho)$ for square plate having clamped edges at $|y| = \pi/2$
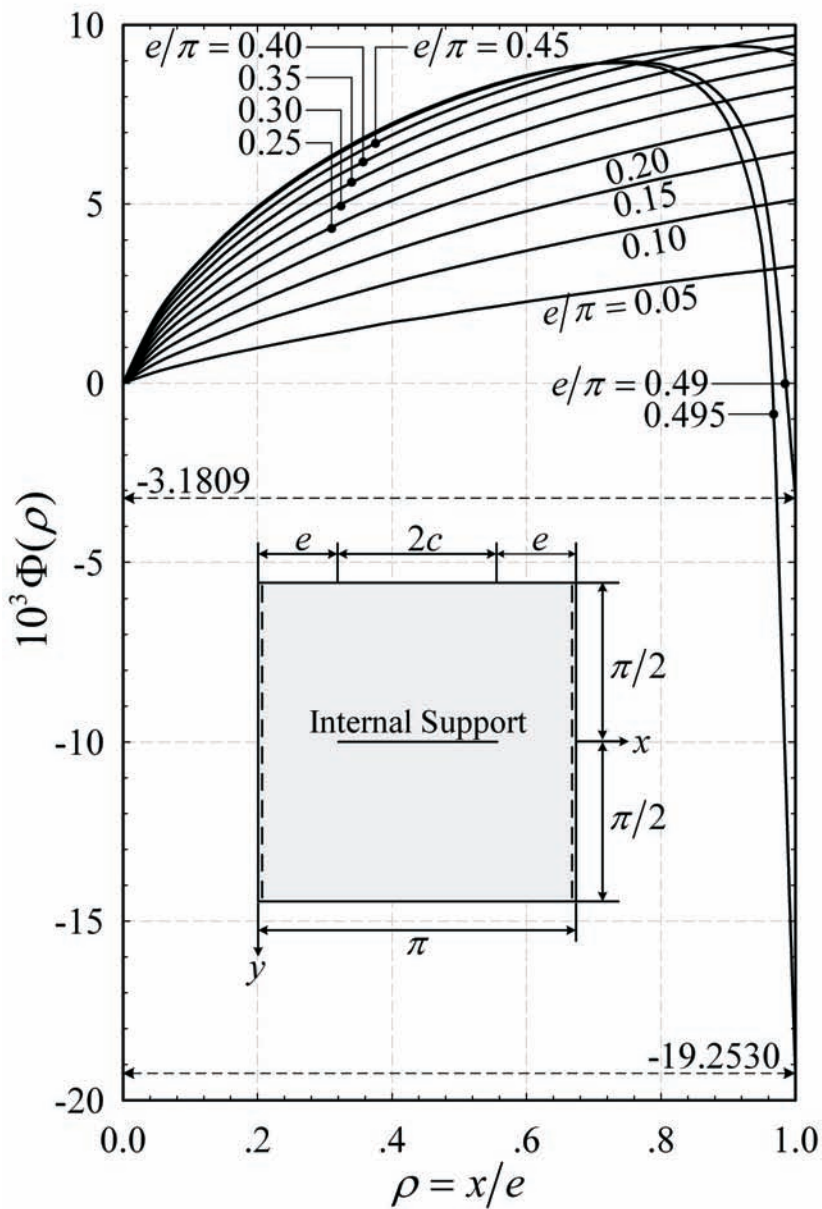
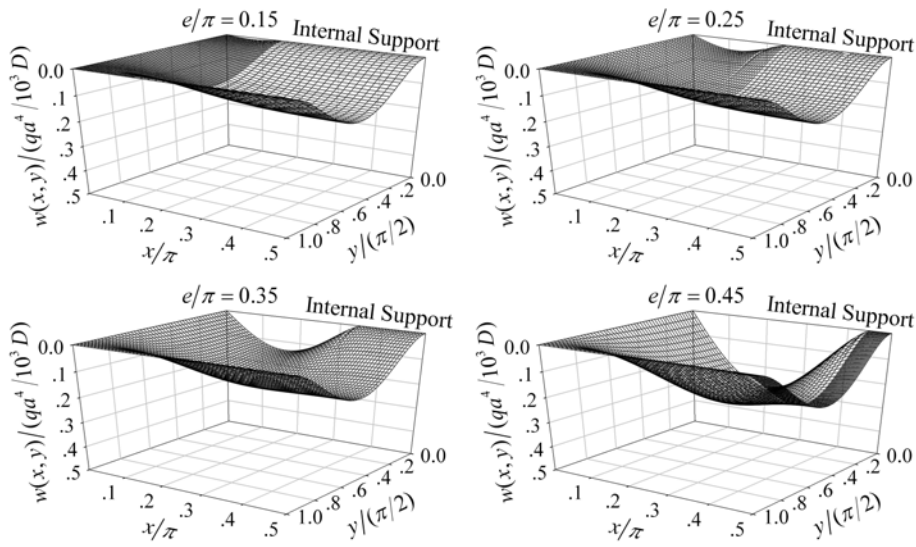Figure 3: Auxiliary function $\Phi(\rho)$ for square plate having free edges at $|y| = \pi/2$

Figure 4: Deflection surfaces for a quarter segment $(0 \leq x, y \leq \pi/2)$ of square plate having clamped edges at $|y| = \pi/2$
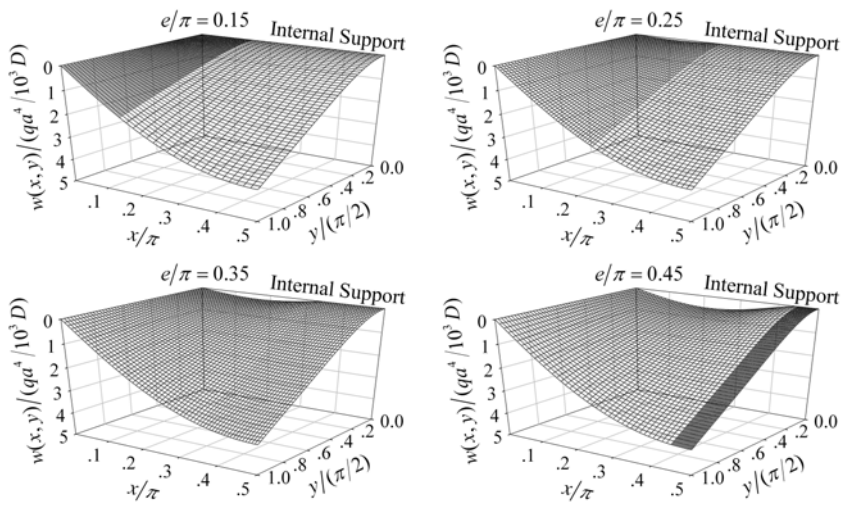

Figure 5: Deflection surfaces for a quarter segment $(0 \leq x, y \leq \pi/2)$ of square plate having free edges at $|y| = \pi/2$
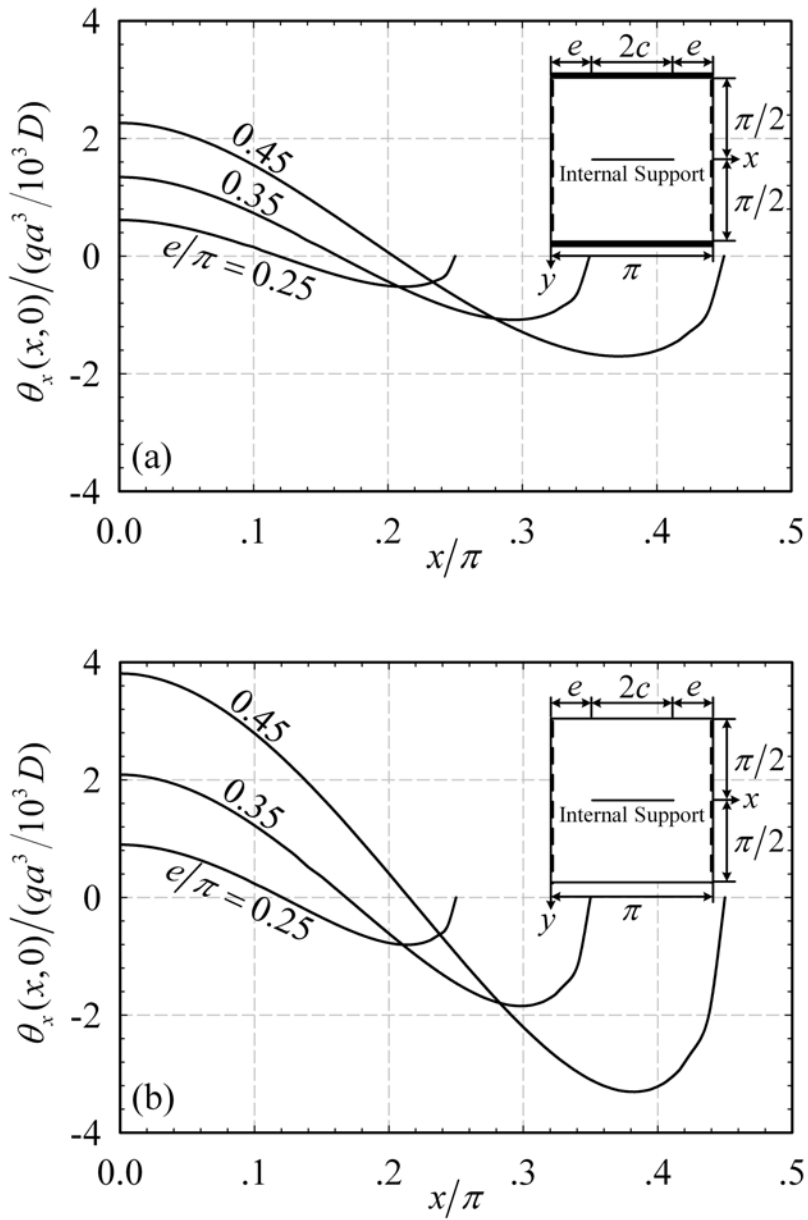
Figure 6: Slopes $\theta_x(x,0)$ for square plate (a) clamped edges at $|y| = \pi/2$ and (b) free edges at $|y| = \pi/2$.
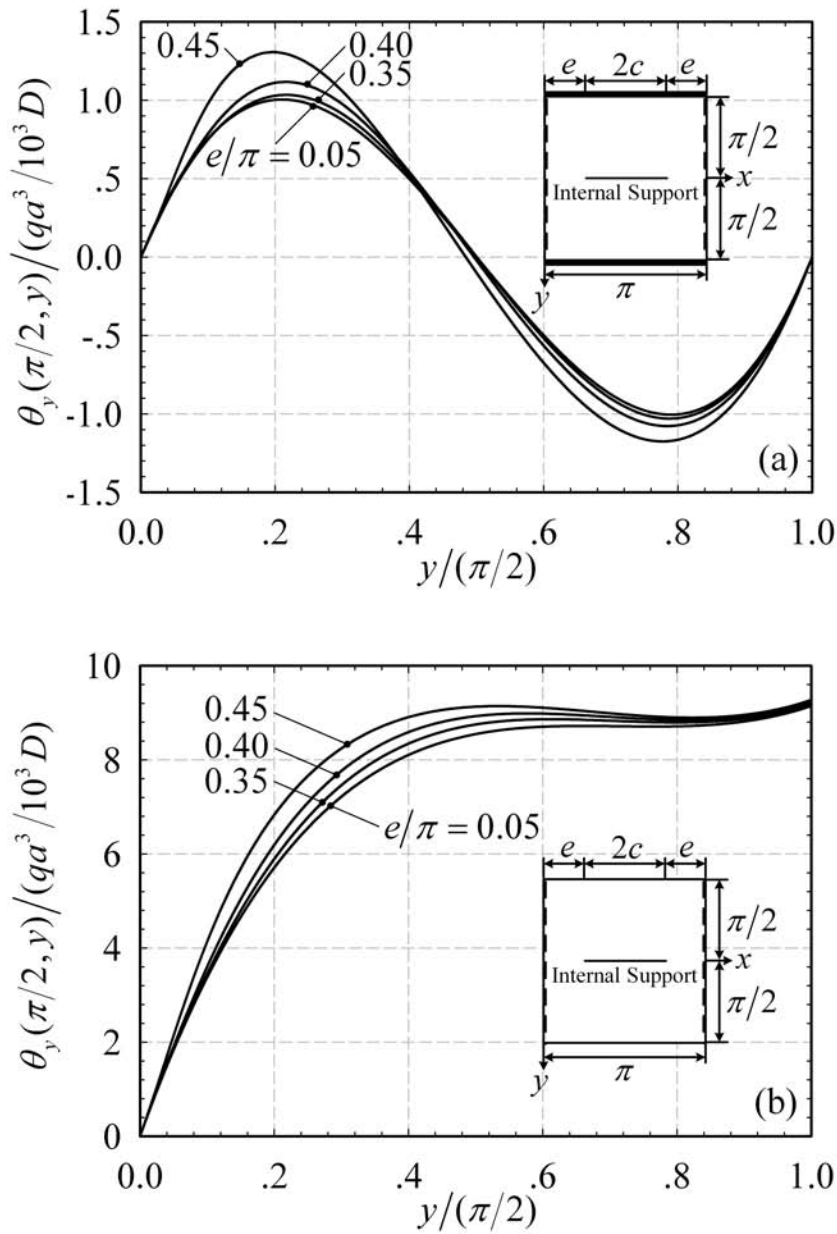
Figure 7: Slopes $\theta_y(\pi/2, y)$ for square plate (a) clamped edges at $|y| = \pi/2$ and (b) free edges at $|y| = \pi/2$.

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Jednačine dualnih redova za statičku deformaciju ploča sa delimičnim linijskim oslanjanjem

Data je primena jednačina dualnih redova na problem pravougaonih ploča koje imaju bar dve paralelne prosto oslonjene ivice i delimični unutrašnji linijski oslonac smešten u centru. Pri tome dužina unutrašnjeg linijskog oslonca može da se menja simetrično sa uniformno rasporedjenim opterećenjem. Izborom podesne Hankel-ove transformacije jednačine dualnih redova mogu da se redukuju na oblik Fredholm-ove integralne jednačine koja se rešava standardnim postupcima. Rešenja ove jednačine i deformacije za svaki posebni slučaj su dati i detaljno obrazloženi.