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On the instability of equilibrium of a mechanical system with nonconservative forces

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Abstract

In this paper the stability of equilibrium of nonholonomic systems, on which dissipative and nonconservative positional forces act, is considered. We have proved the theorems on the instability of equilibrium under the assumptions that: the kinetic energy, the Rayleigh's dissipation function and the positional forces are infinitely differentiable functions;the projection of the positional force component which represents the first nontrivial form of Maclaurin's series of that positional force to the plane, which is normal to the vectors of nonholonomic constraints in the equilibrium position, is central and repulsive (with its centre of action in the equilibrium position).

The suggested theorems are generalization of the results from [V.V. Kozlov, Prikl. Math. Mekh. (PMM), T58, V5, (1994), 31-36] and [M.M. Veskovic, Theoretical and Applied Mechanics, 24, (1998), 139-154]. The result obtained is analogous to the result from [D.R. Merkin, Introduction to theory of the stability of motion, Nauka, Moscow (1987)], which refers to the impossibility of equilibrium stabilization in a holonomic conservative system by dissipative and nonconservative positional forces in case when the potential energy in the equilibrium position has the maximum. The proving technique will be similar to that used in the paper

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[V.V. Kozlov, Prikl. Math. Mekh. (PMM), T58, V5, (1994), 31-36].

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1 Introduction

Let us consider a scleronomic mechanical system in an n-dimensional configuration space. Let $\mathbf{x} = (x^1, ..., x^{\mathrm{n}})^{\mathrm{T}}$ be the vector of generalized coordinates (the index T denotes transposition),

$$T = (1/2)\, \dot{\mathbf{x}}^T A\,(\mathbf{x})\, \dot{\mathbf{x}} \tag{1}$$

is the kinetic energy (A($\mathbf{x}$) is the symmetric positively definite matrix nxn for each $\mathbf{x} \in \mathrm{R}^{\mathrm{n}}$) and

$$\mathbf{X}\,(\mathbf{x}, \dot{\mathbf{x}}) = (X_1\,(\mathbf{x}, \dot{\mathbf{x}})\,, ..., X_n\,(\mathbf{x}, \dot{\mathbf{x}})) \tag{2}$$

are the generalized forces. Let the motion of the described system be subjected to m ($m < n$) ideal constraints, linear in the velocities:

$$B^T(\mathbf{x})\dot{\mathbf{x}} = 0, \tag{3}$$

where B($\mathbf{x}$) is the $n \times m$ matrix of rank m. The differential equations of motion of the system described can be expressed in the form of Lagrangian equations with the multipliers of constraints:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{x}}} - \frac{\partial T}{\partial \mathbf{x}} = \mathbf{X}\,(\mathbf{x}, \dot{\mathbf{x}}) + B\,(\mathbf{x})\,\lambda, \quad B^T\,(\mathbf{x})\,\dot{\mathbf{x}} = 0, \tag{4}$$

where $\lambda = (\lambda_1, ..., \lambda_{\mathrm{m}})^{\mathrm{T}}$ denotes the vector of arbitrary coefficients. If $\mathbf{X(0,0)} = \mathbf{0}$ then $\mathbf{x} = \mathbf{0}$ is the equilibrium position of the described nonholonomic system (reactions of constraints are equal to zero, and

$\mathbf{X}(\mathbf{0},\mathbf{0})=\mathbf{0})$. The reverse statement does not hold: $\mathbf{x}=\mathbf{0}$ is also the equilibrium position in the case when $\mathbf{X}(\mathbf{0},\mathbf{0})=-B(\mathbf{0})\lambda(\mathbf{0},\mathbf{0})$ (reactions of constraints are different from zero, and $\mathbf{X}(\mathbf{0},\mathbf{0})\neq\mathbf{0}$). In the first case, the point $\mathbf{x}=\mathbf{0}$ is said to be the equilibrium position of the II kind, and in the second case it is of the I kind. We shall consider that (2) can be resolved into a positional nonconservative component and a viscose one:

$$\mathbf{X}\left(\mathbf{x},\dot{\mathbf{x}}\right)=\mathbf{X}\left(\mathbf{x}\right)-\frac{\partial\Phi}{\partial\dot{\mathbf{x}}},\tag{5}$$

where

$$\Phi=\frac{1}{2}\dot{\mathbf{x}}^{\mathrm{T}}D(\mathbf{x})\dot{\mathbf{x}},$$

the Rayleigh's dissipation function ($D-$ is the symmetrical $n\times n$ matrix).

Let us assume (**hypothesis H**): *the* $\mathbf{X}(\mathbf{x}),\mathbf{A}(\mathbf{x}),B(\mathbf{x})$ *and* $D(\mathbf{x})$ *are infinitely differentiable functions.*

Let us note that the viscose force does not affect the equilibrium conditions: if Φ=0, then $\mathbf{x}$=$\mathbf{0}$ is the equilibrium position if and only if $\mathbf{X}(\mathbf{0})$=$\mathbf{0}$ or $\mathbf{X}(\mathbf{0})$=-B($\mathbf{0}$)λ($\mathbf{0}$,$\mathbf{0}$).

2 The instability of equilibrium of non-holonomic systems

Let $\mathbf{x}$=$\mathbf{0}$ be the equilibrium position of the second kind: then the reaction of the constraint is equal to zero, but $\mathbf{X}(\mathbf{0})$=$\mathbf{0}$. Let π be the $(n-m)$-dimensional plane given in the following way:

$$\pi=\{\mathbf{x}\in R^{n}:B_0^{T}\mathbf{x}=0\},\tag{6}$$

where B_0=B($\mathbf{0}$). Expand $\mathbf{X}(\mathbf{x})$ in a Maclaurin's series:

$$\mathbf{X}\left(\mathbf{x}\right)=\mathbf{X}_{\mathrm{p}-1}\left(\mathbf{x}\right)+\mathbf{X}_{\mathrm{p}}\left(\mathbf{x}\right)+...\quad .\tag{7}$$

Here $\mathbf{X}_{\mathrm{s}}(\mathbf{x})$ is a homogeneus form of degree s, so that p$\geqslant$2. Let us denote the projection of the form $\mathbf{X}_{\mathrm{p}-1}(\mathbf{x})$ on the plane π by $\mathbf{X}^*_{p-1}(\mathbf{x})$. It is clear that $\mathbf{X}^*_{p-1}(\mathbf{x})$ is also the homogeneous form of the p-1 degree.

Theorem 1. *Let the following conditions be fulfilled:*

a). *the hypothesis* (**H**) *holds,*

b). *the matrix* D($\mathbf{x}$) *is positive definite for each* $\mathbf{x}$,

c). *there is a c-vector whose beginning is at the tht point* $\mathbf{x}$=$\mathbf{0}$*, which lies in the plane* π *so that*

$$\mathbf{X}^*_{p-1}(\mathbf{c}) = \kappa\mathbf{c}, \quad \kappa > 0. \tag{8}$$

Then there is a solution of the equations (4) so that

$$\|\mathbf{x}(t) \oplus \dot{\mathbf{x}}(t)\| \to \mathbf{0} \tag{9}$$

when $t \to -\infty$*. Especially,* $\mathbf{x} = \mathbf{0}$, $\dot{\mathbf{x}} = \mathbf{0}$ *is unstable.*

Theorem 1 generalizes the result from [10], which refers to the instability of equlibrium of nonholonomic dissipative systems. Theorem 1 holds in the case when the constraints (3) are integrable. Especially, if the constraints (3) are absent, if the positional forces are conservative and if the first form in Maclaurin's series for the potential energy in equilibrium position does not have the minimum, Theorem 1 coincides with the result from paper [1], referring to the instability of equilibrium of a holonomic dissipative system.

For p=2, the problem of existence of the asymptotic solution (9) is solved by means of the first Liapunov's method. The solution is looked for in the form of the series

$$\mathbf{x}(t) = \lambda\mathbf{c}e^{\kappa t} + \sum_{i=2}^{\infty}\mathbf{h}_i(t)e^{i\kappa t}, \;\; \kappa = const > 0, \quad \lambda = const \tag{10}$$

where $\mathbf{h}_{\mathrm{i}}(t)$ – the polynomial vector functions. In the analytical case, the series (10) are convergent. If the conditions of the hypothesis (**H**) hold, the series (10) are, as a rule divergent. However, on the basis of Kuznetsov's theorem [7], each series (10) which formally fulfils the equations (4) has its corresponding real solution of the form (9).

Let us assume that p⩾3. Before proving Theorem 1, let us consider in this case certain results on the instability under the conditions more general than those in Theorem 1. Note the equation

$$\dot{\mathbf{x}} = \mathbf{f}_{p-1}(\mathbf{x}), p \geqslant 3, \tag{11}$$

where $\mathbf{f}_{\mathrm{p}-1}(\mathbf{x})$ is a n-vector the coordinates of which are of the homogenous form of degree p-1⩾2 and $\mathbf{f}_{\mathrm{p}-1}(\mathbf{0})=\mathbf{0}.$

Let us assume the solution of the equations (11) is in the form

$$\mathbf{x}(t) = \lambda \mathbf{c}(-t)^{-\mu}, \tag{12}$$

where **c** - is a n-vector different from zero; μ - is a scalar. If such a solution exists, it is necessary that vector **c -** is a nontrivial solution of the algebraic equation

$$\mu \mathbf{c} = \mathbf{f}_{p-1}(\mathbf{c}), \tag{13}$$

whereby μ=1/(p-2). Note the equation

$$\dot{\mathbf{x}} = \mathbf{f}_{p-1}(\mathbf{x}) + \mathbf{O}(\parallel \ddot{\mathbf{x}} \parallel) + \mathbf{O}(\parallel \dot{\mathbf{x}} \parallel^2) + \mathbf{O}(\parallel \mathbf{x} \parallel^p). \tag{14}$$

when $\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}} \to \mathbf{0}$. Let us assume that $\mathbf{O}(\parallel \bullet \parallel)$ are infinitely differentiable functions.

Theorem 2. *If there exists the point* **c** *upon the unit sphere* $\|\mathbf{x}\| = \|\mathbf{c}\|$, $\|\mathbf{u}\| = (\mathbf{u}^{\mathrm{T}} I \mathbf{u})^{1/2}$ (I-*unit* $n \times n$ *matrix), so that*

$$\mathbf{f}_{p-1}(\mathbf{c}) = \kappa \mathbf{c}, \quad \kappa = const. > 0$$

then, the equations (11 have the solution which asymptotically approaches, so that

$$\| \mathbf{x}(t) \oplus \dot{\mathbf{x}}(t) \| \rightarrow \mathbf{0}$$

when $t \rightarrow -\infty$.

Corollary. *Let the equations (14) describe the dynamics of a mechanical system. Under the conditions of Theorem 2,* $\mathbf{x} = 0,\ \dot{\mathbf{x}} = 0$ *is an unstable equilibrium.*
The proof of Theorem 2 is contained in a hidden form in [1].

Proof of Theorem 1

We shall firstly show that the differential equations of motion (4) can be expressed in the form (14). For that purpose, let us express (4) in the following way:

$$A\ddot{\mathbf{x}} + \Gamma(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{X}(\mathbf{x}) - D(\mathbf{x})\dot{\mathbf{x}} + B(\mathbf{x})\lambda, \quad B^T(\mathbf{x})\dot{\mathbf{x}} = \mathbf{0} \tag{15}$$

$\Gamma(\mathbf{x}, \dot{\mathbf{x}})$ denotes the square n-vector function, with respect to $\dot{\mathbf{x}}$. Let us eliminate vector $\lambda \in \mathrm{R}^{\mathrm{m}}$ in the following way: let us find $\dot{\mathbf{x}}$ from (15) (it is possible, since $det D \neq 0$) and then substitute the equations thus obtained in the remaining m equations. In that way, we obtain a system of n equations of the form

$$\dot{\mathbf{x}} = D^{-1}(I - G(\mathbf{x}))\mathbf{X}(\mathbf{x}) - D^{-1}\Gamma(\mathbf{x}, \dot{\mathbf{x}}) + \mathbf{O}(\| \ddot{\mathbf{x}} \|), \tag{16}$$

where $\mathrm{G}(\mathbf{x}) = \mathrm{B}(\mathbf{x})(\mathrm{B}^{\mathrm{T}}(\mathbf{x})\mathrm{D}^{-1}(\mathbf{x})\mathrm{B}(\mathbf{x}))^{-1}\mathrm{B}^{\mathrm{T}}(\mathbf{x})\mathrm{D}^{-1}(\mathbf{x})$. Without reducing the generality, we can assume that the generalized coordinates are chosen so that

$$D(\mathbf{x}) = I + \bar{D}(\mathbf{x}), \quad \bar{D}(\mathbf{0}) = 0. \tag{17}$$

Taking also into account (16) as well as the fact that $\mathbf{X}(\mathbf{x})$, $\mathrm{D}(\mathbf{x})$, $\mathrm{B}(\mathbf{x})$, $\mathrm{A}(\mathbf{x})$ are infinitely differentiable functions, the equations of motion (16) can be expresed in the form of the equations (14), i.e.

$$\dot{\mathbf{x}} = (I - G_0)\mathbf{X}_{\mathrm{p}-1}(\mathbf{x}) + \mathbf{O}(\| \ddot{\mathbf{x}} \|) + \mathbf{O}(\| \dot{\mathbf{x}} \|^2) + \mathbf{O}(\| \mathbf{x} \|^p), \tag{18}$$

where $G_0 = G(\mathbf{0})$. The right hand side of the equations (18) is infinitely differentiable. Since

$$D^{-1}(\mathbf{0}) = I \tag{19}$$

then

$$G_0 = B_0(B_0^T B_0)^{-1} B_0^T, \tag{20}$$

where $B_0 = B(\mathbf{0})$. Let us note that G_0 is the matrix of the symmetrical operator.

It is important to note that I-G_0 is the projector on the hyper plane π. Hence, we obtain

$$(I - G_0)\,\mathbf{X}_{p-1}\,(\mathbf{x}) = \mathbf{X}^*_{p-1}\,(\mathbf{x})\,, \tag{21}$$

and therefrom, for the vector $\mathbf{c}\in\pi$ from the conditions (b) of Theorem 2,

$$(I - G_0)\,\mathbf{X}_{p-1}\,(\mathbf{c}) = \mathbf{X}^*_{p-1}\,(\mathbf{c}) = \kappa\cdot\mathbf{c}. \tag{22}$$

As differential equations of motion are autonomous, from the existence of asymptotic solution with the described behaviour it follows that the mentioned equilibrium state is unstable. Theorem 2 holds. At p$\geqslant$3 (see [1]), the following asymptotic series corresponds to the mentioned solution $\mathbf{x}$(t) with asymptotic behaviour

$$\mathbf{x}(t) = \left(\frac{1}{\kappa(p-2)}\right)^{\frac{1}{p-2}} (-t)^{-\frac{1}{p-2}}\cdot\mathbf{c} + \sum_{i=2}^{\infty} a_i\cdot(-t)^{-\frac{1}{p-2}i} \tag{23}$$

when t$\to -\infty$, where $\boldsymbol{a}_{\mathrm{i}}$- the polynomial vector functions of ln(-t). Then

$$\mathbf{x}_0(t) = \left(\frac{1}{\kappa(p-2)}\right)^{\frac{1}{p-2}} (-t)^{-\frac{1}{p-2}}\cdot\mathbf{c} \tag{24}$$

is the solution to the "simplified" equation,

$$\dot{\mathbf{x}} = (I - G_0)\mathbf{X}_{p-1}(\mathbf{x}). \tag{25}$$

Note 1. In the case when in (4) we put that Φ=0**,** we obtain a mechanical system, as we have shown in the introduction, with the same equilibrium positions, and Theorem 3 coincides with the result from the paper [3], referring to the nonholonomic system. The paper [1] has shown that the asymptotic series (23) when D($\mathbf{x}$)→0 is not reduced to the asymptotic series which corresponds to the asymptotic solution of the system at D($\mathbf{x}$)=0.

Let us consider the conditions under which the equation

$$\mathbf{X}^*_{p-1}(\mathbf{c}) = \kappa\mathbf{c}, \quad \kappa > 0 \tag{26}$$

has nontrivial solutions, i.e. solutions $\mathbf{c}\neq\mathbf{0}$. The following proposition holds (see[2] or [3]):

> **Lemma 1.** *Let* $\mathbf{x}$=$\mathbf{0}$ *be isolated zero of the vector field* $\mathbf{X}^*_{p-1}(\mathbf{x})$, *and let the dimension* n *of the configuration space be odd. Then the equation* $\mathbf{X}^*_{p-1}(c) = \kappa c$, $\kappa \neq 0$ *always has a non-trivial solution.*

This statement is based on the fact known from topology that every vector field on the sphere of even dimension has at least one singular point. It is true that the κ does not always have to be positive.

It is clear that the Lemma 1 holds even when the constraints (3) are integrable. Especially, if the constraints (3) are absent, m=0, and $\mathbf{X}^*_{\mathrm{p}-1}(\mathbf{x})$= $\mathbf{X}_{\mathrm{p}-1}(\mathbf{x})$. In this case, the results of Lemma 1 refer to holonomic systems with n-degrees of freedom.

3 Examples

Let us consider the cases when conservative and circulatory forces, as well as forces of viscose friction, act on the system simultaneously. Let

$$\mathbf{X}\left(\mathbf{x}\right) = -\Pi'(\mathbf{x}) + \mathbf{C}\left(\mathbf{x}\right), \tag{27}$$

where $\Pi(\mathbf{x})$ - the potential energy and $\mathbf{C}(\mathbf{x})$ denotes the circulatory force, i.e.

$$\mathbf{x}^T\mathbf{C}\left(\mathbf{x}\right) \equiv 0 \quad . \tag{28}$$

Let $\mathbf{x}$=0 be the equilibrium position of the II kind, i.e. let the following conditions be fulfilled $-\Pi'(\mathbf{0}) + \mathbf{C}(\mathbf{0}) = \mathbf{0}.$

In addition to this, we shall consider that the equilibrium position $\mathbf{x}$=0 is a stationary point of the potential energy $\Pi(\mathbf{x})$ and $\Pi(\mathbf{0}) = 0$. In accordance with this, the following holds

$$\Pi'(\mathbf{0}) = \mathbf{0}, \mathbf{C}(\mathbf{0}) = \mathbf{0}. \tag{29}$$

If the conditions of the hypothesis ($\mathbf{H}$) are fulfilled, then $\Pi(\mathbf{x})$ and $\mathbf{C}(\mathbf{x})$ are also infinitely differentiable functions. Let

$$\Pi\left(\mathbf{x}\right) = \Pi_{\mathrm{p}}\left(\mathbf{x}\right) + \Pi_{p+1}\left(\mathbf{x}\right) + ... \quad p \geqslant 3 \tag{30}$$

and

$$\mathbf{C}(\mathbf{x}) = \mathbf{C}_{P-1}(\mathbf{x}) + \mathbf{C}_{p}(\mathbf{x}) + ... \quad p \geqslant 3 \tag{31}$$

are the Maclaurin series for the function $\Pi(\mathbf{x})$, that is $\mathbf{C}(\mathbf{x})$, where $\Pi_{\mathrm{s}}(\mathbf{x})$, $\mathbf{C}_{\mathrm{s}}(\mathbf{x})$ - denote homogeneous forms of the s degree. The series (30) and (31), taking into account the hypothesis ($\mathbf{H}$), are not convergent, as a rule. From (28) it follows that:

$$\mathbf{x}^T \mathbf{C}_{p-1}\left(\mathbf{0}\right) \equiv \mathbf{0}. \tag{32}$$

The differential equations of motion (18) now obtain the following form:

$$\dot{\mathbf{x}} = (I - G_0)\Pi_p'(\mathbf{x}) + (I - G_0)\mathbf{C}_{p-1}(\mathbf{x}) + \mathbf{O}(\parallel \ddot{\mathbf{x}} \parallel) + \mathbf{O}(\parallel \dot{\mathbf{x}} \parallel^2) + \mathbf{O}(\parallel \mathbf{x} \parallel^p) \tag{33}$$

Let us first consider the case when

$$\mathbf{C}_{p-1}(\mathbf{x}) \not\equiv \mathbf{0}. \tag{34}$$

Let us use $\hat{\Pi}(\mathbf{x})$ to denote restriction of the function $\Pi(\mathbf{x})$ on the hyperplane π. It is clear that $\hat{\Pi}_p(\mathbf{x})$ is also a homogeneous form of the p homogeneity degree. The following holds:

Theorem 3. *Let the following conditions be fulfilled:*

a) *the matrix D is positively definite for each* $\mathbf{x}$;

b) *the point* $\mathbf{x}=\mathbf{0}$ *is the maximum of the function* $\Pi(\mathbf{x})$ *(it does not have to be strict);*

c) $\mathbf{x}=\mathbf{0}$ *is the isolated solution of the equation* $-(I-G_0)(\Pi_p'(\mathbf{x})-\mathbf{C}_{p-1}(\mathbf{x}))=\mathbf{0}$;

d) $\hat{\Pi}_p(\mathbf{x})\mathrm{T}\mathbf{0}$;

e) *the dimension n of the configuration space is odd.*

Then $\mathbf{x}=\mathbf{0}, \dot{\mathbf{x}}=\mathbf{0}$ *is the unstable equilibrium state of the mechanical system whose motion is described with differential equations (18).*

Note 2. The conclusions of Note 1 hold for Φ=0. If $\Phi=0$ and $\mathbf{C}=\mathbf{0}$, the system is conservative and, taking into account (29), the point $\mathbf{x}=\mathbf{0}$ is the equilibrium position, and the corresponding equilibrium state $\mathbf{x}=\mathbf{0}$, $\dot{\mathbf{x}}=\mathbf{0}$ is unstable. It has been established in [4] and [11]. In accordance with this, Theorem 3 can also be formulated in the following way: *circulatory forces and forces of viscose friction cannot stabilise the equilibrium* $\mathbf{x}=\mathbf{0}$, $\dot{\mathbf{x}}=\mathbf{0}$ *of the conservative system obtained from the system (18) at* D($\mathbf{x}$)=0 *and* $\mathbf{C}(\mathbf{x})=\mathbf{0}$*, if the conditions of Theorem 3 are fulfilled.* Theorem 3 is analogous to the result from the reference [9], which refers to the impossibility of equilibrium stabilisation in a holonomic conservative system by dissipative and positional nonconservative forces, and in case when the potential energy in the equilibrium position has the maximum (see[9], p.204). Theorem 3 holds even when the constraints (3) are integrable. Especially, if m=0 then $\hat{\Pi}_p(\mathbf{x})=\Pi_p(\mathbf{x})$ and $G(\mathbf{x})\equiv\mathbf{0}$.

Proof of Theorem 3.

We shall use Theorem 1 for the purpose of proving. On the basis of the Lemma 1, it follows that the nontrivial solution of the equation

$$-(I-G_0)(\Pi_p'(\mathbf{c})-\mathbf{C}_{p-1}(\mathbf{c}))=\kappa\cdot\mathbf{c}, \quad \kappa\neq 0 \tag{35}$$

exists. Let us prove that $\mathbf{c} \in \pi$ and $\kappa > 0$. In order to prove that, let us first multiply the equation (15) by B_0^T. After multiplication, we obtain

$$-B_0^T(I - G_0)(\Pi_p'(\mathbf{c}) - \mathbf{C}_{p-1}(\mathbf{c})) = \kappa B_0^T \mathbf{c}, \tag{36}$$

that is, taking into account (20),

$$B_0^T \mathbf{c} = \mathbf{0}. \tag{37}$$

So, $\mathbf{c} \in \pi$. After we multiply the equation (15) by $\mathbf{c}^{\mathrm{T}}$ and solve it by κ, we obtain:

$$\kappa = -(\mathbf{c}^T \mathbf{c})^{-1} \mathbf{c}^T (I - G_0)(\Pi_p'(\mathbf{c}) - \mathbf{C}_{p-1}(\mathbf{c})). \tag{38}$$

On the basis of the fact that $\mathbf{c} \in \pi$, the following holds

$$\mathbf{c}^T G_0 = \mathbf{0}. \tag{39}$$

By using (32) and (39), we obtain $\kappa = -(\mathbf{c}^{\mathrm{T}}\mathbf{c})^{-1}\mathbf{c}^{\mathrm{T}}\Pi_{\mathrm{p}}'(\mathbf{c})$ and therefrom, as Π_{p}' is a homogeneous function,

$$\kappa = -(\mathbf{c}^T \mathbf{c})^{-1} p \Pi_p(\mathbf{c}).$$

The following holds from the conditions (b) and (d):

$$\Pi_p(\mathbf{c}) < 0,$$

there from

$$\kappa = -(\mathbf{c}^T \mathbf{c})^{-1} p \hat{\Pi}_p(\mathbf{c}) > 0$$

Theorem 3. is proved.

Let now $\mathbf{C}_{\mathrm{p}-1}(\mathbf{x}) \equiv \mathbf{0}$. The differential equations of motion (15) have the form:

$$A\ddot{\mathbf{x}} + \Gamma(\mathbf{x}, \dot{\mathbf{x}}) = -\Pi_p'(\mathbf{x}) + \mathbf{V}(\mathbf{x}) - D(\mathbf{x})\dot{\mathbf{x}} + B(\mathbf{x})\lambda, \quad B^T(\mathbf{x})\dot{\mathbf{x}} = 0,$$

where

$V(\mathbf{x}) = \mathbf{O}(\,\|\,\mathbf{x}\,\|^p)$ when $\mathbf{x} \to \mathbf{0}$.

We shall consider that the equilibrium position $\mathbf{x} = \mathbf{0}$ is of the II kind and that $\Pi'(\mathbf{0}) = \mathbf{0}$ (if $\Pi'(\mathbf{0}) = \mathbf{0}$ then $\mathbf{C}(\mathbf{0}) = \mathbf{0}$).

Theorem 4. *Let the following conditions be fulfilled:*

a) *the functions* T,D, Π *and* $\mathbf{C}$ *are infinitely differentiable for each* $\mathbf{x}$ *and* $\dot{\mathbf{x}}$;

b) *the matrix* D *is positively definite for each* $\mathbf{x}$;

c) *at the point* $\mathbf{x} = \mathbf{0} \in \pi, \hat{\Pi}_p(\mathbf{x})$ *does not have the minimum.*

Then $\mathbf{x} = \mathbf{0}$, $\dot{\mathbf{x}} = \mathbf{0}$ *is the unstable equilibrium state.*

Theorem 5. *Let the conditions a) and c) of Theorem 4 be fulfilled and let the equilibrium position* $\mathbf{x}$=$\mathbf{0}$ *is the isolated maximum (not necessarily strict) of the potential energy* $\Pi(\mathbf{x})$, *and* $\hat{\Pi}_p(\mathbf{x})\,\mathrm{T}\,0$. *Then* $\mathbf{x} = \mathbf{0}$, $\dot{\mathbf{x}} = \mathbf{0}$ *is the unstable equilibrium state.*

Theorem 5 is a corollary of Theorem 4. Theorem 4 holds under the condition that $\mathbf{C}(\mathbf{x})$=$\mathbf{0}$. It is the result from [10]. Under the condition that $\mathbf{C}(\mathbf{x})$=$\mathbf{0}$, Theorem 4 extends to nonholonomic systems result from reference [8], and it refers to the impossibility of stabilisation of equilibrium by means of forces of viscose friction in holonomic systems. Theorem 5 is generalization of the result referring to the impossibility of stabilization of the equilibrium by means of circulatory forces and forces of viscose friction in the case when the potential energy has the maximum (see [9], p.204).

4 Conclusion

Theorems of equilibrium instability have been proved by using the I Liapunov‘s method under assumption that a force can be resolved into conservative, circulatory and dissipative components. Theorem 1 generalizes Kozlov‘s result [1] and the result from Ref.[10]. The results obtained is analogous to the results referring to the impossibility of stabilization of the equilibrium by means of circulatory forces and forces of viscose friction in the case when the potential energy has the maximum (see [9], [8]). The proving technique will be similar to that used in the paper [1].Those results develop and spread the ideas of the I Liapunov‘s method.

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O nestabilnosti ravnoteže mehaničkih sistema u polju nekonzervativnih sila

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U ovom radu razmatra se problem nestabilnosti ravnoteže skleronomnih neholonomnih mehaničkih sistema pri istovremenom dejstvu pozicionih sila i sila viskoznog trenja. Uvedene su sledeće pretpostavke: u okolini ravnotežnog položaja pozicione sile, koeficijenti kinetičke energije i Relijeve funkcije disipacije su beskonačno diferencijabilne funkcije; postoji pravac, koji prolazi kroz ravnotežni položaj, duž koga ortogonalna projekcija prve netrivijalne forme pozicione sile, na ravan koja je normalna na vektore neholonomnih veza u ravnotežnom položaju, ima karakter centralne i odbojne sile. Predložene teoreme uopštavaju rezultate iz radova [1], [10], [9]. Zaključak o nestabilnosti izvodi se iz egzistencije rešenja diferencijalnih jednačina kretanja, koja asimptotski teže ka ravnotežnom položaju kad $t \to \infty$.