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Elastic - plastic analysis of crack on bimaterial interface

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Abstract

In this paper are presented solutions for the stress and displacement fields for a crack that lies along the interface of an elastic and elastic - plastic material and for a crack between two different elastic - plastic materials. These solutions are obtained using the J_2-deformation theory with the power - law strain hardening. In this paper results are described for a small scale yielding at the crack tip. The near tip fields do not have a separable singular form, of the HRR type fields, as in homogeneous media, they do, however bare interesting similarities to certain mixed - mode HRR fields. Under the small scale yielding the elastic fields are specified by a complex stress intensity factor and phase angle loading, while plastic field is characterized by a new phase angle. The size of plastic zone in plane strain and plane stress and displacement fields at the crack tip for the new phase angle are obtained. The crack tip opens smoothly and the crack opening displacement is scaled by the J - integral. The whole analysis is performed by application of the Mathematica symbolic programming routine.

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1 Introduction

There are not many papers, which studied the elastic - plastic analysis of an interface crack. This problem was analyzed by Shih and Asaro (1988, 1989) and Shih, Asaro and O′Dowd (1991). The purpose of this paper is to obtain solutions for problem of an interfacial crack between the two elastic - plastic materials within the framework defined by the non-linear fracture mechanics, Hutchinson (1991).

In this paper are described structures of the stress and displacement fields at the tip of a crack lying on the interface between the elastic - plastic materials under the small scale yielding. In that sense, in Figure 1 is shown the centered crack in plane strain between two elastic - plastic materials.

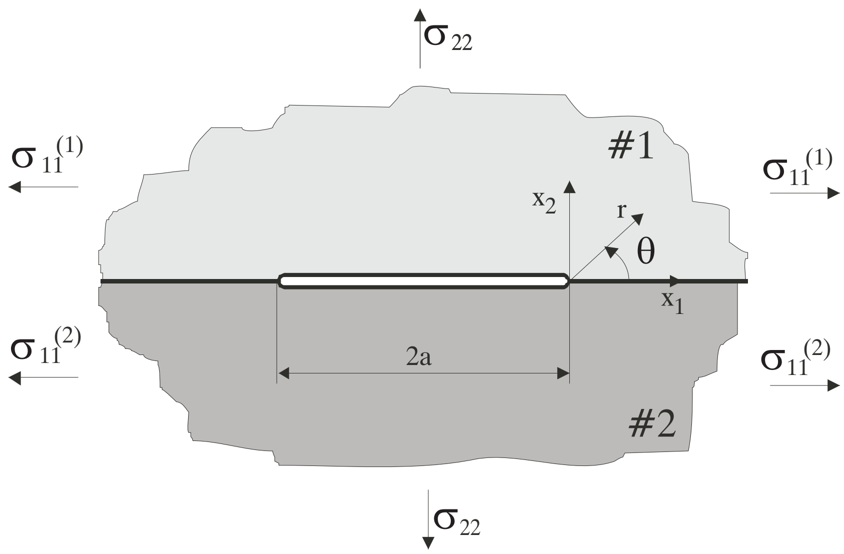

Figure 1: Interface crack between the two elastic - plastic materials

We refer to Figure 1, which shows a crack lying along an interface separating two elastic - plastic materials whose behavior is described by the J_2 - deformation theory. This problem is the starting point for solving several problems interesting for practice, such as micro cracking and microscopic fracture.

The fields at the crack tip will be analyzed for the bonding of two elastic materials, elastic and an elastic - plastic material and the bonding of the two elastic - plastic materials, under the small scale yielding conditions. Interfacial crack tip behavior is more complex than one found for cracks in homogeneous media for which the proportional remote loading induces proportional stressing near the crack tip.

2 Structure of fields under small scale yielding

For the purpose of developing the structure of the fields under small scale yielding, it is helpful to write the complex stress intensity factor as a magnitude and a phase, i.e.:

$$K = K_1 + iK_2 = |K|\, e^{i\psi} \tag{1}$$

Traction on the bonding surface near the tip can be written as:

$$\sigma_{22} + i\sigma_{12} = \frac{K}{\sqrt{2\pi r}} \left(\frac{r}{L}\right)^{i\varepsilon} = \frac{|K|}{\sqrt{2\pi r}} e^{i\left(\psi + \varepsilon \ln\left(\frac{r}{L}\right)\right)} \tag{2}$$

In this expression the length of the crack, denoted by L, may be associated with the length 2a indicated in Figure 1. The variation of the near - tip stresses along any radial line has the form given by equation (2). In the small scale yielding problem, the actual crack problem is replaced by a semi - infinite crack in an infinite medium with asymptotic boundary condition that at large r the field approaches the form given by (2). If one or both of the bonded solids deform plastically, the plastic zone size and shape will depend on the elastic and plastic properties of the two solids and stress intensity factors. The deformable medium is taken to be described by the J_2 - deformation theory.

Let σ_{01} and σ_{02} be the yield strengths of the materials 1 and 2, respectively. It is convenient to denote the yield strength of the weaker material by $\sigma_0 = min(\sigma_{01}, \sigma_{02})$. The strain - hardening exponents are n_1

and n_2, Young's modulus and Poisson's ratios are E_1, ν_1 and E_2, ν_2, respectively. Under small scale yielding, the stresses depend on the stress intensity factor **K;** and material properties on the dimensionless ratios $\sigma_{01}/\sigma_{02}, E_1/E_2, \nu_1/\nu_2$ and n_1/n_2. From dimensional considerations, equilibrium condition and equation (1), Shih and Asaro (1988, 1989), one obtains:

$$\sigma_{ij} = \sigma_0 f_{ij}\left(\frac{r\sigma_0^2}{K\bar{K}},\ \theta,\ phase\left\{K\left(\frac{r}{L}\right)^{i\varepsilon}\right\},\ dimensionless\ parameters\right), \tag{3}$$

where f_{ij} is a dimensionless function of dimensionless material properties. The dependence of σ_{ij} on **K** and distance from the crack tip, Shih and Asaro (1988, 1989) is:

$$\sigma_{ij} = \sigma_0 f_{ij}\left(\frac{r\sigma_0^2}{K\bar{K}},\ \theta,\ \psi + \varepsilon \ln\left(\frac{r}{L}\right)\right), \tag{4}$$

where $\psi + \varepsilon \ln(r/L)$ phase $K(r/L)^{i\varepsilon}$.

Let ξ denote the new phase parameter, defined by Shih and Asaro (1988, 1989):

$$\xi = \psi + \varepsilon \ln\left(\frac{K\bar{K}}{L\sigma_0^2}\right), \tag{5}$$

Using equation (5) equation (4) can be written as:

$$\sigma_{ij} = \sigma_0 f_{ij}\left(\frac{r\sigma_0^2}{K\bar{K}},\ \theta,\ phase\left\{\left(\frac{r\sigma_0^2}{K\bar{K}}\right)^{i\Sigma}\right\},\ \xi\right). \tag{6}$$

Since ξ is the phase angle of a complex quantity, f_{ij} has periodicity of 2π with respect to the argument ξ, i.e.:

$$f_{ij}(...,\ \xi) = f_{ij}(...,\ \xi + m\pi), \quad m = 2, 4, 6, ... \tag{7}$$

Due to the linearity of the equilibrium and strain - displacement equations, it also holds:

$$f_{ij}(...,\ \xi) = f_{ij}(...,\ \xi + m\pi), \quad m = 1, 3, 5, ... \tag{8}$$

Thus, ξ serves as the phase parameter of the fields in the small scale yielding formulation just as ψ is the phase angle of the linear elastic singular fields.

The mathematical structure of the fields, expressed by equations (6) and (7), was derived regardless to contact between the crack faces. In fact, displacement jumps across the crack faces must have the form:

$$\Delta u_i = \varepsilon_0 \frac{K\bar{K}}{\sigma_0^2} g_i \left(\frac{r\sigma_0^2}{K\bar{K}},\ phase\left\{ \left(\frac{r\sigma_0^2}{K\bar{K}} \right)^{i\Sigma} \right\},\ \xi \right), \tag{9}$$

where ε_0 is the yield strain and the dimensionless function g_i has the periodic structure, expressed by (7) and (8). In this work our interest is restricted to the range of ξ where the crack faces are not in contact.

The effective stress σ_e has the form:

$$\sigma_e = \sigma_{ij} f_e g \left(\frac{r\sigma_0^2}{K\bar{K}},\ \theta,\ phase\left\{ \left(\frac{r\sigma_0^2}{K\bar{K}} \right)^{i\Sigma} \right\},\ \xi \right). \tag{10}$$

The elastic - plastic boundary in the weaker material is the locus of points where σ_e equals σ_0. Substituting these values in (10) and rearranging leads immediately to the following results for the plastic zone, Shih and Asaro (1989):

$$r_p(\theta) = \frac{K\bar{K}}{\sigma_0^2} R(\theta,\ \xi). \tag{11}$$

Here $R(\theta, \xi)$ is a dimensionless angular function, which depends on ξ and on dimensionless material properties. Since σ_e is quadratic in the stress components which have periodicity expressed by (7) and (8), the angular function R has a periodicity of π with respect to ξ:

$$R(\theta,\ \xi) = R(\theta,\ \xi + m\pi) \quad m = 1, 2, 3, ... \tag{12}$$

Rice (1988), obtained same relation for size of the plastic zone, by dimensional consideration:

$$r_p(\theta) = \frac{K\bar{K}}{\sigma_0^2}\tilde{R}\left[\phi - \varepsilon \ln\left(\frac{L}{r_p}\right)\right]. \tag{13}$$

3 Mixed - mode crack tip fields at the tip of interface crack

Application of the J- integral to the mixed - mode small scale yielding problem reveals that the asymptotic behavior of the stresses, strains and displacements near the crack tip has the form:

$$\begin{array}{l} \sigma_{ij} = \sigma_0 K^p r^{-\frac{1}{n+1}}\,\tilde{\sigma}_{ij}(\theta,\ M^p,\ n) \\ \varepsilon_{ij} = \frac{\alpha\sigma_0}{E}(K^p)^n r^{-\frac{n}{n+1}}\,\tilde{\varepsilon}_{ij}(\theta,\ M^p,\ n) \\ u_i = \frac{\alpha\sigma_0}{E}(K^p)^n r^{\frac{1}{n+1}}\,\tilde{u}_i(\theta,\ M^p,\ n) \\ \sigma_e = \sigma_0 K^p r^{-\frac{1}{n+1}}\,\tilde{\sigma}_e(\theta,\ M^p,\ n) \end{array} \tag{14}$$

In equation (14) the dimensionless angular functions $\tilde{\sigma}_{ij}$, $\tilde{\varepsilon}_{ij}$ $\tilde{u}_i$, $\tilde{\sigma}_e$ depend parametrically on the plastic mixity parameter M^p, and the hardening exponent, n. *Angular functions $\tilde{\sigma}_{ij}$ are defined in Appendix by equation (A1).* The plastic mixity is determined as, Shih (1974):

$$M^p = \frac{2}{\pi} arctg \left| \lim_{r\to 0} \frac{\sigma_{\theta\theta}(r,\ \theta = 0)}{\sigma_{r\theta}(r,\ \theta = 0)} \right|, \tag{15}$$

such that $M^p = 1$ for pure mode I and $M^p = 0$ for pure mode II. The amplitude of the HRR singularity field, K^p (plastic stress intensity factor) is defined by Shih (1974), such that the angular distribution, attains a maximum value of unity with this definition, K^p is related to the value of J by:

$$J = \frac{\alpha\sigma_0^2}{E} I_n (K^p)^{n+1}. \tag{16}$$

The factor I_n depends on the degree of mixity, M^p and n. For a wide range of these parameters the factor I_n has been determined by

Shih (1974). For purposes of this analysis, it is convenient to rescale K^p by setting $I_n \cong 1$.

For linear elasticity, the mixity parameter can be reinterpreted as:

$$M^p \to M^e \equiv \frac{2}{\pi} arctg \left| \frac{K_I}{K_{II}} \right| . \tag{17}$$

For small scale yielding, where the stresses beyond the plastic zone are those of the elastic field, M^e is also defined by:

$$M^e \equiv \frac{2}{\pi} arctg \left| \frac{\sigma_{\theta\theta}(r*, \theta = 0)}{\sigma_{r\theta}(r*, \theta = 0)} \right| , \tag{18}$$

where $r*$ is within the zone of dominance of the elastic field. In this case J is given by:

$$J = \frac{(1 - \nu^2)}{E} (K_I^2 + K_{II}^2), \tag{19}$$

for homogeneous case, and

$$J = \frac{1}{ch^2(\pi\varepsilon)} \cdot \frac{|K|^2}{E*}, \tag{20}$$

for bimaterial case, where $2/E* = 1/E_1' + 1/E_2'$ and $E' = E/(1 - \nu^2)$ for plane strain and $E' = E$ for plane stress.

4 Results and discussion

Figures 2, 3 and 4 show the angular stress distribution ahead of the crack tip along the interface between the two elastic - plastic materials *for different hardening exponents.*

There, problem considered is the plane strain interfacial crack. Materials 1 and 2 are elastic - plastic with elastic properties E_1 and E_2 where $E_2/E_1 = 2.5$ and Poisson's ratios are $\nu_1 = \nu_2 = 0.3$. For this bimaterial combination oscillatory index is $\varepsilon = 0.028$. Plastic properties of the material 1 are characterized by yield stress σ_0, strain - hardening exponent n which equals *n= 1, 3, 10* and material constant $\alpha = 0.1$.

Material 2 has yield stress σ_{02}, which is four times higher than the yield stress of material 1. The strain hardening exponent of material 2, n_2 is 10 and α_2 is 0.1. Loading conditions are defined by plastic phase angle in small scale yielding ξ.

The stress distribution σ_{ij} versus angle θ, according to equation (14), is shown for the radial distance, which is chosen to amount about 5 % of the maximum plastic zone extension. From Figures 2, 3, and 4, one can see that the maximum stress $\sigma_{\theta\theta}$ is for $\xi = 0.524$ in material 1, while for $\xi = 0$ it is on the interface. This completely agrees with the linearly elastic solution for the same bi-material combination.

One very significant characteristics of the fields, defined by (14) is that the normal stresses ahead of the crack tip are smaller for the case of the mixed Mode than that for the pure Mode I load. Another characteristics of this solution is the large shear stress ahead of the crack tip.

By comparing figures 2, 3 and 4 one can notice that the stress field in the crack tip is determined by the material with the lower strain-hardening coefficient.

The whole analysis was done by the *Mathematica* symbolic programming routine.

The comparative presentation of results of numerical procedure, defined in papers by Shih and Asaro (1988, 1989),and results obtained by application of equation (14) with use of the angular stress functions (A1) is given in Figure 5.

From Figure 5 can be noticed that results of numerical procedure (Shih and Asaro (1988, 1989) - black dotted lines) differ from analytical results (colored lines), based on application of equation (14), less than 5 %. This leads to a conclusion that the angular stress functions, defined by equations (A1) describe satisfactorily the problem of a crack tip on the interface between the two elastic-plastic materials.

By application of equations (A1) the problem of the crack along the interface between the two elastic-plastic materials can be solved in the same way as the problem of the crack along the interface of the two elastic materials, equations (1.3), Veljkovic (1998).

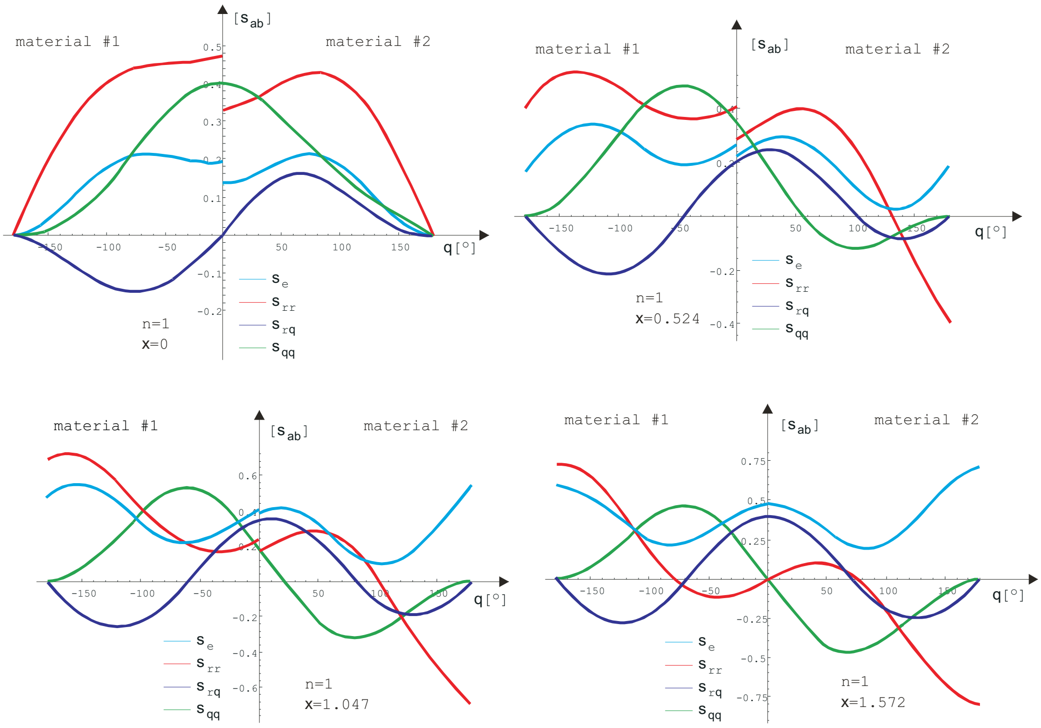

Figure 2: Angular variation of stress for a bimaterial combination for different loading conditions and strain hardening exponent n=1.

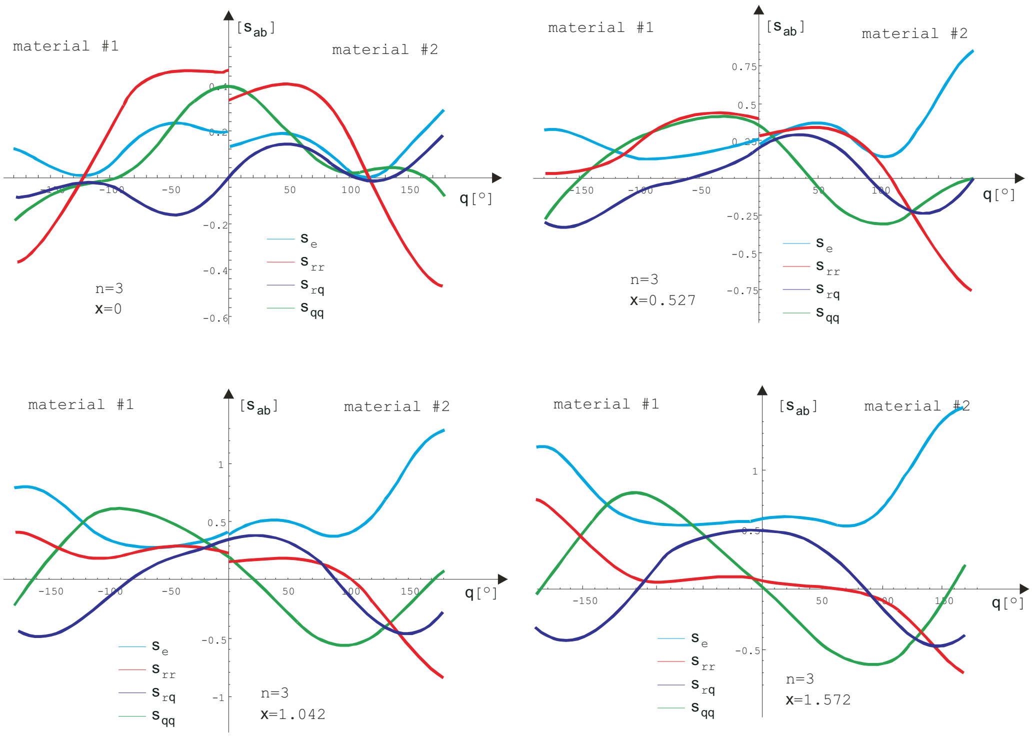

Figure 3: Angular variation of stress for a bimaterial combination for different loading conditions and strain hardening exponent n=3.

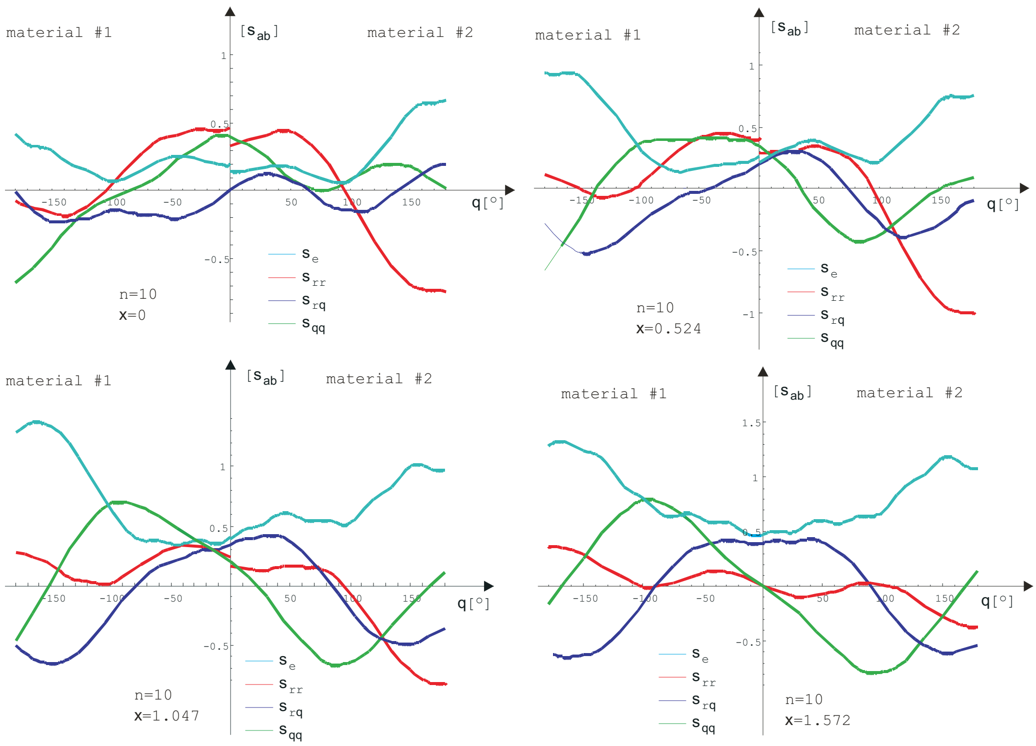

Figure 4: Angular variation of stress for a bimaterial combination for different loading conditions and strain hardening exponent n=10.

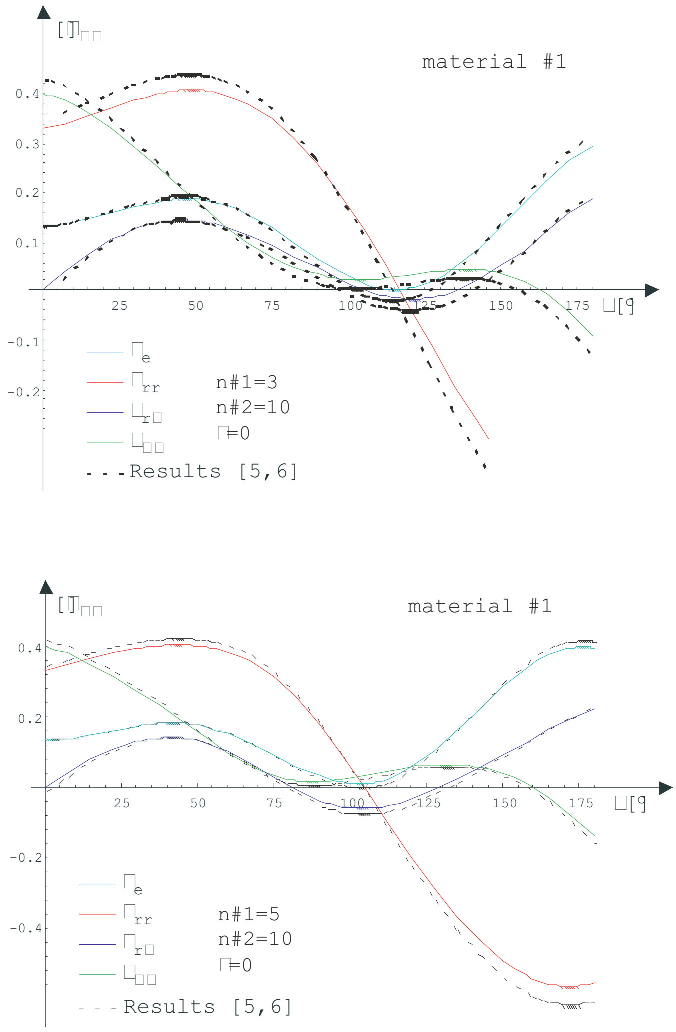

Figure 5: Comparative presentation of results obtained from equation (14) and results from papers by Shih and Asaro (1988, 1989) (black dotted lines).

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Appendix [1]

Angular functions $\tilde{\sigma}_{\alpha\beta}^{I,II}(\theta,\ n)$ for material 1, have the form:

$$
\begin{aligned}
\tilde{\sigma}_{rr}^{I}(\theta) &= -\tfrac{sh\varepsilon(\pi-\theta)}{ch\varepsilon\pi}\cos\tfrac{(n+2)\theta}{(n+1)} + \tfrac{e^{-\varepsilon(\pi-\theta)}}{ch\varepsilon\pi}\cos\tfrac{n\theta}{(n+1)}(1+\sin^2\tfrac{n\theta}{(n+1)} + \varepsilon\sin n\theta)\\
\tilde{\sigma}_{\theta\theta}^{I}(\theta) &= \tfrac{sh\varepsilon(\pi-\theta)}{ch\varepsilon\pi}\cos\tfrac{(n+2)\theta}{(n+1)} + \tfrac{e^{-\varepsilon(\pi-\theta)}}{ch\varepsilon\pi}\cos\tfrac{n\theta}{(n+1)}(\cos^2\tfrac{n\theta}{(n+1)} - \varepsilon\sin n\theta)\\
\tilde{\sigma}_{r\theta}^{I}(\theta) &= \tfrac{sh\varepsilon(\pi-\theta)}{ch\varepsilon\pi}\sin\tfrac{(n+2)\theta}{(n+1)} + \tfrac{e^{-\varepsilon(\pi-\theta)}}{ch\varepsilon\pi}\sin\tfrac{n\theta}{(n+1)}(\cos^2\tfrac{n\theta}{(n+1)} - \varepsilon\sin n\theta)\\
\tilde{\sigma}_{rr}^{II}(\theta) &= \tfrac{ch\varepsilon(\pi-\theta)}{ch\varepsilon\pi}\sin\tfrac{(n+2)\theta}{(n+1)} - \tfrac{e^{-\varepsilon(\pi-\theta)}}{ch\varepsilon\pi}\sin\tfrac{n\theta}{(n+1)}(1+\cos^2\tfrac{n\theta}{(n+1)} - \varepsilon\sin n\theta)\\
\tilde{\sigma}_{\theta\theta}^{II}(\theta) &= -\tfrac{ch\varepsilon(\pi-\theta)}{ch\varepsilon\pi}\sin\tfrac{(n+2)\theta}{(n+1)} - \tfrac{e^{-\varepsilon(\pi-\theta)}}{ch\varepsilon\pi}\sin\tfrac{n\theta}{(n+1)}(\sin^2\tfrac{n\theta}{(n+1)} + \varepsilon\sin n\theta)\\
\tilde{\sigma}_{r\theta}^{II}(\theta) &= \tfrac{ch\varepsilon(\pi-\theta)}{ch\varepsilon\pi}\cos\tfrac{(n+2)\theta}{(n+1)} + \tfrac{e^{-\varepsilon(\pi-\theta)}}{ch\varepsilon\pi}\cos\tfrac{n\theta}{(n+1)}(\sin^2\tfrac{n\theta}{(n+1)} + \varepsilon\sin n\theta)
\end{aligned}
\tag{A.1}
$$

Angular functions $\tilde{\sigma}_{\alpha\beta}^{I,II}(\theta,\ n)$ for material 2 have the same form as equation (A.1), one only needs to substitute $-\pi$ with π, and vice versa.

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Elastoplastična analiza prsline na bimaterijalnom interfejsu

UDK 539.42, 539.421

U ovom radu su predstavljena rešenja za polje napona i deformacije za prslinu koja leži na interfejsu elastičnog i elastično-plastičnog materijala i za prslinu izmedju dva različita elastično-plastična materijala.

[1]This appendix is given for easier understanding of the performed analysis, and is not included in the core text due to the limited length.

Ova rešenja su dobijena korišćenjem J2 - deformacijske teorije sa stepenim zakonom ojačanja. U radu su opisani rezultati za slučaj malog tečenja oko vrha prsline. Polja oko vrha prsline nemaju poseban singularan oblik tipa HRR polja, kao kod homogenih sredina, mada imaju interesantne sličnosti sa nekim HRR poljima za kombinovani mod. U uslovima malog tečenja elastična polja se karakterišu kompleksnim faktorom intenziteta napona i faznim uglom opterećenja, dok se plastična polja karakterišu novim faznim uglom. Dobijeni su veličina plastične zone pri ravanskom stanju deformacije i napona i polja pomeranja u vrhu prsline za nove fazne uglove. Vrh prsline se otvara ravnomerno i pomeranje otvora prsline je normalizovano J - integralom. Kompletna analiza je uradjena korišćenjem Mathematica paketa za simboličko programiranje.