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A polynomial approach to determine stress in an elastic body of circular section indented by conformable contact

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Abstract

A polynomial of degree greater than two that describes the indenter concavity shape is proposed. From the proposed polynomial, the gradient of the displacement is derived and combined with that one determined by Timoshenko and Goodier to obtain the polynomial distribution of the pressure in the cross direction of wire. By using the polynomial pressure in the "stress function" proposed by Flamant, a set of equations serving to know the stresses state in the wire section is obtained. To extend the analysis to two opposite indenters, all contributions to total stress are considered, to knowing: the stresses being produced by each one of the indenters; the biaxial tension to balance the free area of pressure. Finally, by using all contributions to total stress and determining the principal stresses, the magnitude of maximum-shear-stress at each point of elastic body it could be obtained. In order to confront the model with the reality, by associating to each point

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its maximum-shear-stress respective, patterns of lines representing isostresses were obtained; such patterns were compared with a photo-elasticity image, showing a good agreement.

Keywords: Indentation process; conformable contact; stress function; polynomial pressure; photo-elasticity technique; elastic regime.

1 Introduction

An important mechanical parameter in the indentation process is the load required as well as the manner in that this load will be applied to the sample. To determine this load, an analysis of how the pressure would be distributed along the contact surface is mandatory. For a circular section, the problem is something complex; however, based on the *continuum mechanics* theory and by considering certain restrictions as: conformable contact; frictionless indentation; low strain rates; rigid indenters, the difficulty can be overcome and the load can be determined. Some models have been proposed [1] but none of them is capable of predicting with accuracy the value of the load for the case of a circular section. The Hertz's model for example, is not applicable to this case since this one does not predicts the displacement of points staying apart of the contact point. Accordingly, in this work we propose a more adequate distribution of the pressure in the cross direction, by leaving the lengthwise direction without modification, so that the model can agree with the experimental results. For the analysis, by using a polynomial of degree greater than two, a polynomial distribution of the pressure (polynomial pressure) in the cross direction along an arc of circle for producing a homogeneous strain was considered; remarking that this polynomial distribution of the pressure is limited at indenter's wedges.

2 Theoretical part

2.1 Geometrical assumptions and the polynomial model

Figure 1. shows schematically a transversal cut of the elements that act in the indentation process. In the scheme, it can see that single an

indenter was depicted. In this figure, B represents the deformable solid that is tested under load, whereas A represents the rigid indenter. In this manner, the model is developed for a single indenter and later extended for two opposite indenters.

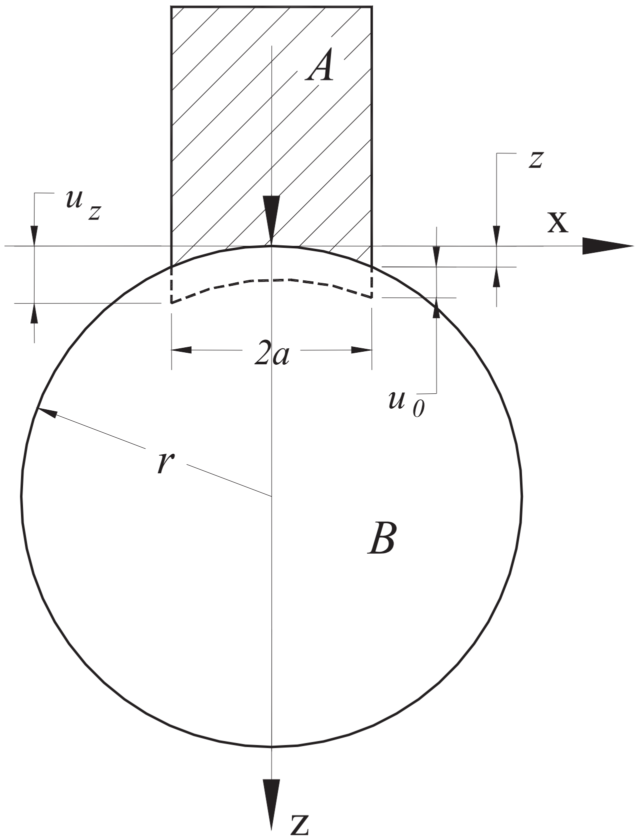

Figure 1: Schematic diagram showing a transversal cut of the indenter (A) and elastic body (B).

By taking into account the geometrical parameters, if is assumed that the secant of the arc of circle of the indenter is $2a$ long, the dimensionless polynomial curve which coincides with the indenter cavity profile on the plane (X, Z) is described by

$$Z = \frac{X^4}{6R^3} + \frac{X^2}{2R} \tag{1}$$

where $R = r/a$, $Z = z/a$ and $X = x/a$.

Since the strain will be homogeneous, every point in the surface will experiment the same displacement u_0; in this way, the displacement in the z-direction u_Z will be

$$u_Z = u_0 + \frac{X^4}{6R^3} + \frac{X^2}{2R} \tag{2}$$

Last equation is the dimensionless equation-of-displacement of the contact surface. From equation (2), any variation of the displacement in the surface in the x-direction can be determined by differentiation of this equation with respect to X

$$\frac{\partial u_z}{\partial X} = \frac{2X^3}{3R^3} + \frac{X}{R} \tag{3}$$

By using the "stress function" proposed by Flamant, for the case of the unitary pressure that is produced by a concentrated normal force P, in the Hooke's law, Timoshenko and Goodier [2] determined the displacementu_z

$$u_z = \frac{(1-\nu^2)}{\pi E} 2P \ln|x| - \frac{(1+\nu)}{\pi E} P \tag{4}$$

In our case, as proposed, a nonuniform distribution of pressure is acting. This condition will be satisfied if it is replaced the concentrated normal force P by $\int_{-a}^{a} p(s)ds$ in the last expression. On this condition, the displacement results in

$$u_z = \frac{2\,(1-\nu^2)}{\pi E} \int_{-a}^{a} p(s) \ln|x-s|\, ds - \frac{(1+\nu)}{\pi E} \int_{-a}^{a} p(s)ds \tag{5}$$

and, the gradient of displacement at contact surface [3] is

$$\frac{\partial u_z}{\partial x} = -\frac{2\,(1-\nu^2)}{\pi E} \int_{-a}^{a} \frac{p(s)}{x-s} ds \tag{6}$$

As is noted, the $x-$variable has been changed by $x - s$ in order to restrict the integration to the load zone.

Since the indentation process is restricted to conformable contact, as mentioned in the introduction section, equations (3) and (6) (in dimensionless form) can be combined

$$\int_{-1}^{1} \frac{p(S)}{X-S} dS = -\frac{\pi E}{2(1-\nu^2)} \left(\frac{2S^3}{3R^3} + \frac{S}{R} \right) \tag{7}$$

where $S = s/a$.

Equation (7) is a singular integral equation [4], which has the solution

$$\begin{aligned} p(X) &= -\frac{E}{2\pi\left(1-\nu^2\right)\left(1-X^2\right)^{1/2}} \int_{-1}^{1} \frac{\left(1-S^2\right)^{1/2}}{S-X}\left[\frac{2S^3}{3R^3}+\frac{S}{R}\right] dS \\ &\quad + \frac{2D}{a\left(1-X^2\right)^{1/2}} \end{aligned} \tag{8}$$

Using the Cauchy's principal value for the integral in equation (8), we get

$$\begin{aligned} p(X) &= -\frac{E}{2\left(1-\nu^2\right)\left(1-X^2\right)^{1/2}}\left[\frac{2X^4}{3R^3}-\frac{X^2}{3R^3}+\frac{X^2}{R}-\frac{1}{12R^3}-\frac{1}{2R}\right] \\ &\quad + \frac{2D}{a\left(1-X^2\right)^{1/2}} \end{aligned} \tag{9}$$

If $W$is the applied load and taking into account that $\frac{W}{a} = \int_{-1}^{1} p\left(X\right) dX$, the D constant can be determined

$$D = \frac{W}{2\pi} \tag{10}$$

Substituting equation (10) into equation (9) results in

$$\begin{aligned} p(X) &= -\frac{E}{2\left(1-\nu^2\right)\left(1-X^2\right)^{1/2}}\left[\frac{2X^4}{3R^3}-\frac{X^2}{3R^3}+\frac{X^2}{R}-\frac{1}{12R^3}-\frac{1}{2R}\right] \\ &\quad + \frac{W}{\pi a\left(1-X^2\right)^{1/2}} \end{aligned} \tag{11}$$

Last equation is the *polynomial pressure* in the cross direction of cylindrical body as a function of E and W. To obtain the polynomial pressure as a function of W only, the symmetry condition of the distribution $p(X)$ must be used, that is $\frac{\partial p(X)}{\partial X} = 0$ at $X = 0$ since the pressure distribution have a point of inflection in this point. Therefore, the *polynomial pressure* as a function of W only is given by

$$p(X) = \frac{4W}{9\pi a}\left[\frac{2X^2+3R^2+1}{2R^2-1}\right]\left(1-X^2\right)^{1/2} + \frac{2W}{3\pi a\left(1-X^2\right)^{1/2}}\left(\frac{3R^2-2}{2R^2-1}\right) \tag{12}$$

To visualize graphically this result, Figure 2. schematically shows the pressure distribution corresponding to equation (12).

2.2 Stresses generated in the body of circular section

The polynomial pressure acting on the surface (Figure 2) produces stresses in the elastic body by reaction (Newton's law). These stresses can be determined by substituting the polynomial pressure (equation (12)) into σ_x, σ_z, τ_{xz} of the function of stress proposed by Flamant. Thus, if it is replaced $X = x/a$, $S = s/a$, $Z = z/a$ and changed X *by* X-S in the resultant equations, the dimensionless stresses acting in the elastic body are given by

$$\begin{aligned}
\sigma_x &= \\
&\frac{-2Z}{\pi}\left[\frac{4W}{9\pi a\left(2R^2-1\right)}\int_{-1}^{1}\frac{\left(2X^2+3R^2+1\right)\left(1-S^2\right)^{1/2}(X-S)^2\,dS}{\left[(X-S)^2+Z^2\right]^2}\right. \\
&\left.+\frac{2W}{3\pi a}\left[\frac{3R^2-2}{2R^2-1}\right]\int_{-1}^{1}\frac{(X-S)^2}{\left(1-S^2\right)^{1/2}\left[(X-S)^2+Z^2\right]^2}dS\right] \\
\sigma_z &= \\
&\frac{-2Z^3}{\pi}\left[\frac{4W}{9\pi a\left(2R^2-1\right)}\int_{-1}^{1}\frac{\left(2X^2+3R^2+1\right)\left(1-S^2\right)^{1/2}dS}{\left[(X-S)^2+Z^2\right]^2}\right. \\
&\left.+\frac{2W}{3\pi a}\left[\frac{3R^2-2}{2R^2-1}\right]\int_{-1}^{1}\frac{1}{\left(1-S^2\right)^{1/2}\left[(X-S)^2+Z^2\right]^2}dS\right] \qquad (13)\\
\tau_{XZ} &= \\
&\frac{-2Z^2}{\pi}\left[\frac{4W}{9\pi a\left(2R^2-1\right)}\int_{-1}^{1}\frac{\left(2X^2+3R^2+1\right)\left(1-S^2\right)^{1/2}(X-S)\,dS}{\left[(X-S)^2+Z^2\right]^2}\right. \\
&\left.+\frac{2W}{3\pi a}\left[\frac{3R^2-2}{2R^2-1}\right]\int_{-1}^{1}\frac{(X-S)}{\left(1-S^2\right)^{1/2}\left[(X-S)^2+Z^2\right]^2}\,dS\right]
\end{aligned}$$

Last relations are the equations serving to determine the stresses in elastic body of circular section when a polynomial pressure in the cross direction is applied.

For the case of two symmetric polynomial pressures acting oppositely to each other as depicted in Figure 3, the total stress can be determined by superposition of all the contributions; to knowing

1. the contribution produced by the upper indenter (equations (13)),
2. the contribution produced by the lower indenter (equations (13), replacing Z by "$2R - Z$"),
3. the necessary biaxial tension to balance the free area of pressure

$$\sigma_{biaxial} = -2\ \sigma_Z\big|_{Y=1.25}^{X=1.25} = k(Z), \qquad \text{for } Z > 0.5 \tag{14}$$

By using all contributions to total stress and determining the principal stresses, the magnitude of maximum-shear-stress at each point of elastic body it could be obtained. A collection of shear stress values has been obtained and is summarized in Table 1. To visualize these values, in the image of Figure 4. is shown an idealized circular section containing patterns of lines. These lines are representing isostresses.

3 Experimental part

To test if proposed model of polynomial pressure predicts the elastic behavior of a body of circular section when being indented, the photo-elasticity technique was employed [5]. A disk manufactured in polymeric material of 100 mm in diameter and 25.4 mm thickness was tested under diametrical compression. Two opposite indenters manufactured in austenitic steel were utilized. At start, the radius of the cavities and the radius of the disk were equals and the ratio$r/a$was kept constant. An optical photograph of disk by using the mentioned technique at moment at which the disk is compressed was obtained.

4 Results and discussion

The photographic image obtained "*in situ*" by using the photo-elasticity technique is shown in Figure 5. In the illuminated zone, contrasted bands (orange) are clearly seen. These bands are associated with critical variations of the internal stresses with respect to the yellow regions. In the dark zone, the profiles of the upper indenter and the lower indenter can be distinguished. In front of the indenter's wedges, inside the illuminated zone, the orange bands delineate elongated and concentric small curves, which indicates a stress concentration in these points. By observing the patterns of lines which were built using the polynomial approach (Figure 4), a great similarity exist between the lines and the orange bands of the photographic image for the considered points. However, a discrepancy is noted in the central zone when comparing the images; about this, there is work in process.

It is worth to mention that the indenter design plays an important role, since, in our case, the indenter concavity shape allowed reaching the theoretical pressure of model. As for the analysis, the application of the components σ_x, σ_z, τ_{xz} of the function of stress proposed by Flamant to a non-planar surface is justified, since the direction of stress remain valid even in this case. Moreover, the substitution of the normal force by a distribution of pressure is justified, since such distribution can be considered as a collection of concentrated normal forces with magnitudes depending on position. For our case, it is clear that the forces of such collection located at $X \neq 0$ are not normal, but thinking that such forces have a normal component to contact surface, the problem is overcome.

Finally, the selection of the polynomial was made by numerical method of manner that the polynomial proposed could describe, possible closest, the profile of the cavity of the indenter.

5 Conclusions

Based on the theoretical reasoning established, it is concluded that, the function of stress proposed by Flamant, for the case of the unitary pressure that is produced by a concentrated normal force in elastic half-space, it is applicable to a circular section.

The similarity between the patterns of lines predicted by the polynomial model and the photo-elasticity image indicates that the model agrees, at least qualitatively (which is an advantage over other models), with the experiment for the elastic regime.

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Polinomijalni pristup odredjivanju napona u elastičnom telu kružnog poprečnog preseka utiskivanjem konformabilnim kontaktom

UDK 620.171.5

Polinom stepena višeg od dva koji opisuje konkavnost utiskivača je predložen. Time je gradijent pomeranja izveden i kombinovan sa odgovarajućim predloženim od Timošenka i Gudijera u cilju dobijanja polinomijalnog rasporeda pritiska u poprečnom pravcu žice. Korišćenjem "polinomijalnog" pritiska u naponskoj funkciji predloženoj od Flamanta dobijaju se jednačine naponskog stanja u poprečnom preseku žice. Za proširenje analize na dva suprotna utiskivača posmatraju se: naponi izazvani svakim utiskivačem kao i dvoosni napon za balansiranje slobodne površine pritiska. Konačno, korišćenjem svih doprinosećih ļanova totalnom naponu kao i odredjivanjem glavnih napona dobija se maksimalna vrednost smičuég napona. U cilju provere realnosti modela mreže linija jednakih napona se uporedjuju sa fotoelastičnom slikom pokazujući dobro slaganje.

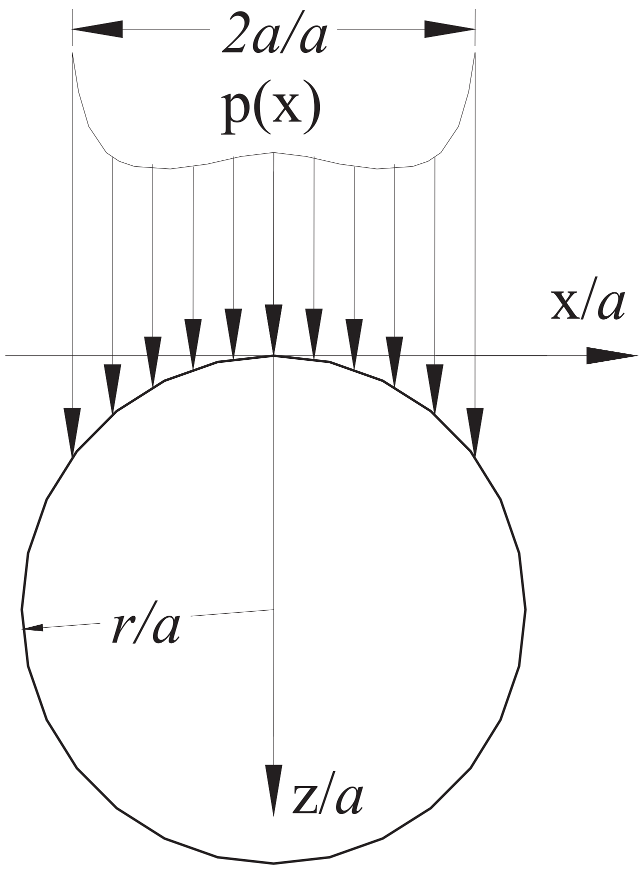

Figure 2: Schematic diagram showing the polynomial distribution of pressure along the arch of circle. Note the symmetrical character of the pressure distribution.

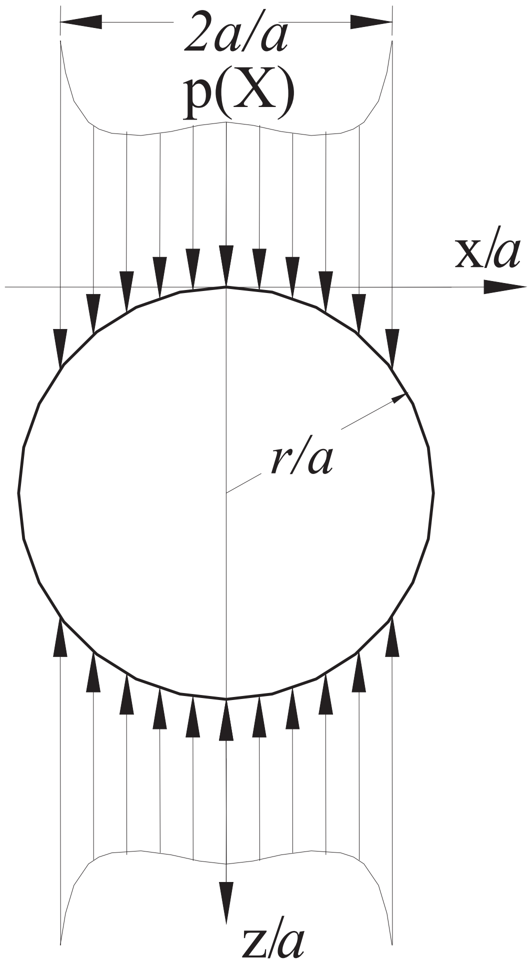

Figure 3: Schematic diagram showing two opposite pressures with polynomial distributions.

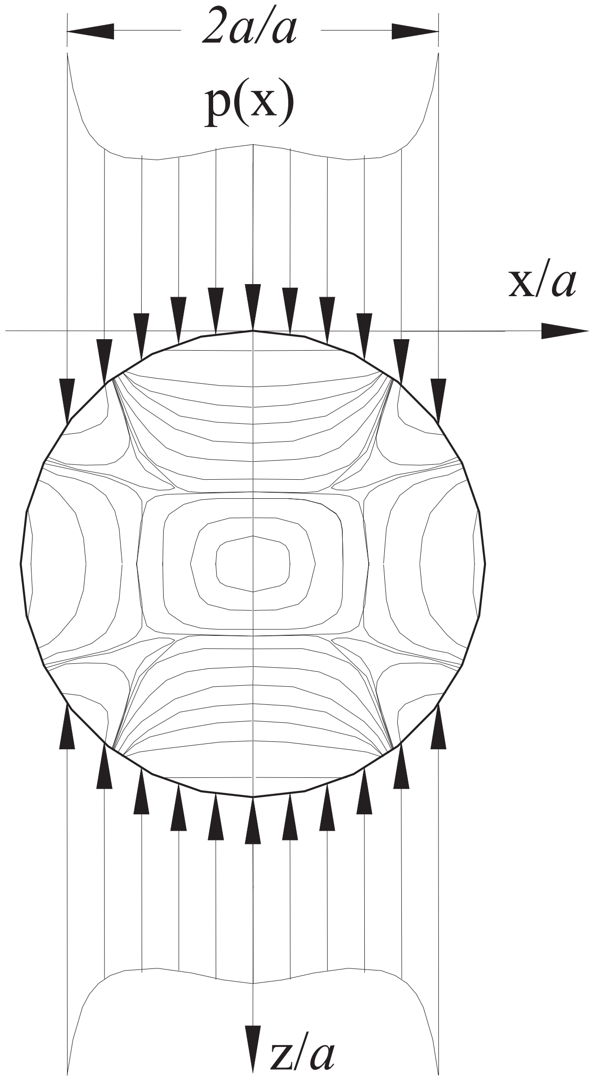

Figure 4: Schematic diagram showing patterns of lines inside the circular section for the case of two opposite pressures with polynomial distributions. The lines are representing isostresses.

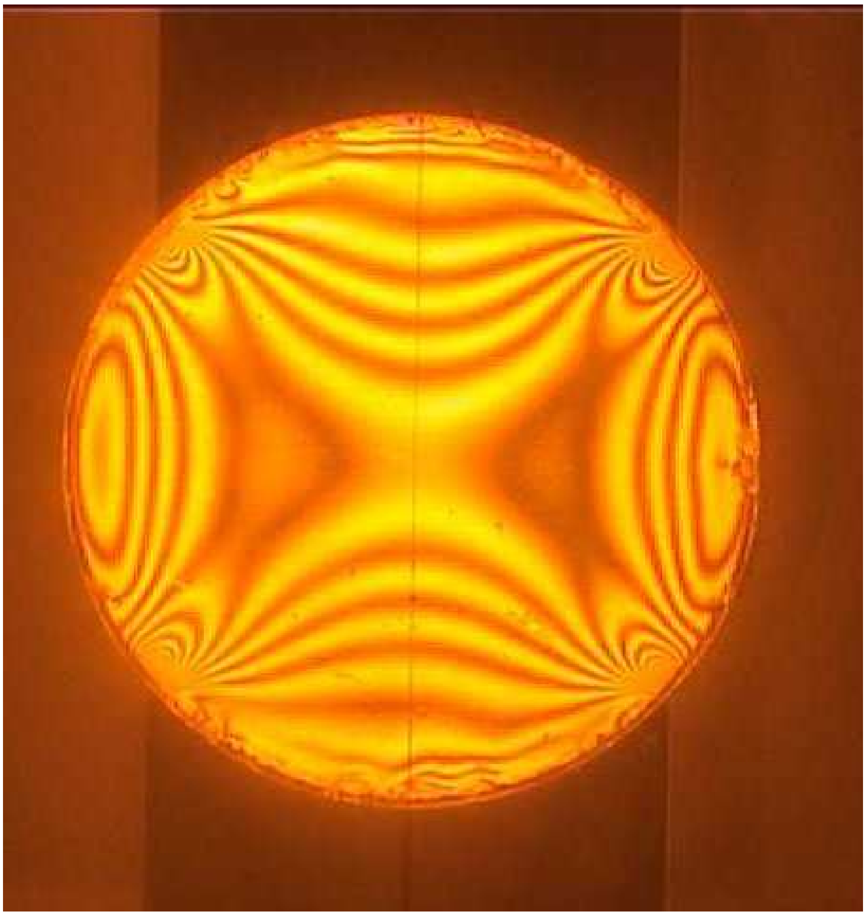

Figure 5: Photo-elasticity image showing patterns of bands (orange) at interior of illuminated zone. The bands indicate critical variations of the internal stresses in the body of polymeric material when is compressed.

	$x/a =$ 0	$x/a =$ 0.2	$x/a =$ 0.4	$x/a =$ 0.6	$x/a =$ 0.8	$x/a =$ 1	$x/a =$ 1.25
$z/a = 0$	0.246	0.243	0.235				
$z/a = 0.1$	0.302	0.302	0.3	0.304			
$z/a = 0.2$	0.361	0.362	0.368	0.394			
$z/a = 0.3$	0.417	0.421	0.436	0.491	0.719		
$z/a = 0.4$	0.469	0.474	0.495	0.557	0.728		
$z/a = 0.5$	0.115	0.52	0.541	0.593	0.699	0.806	
$z/a = 0.6$	0.554	0.558	0.574	0.609	0.664	0.702	
$z/a = 0.7$	0.585	0.588	0.598	0.615	0.635	0.635	
$z/a = 0.8$	0.61	0.611	0.613	0.616	0.613	0.591	
$z/a = 0.9$	0.628	0.628	0.624	0.615	0.596	0.562	
$z/a = 1$	0.641	0.639	0.631	0.614	0.585	0.543	
$z/a = 1.1$	0.65	0.646	0.635	0.613	0.578	0.531	
$z/a = 1.2$	0.654	0.65	0.637	0.612	0.575	0.526	0.457

Table 1: Values of the dimensionless maximum-shear-stress $\frac{\tau}{p_0}$, for $0 \leq \frac{x}{a} \leq 1.25$ and $0 \leq \frac{z}{a} \leq 1.2$, obtained using a polynomial distribution of the pressure in the cross direction of the circular body. The assumed values for $''r''$ and $''a''$ were 3 and 2.4 respectively. *The value of the function $p(X)$ at $X = 0$ is p_0