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# Modelling of a light elastic beam by a system of rigid bodies

**Slaviša Šalinić** *

**Abstract**

This paper has shown that a light elastic beam, in the case of small elastic deformations, can be modelled by a kinematic chain without branching composed of rigid bodies which are connected by passive revolute or prismatic joints with corresponding springs in them. Elastic properties of the beam are modelled by the springs introduced. The potential energy of the elastic beam is expressed as a function of components of the vector of elastic displacement and the vector of elastic rotation calculated for the elastic centre of the beam, which results in the diagonal stiffness matrix of the beam. As the potential energy of the introduced system of bodies with springs is expressed in the function of relative joint displacements, the diagonal stiffness matrix is obtained. In addition, these two stiffness matrices are equal. The modelling process has been demonstrated on the example of an elastic beam rotating about a fixed vertical axis, with a rigid body whose mass is considerably larger than the beam mass fixed to its free end. Differential equations of motion have been formed for this mechanical system. The modelling technique described here aims at expanding of usage of well developed methods of dynamics of systems of rigid bodies to the analysis of systems with elastic bodies.

**Key words:** elastic beam, elastic centre of beam, fictive rigid body, dynamics of systems of rigid bodies

*University of Kragujevac, Faculty of Mechanical Engineering, Dositejeva 19, 36000 Kraljevo, Serbia and Montenegro, e-mail: salinic.s@maskv.edu.yu

# 1 Introduction

A model in the form of mutually connected rigid bodies is often used for research of technical objects (robots, machine tools, building machines, etc.), where the joints are smooth (without friction) and rigid. However, in many cases modelling in the described way does not provide complete information about behavior of technical objects. Therefore, it is also necessary to take into account the elasticity of the body and the joints in models.

Distinguishing approaches in modelling of a system with elastic elements are those aiming at treating dynamic analysis of such systems by a well developed theory of systems of rigid bodies. So, in [12], modelling of elastic systems has been performed by means of fictive rigid joints with corresponding springs in them, and in references [1, 8, 14, 15] an elastic body is discretized to a set of rigid bodies mutually connected by joints. Corresponding springs which model elastic properties of the body are placed in the joints. In [10, 11], modelling of elastic joints has been performed by decomposing the elastic joint into two rigid joints of the same type by means of a fictive rigid body. By using ideas from these modelling techniques, this paper focuses on modelling light elastic bodies in the form of rectilinear beams.

The problem of modelling of light elastic beams is encountered in robotic manipulators with elastic segments whose characteristic is that the mass of load transferred by the robot is much larger in relation to the mass of segments (see e.g. [3]). In this paper, the light elastic beam is modelled as an open kinematic chain without branching which is formed by a rigid beam representing undeformed configuration of the considered elastic beam and five non-inertial fictive rigid bodies. All bodies in the chain are connected by kinematic pairs of the fifth class (passive revolute or prismatic joints). Corresponding springs which model elastic properties of the body are placed in the joints. It is required that such systems of bodies have the same potential energy as the elastic beam. The stiffness matrix of this system of bodies, in contrast to the stiffness matrix of the elastic beam, represents a diagonal matrix since the potential energy is expressed in the function of relative joint displacements. However, in [4] it is shown that the stiffness matrix of a light elastic beam can obtain a diagonal form if it is calculated for the elastic centre of the beam which is in the middle of undeformed configuration of the beam. This solves the problem regarding the previously given requirement for the equality of potential energies of these two systems so that a part of the kinematic chain composed of fictive bodies should be connected to the middle of the rigid beam span by joint.

The presented procedure for modelling of elastic beams enables treating their dynamics by dynamics of rigid bodies.

# 2 Modelling of a light elastic beam

## 2.1 Potential energy of a light elastic beam

Let us observe the elastic beam ***I*** whose end A is rigidly connected to the vertical axis OO' about which it can rotate, while the rigid body ***II*** is fixed to its free end B (see Fig.1). Such a mechanical system most frequently represents a model of a robot's manipulating arm with a load, where the load manipulated is modelled by the body ***II***.

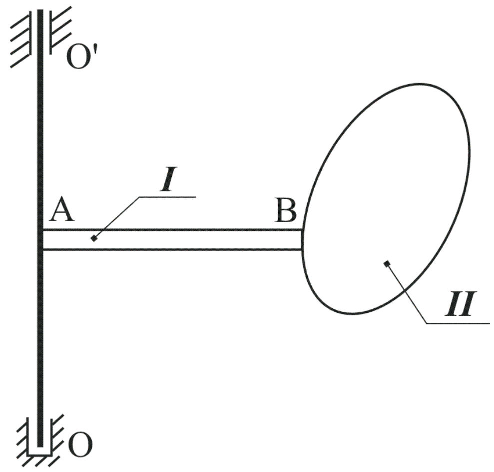


Figure 1: Mechanical system with an elastic beam

Let us assume that the mass of the beam ***I*** can be neglected in relation to the mass of the body ***II***. Taking into account this assumption, in further analyses the beam ***I*** will be considered as a light elastic beam. Let us place, at the B end of the beam, in its undeformed configuration, Descartes' right-angled coordinate system $Bxyz$ whose axis $Bx$ coincides with the beam axis, and let the axis $By$ be parallel to the axis OO'. Due to small elastic deformations of the beam, the coordinate system $Bxyz$ is transformed into the new position $B'x'y'z'$ by translating the point $B$ into the position $B'$ by the vector of elastic displacement $\vec{u} = \overrightarrow{BB'}$:

$$\{\vec{u}\} = (u_x \;\; u_y \;\; u_z)^T \tag{1}$$

and by rotating around the point $B'$ described by the matrix of transformation of the coordinates $[A] \in R^{3\times3}$ from the coordinate system $B'x'y'z'$ into the system $B'xyz$. This rotation can be described by Euler's angles, Bryant's angles, etc. (see e.g. [9], [13]). However, since the analysis is performed within small elastic deformations of the beam, it is most suitable to choose Bryant's angles because during a small displacement of the coordinate system $B'x'y'z'$ in relation to $B'xyz$ around the point $B'$ all three Bryant's angles remain small, which is not fulfilled in Euler's angles (for details see [13]). Let us denote Bryant's angles by $\psi$, $\theta$ and $\varphi$. The angle $\psi$ represents the angle of rotation of the coordinate system $B'xyz$ about the axis $B'x$ and after that it is transformed into the new position $B'x_1y_1z_1$ ($B'x_1 \equiv B'x$). The second angle $\theta$ represents the angle of rotation of the newly formed coordinate system $B'x_1y_1z_1$ about the axis $B'y_1$ and after that it is transformed into the position $B'x_2y_2z_2$ ($B'y_1 \equiv B'y_2$). The third angle $\varphi$ represents the angle of rotation of the system $B'x_2y_2z_2$ about the axis $B'z_2$, which results in obtaining $B'x'y'z'$ ($B'z_2 \equiv B'z'$). On the basis of this, the matrix $[A]$ for the case of Bryant's angles has the form

$$\begin{aligned}[A] &= [A_{x,\psi}][A_{y_1,\theta}][A_{z_2,\varphi}] = \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\varphi & -s_\varphi & 0 \\ s_\varphi & c_\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}\end{aligned} \tag{2}$$

where the following denotations are used: $c_* = \cos(*)$ and $s_* = \sin(*)$. The matrices $[A_{x,\psi}]$, $[A_{y_1,\theta}]$ and $[A_{z_2,\varphi}]$ represent the matrices of transformation of coordinates from $B'x_1y_1z_1$ into $B'xyz$, $B'x_2y_2z_2$ into $B'x_1y_1z_1$ and $B'x'y'z'$ into $B'x_2y_2z_2$, respectively. In the case of small deformations of the beam, $\cos\psi \approx 1$, $\sin\psi \approx \psi$, $\cos\theta \approx 1$, $\sin\theta \approx \theta$, $\cos\varphi \approx 1$ and $\sin\varphi \approx \varphi$ hold for the angles $\psi, \theta$ and $\varphi$, and their mutual products are neglected as small values of a higher order. On the basis of this, the relation (2) becomes

$$\begin{aligned}
[A] &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\psi \\ 0 & \psi & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \theta \\ 0 & 1 & 0 \\ -\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\varphi & 0 \\ \varphi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \\
&= \begin{bmatrix} 1 & -\varphi & \theta \\ \varphi & 1 & -\psi \\ -\theta & \psi & 1 \end{bmatrix}.
\end{aligned} \tag{3}$$

Let us denote the vector of elastic rotation of the coordinate system $B'x'y'z'$ around the point $B'$ by $\vec{\delta}$. As displacements of the coordinate system $B'x'y'z'$ are small in relation to $B'xyz$ (small elastic deformations of the beam), the vector $\vec{\delta}$ can be written as

$$\vec{\delta} = \psi\vec{i} + \theta\vec{j}_1 + \varphi\vec{k}_2, \tag{4}$$

where $\vec{i}$, $\vec{j}_1$ and $\vec{k}_2$ are the unit vectors of the axes $B'x$, $B'y_1$ and $B'z_2$, respectively. The coordinates of these unit vectors in the coordinate system $Bxyz$ (they are the same in $B'xyz$ because this coordinate system is obtained from $Bxyz$ by translation), taking into account (3), are

$$\{\vec{i}\} = (1 \;\; 0 \;\; 0)^T \quad , \tag{5}$$

$$\{\vec{j}_1\} = [A_{x,\psi}](0 \;\; 1 \;\; 0)^T = (0 \;\; 1 \;\; \psi)^T, \tag{6}$$

$$\{\vec{k}_2\} = [A_{x,\psi}][A_{y_1,\theta}](0 \;\; 0 \;\; 1)^T = (\theta \;\; -\psi \;\; 1)^T. \tag{7}$$

Taking into account (5), (6) and (7), the relation (4) becomes

$$\{\vec{\delta}\} = \psi\{\vec{i}\} + \theta\{\vec{j}_1\} + \varphi\{\vec{k}_2\} = (\psi + \varphi\theta \quad \theta - \varphi\psi \quad \theta\psi + \varphi)^T \tag{8}$$

which is, after neglecting small values of a higher order, transformed (8) into

$$\{\vec{\delta}\} = (\psi \quad \theta \quad \varphi)^T. \tag{9}$$

The potential energy of the elastically deformed beam ***I*** can be written in the quadratic form

$$\Pi = \frac{1}{2}(w)[C]\{w\}, \tag{10}$$

where

$$\{w\} = (\ \{\vec{u}\}^T \ \ \{\vec{\delta}\}^T\ )^T, \tag{11}$$

and $[C] \in R^{6\times 6}$ denotes the stiffness matrix of the beam ***I***, which is according to [2] given with

$$[C] = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} \\ 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\ 0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\ 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\ 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} \end{bmatrix}. \tag{12}$$

The expression (12) contains: E-Young's modulus of elasticity; L-the length of the beam; A-the area of the cross section of the beam; G-the shear modulus; J-the polar moment of inertia of the cross section of the beam in relation to the axis x; $I_y$, $I_z$ - the axial moments of inertia for the principal axes y and z of the cross section.

The references for the considered elastic beam introduce the concept of the elastic centre E as a point at which, for the corresponding coordinate system $E\xi\eta\zeta$, the stiffness matrix becomes a diagonal matrix. When the beam is in its undeformed configuration, the elastic centre E coincides with the middle point of the beam span where the coordinate system $E\xi\eta\zeta$ is obtained by translation of the coordinate system $Bxyz$ by the vector $(-L/2)\vec{i}$ (see [4]). The elastic centre of the beam is treated as a point belonging to an imagined rigid whole, whose constituent part is the body ***II***, where this whole is fixed to the free end of the beam. The presented facts relating to the elastic centre will in further elaboration be used for the purpose of introducing components of the vector of elastic displacement $\vec{u}_E$ and the vector of elastic rotation $\vec{\delta}_E$, which correspond to the elastic centre E, into the expression for the potential energy (10).

The relationship between the vectors $\vec{u}$ and $\vec{u}_E$, as well as between $\vec{\delta}$ and $\vec{\delta}_E$, is given in the following relations (see e.g. [3])

$$\vec{u} = \vec{u}_E - \vec{\delta} \times \overrightarrow{BE} \ , \quad \vec{\delta}_E = \vec{\delta} \tag{13}$$

which can also be written in the matrix form

$$\{\vec{u}\} = \{\vec{u}_E\} + [BE^d]\{\vec{\delta}\} \ , \quad \{\vec{\delta}_E\} = \{\vec{\delta}\} \ , \tag{14}$$

where

$$\{\vec{u}_E\} = (u_{Ex} \quad u_{Ey} \quad u_{Ez})^T, \tag{15}$$

and

$$[BE^d] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & L/2 \\ 0 & -L/2 & 0 \end{bmatrix} \tag{16}$$

denotes the dual object of the vector $\overrightarrow{BE}$ which represents a skew-symmetric matrix composed of projections of the vector $\overrightarrow{BE}$ on the axes of the coordinate system $Bxyz$.

The matrix of transformation of the coordinates $[A_{B,E}]$ from the coordinate system $E\xi\eta\zeta$ into the coordinate system $Bxyz$ is equal to the identity matrix, i.e. $[A_{B,E}] = [I]$. If the vectors $\vec{u}_E$ and $\vec{\delta}$ are expressed through their projections in the coordinate system $E\xi\eta\zeta$, which can be written in the matrix form as $\{\vec{u}_E^{(E)}\}$ and $\{\vec{\delta}^{(E)}\}$, then on the basis of (14) we have that

$$\{\vec{u}\} = [A_{B,E}]\{\vec{u}_E^{(E)}\} + [BE^d][A_{B,E}]\{\vec{\delta}^{(E)}\}, \tag{17}$$

$$\{\vec{\delta}\} = [A_{B,E}]\{\vec{\delta}^{(E)}\}. \tag{18}$$

By replacing (17) and (18) into (10), we obtain

$$\Pi = \frac{1}{2}(\{\vec{u}_E^{(E)}\}^T \ \ \{\vec{\delta}^{(E)}\}^T)[R]^T[C][R](\{\vec{u}_E^{(E)}\}^T \ \ \{\vec{\delta}^{(E)}\}^T)^T \tag{19}$$

where

$$[C_E] = [R]^T[C][R] = diag(\frac{AE}{L}, \ \frac{12EI_z}{L^3}, \ \frac{12EI_y}{L^3}, \ \frac{GJ}{L}, \ \frac{EI_y}{L}, \ \frac{EI_z}{L}), \tag{20}$$

$$[R] = \begin{bmatrix} [A_{B,E}] & [BE^d][A_{B,E}] \\ [0]_{3\times 3} & [A_{B,E}] \end{bmatrix} = \begin{bmatrix} [I] & [BE^d] \\ [0]_{3\times 3} & [I] \end{bmatrix}, \tag{21}$$

$[0]_{3\times 3}-$ the zero matrix of the dimension 3x3.

The expression (19) can also be written in a developed form

$$\Pi = \frac{1}{2}(\frac{AE}{L}u_{E\xi}^2+\frac{12EI_z}{L^3}u_{E\eta}^2+\frac{12EI_y}{L^3}u_{E\zeta}^2+\frac{GJ}{L}\psi^2+\frac{EI_y}{L}\theta^2+\frac{EI_z}{L}\varphi^2). \tag{22}$$

## 2.2 Modelling by a system of rigid bodies

The reference [7] shows that motion of a rigid body (translational, plane, spherical and general) can be modelled, by introducing fictive bodies, as motion of an open kinematic chain without branching with the joints between bodies which correspond to kinematic pairs of the fifth class (prismatic and revolute joints). A fictive body means a body whose mass is equal to zero, whose dimensions are arbitrary and whose fictive centre of mass can be arbitrarily chosen.

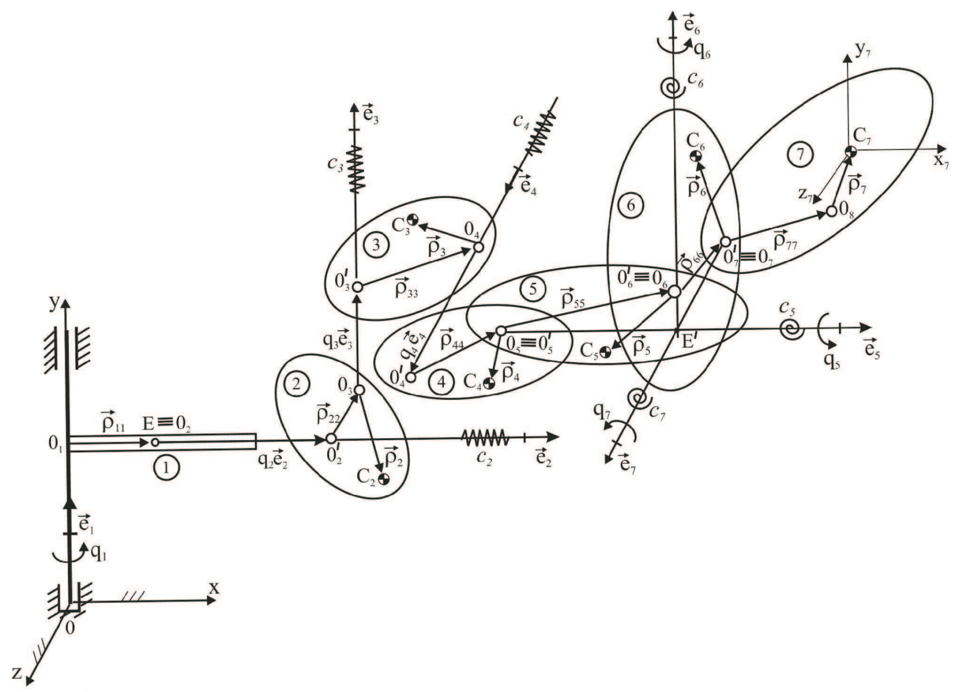

Figure 2: Modelling of an elastic beam by a system of rigid bodies

By using this modelling methodology, motion of the considered mechanical system in Fig.1 can be modelled by an open kinematic chain in which the beam ***I*** is replaced with a rigid beam which coincides with undeformed configuration of the beam ***I***. An open kinematic chain of six bodies interconnected by prismatic or revolute joints is connected to the middle of the span of the introduced beam by a prismatic joint, where the first five bodies of this chain are fictive bodies, and the last body has the same mass, tensor of inertia and the centre of mass as the body ***II***. The configuration of this system of bodies for $q_1 = q_5 = q_6 = q_7 = 0$, $q_2 \neq 0,\ q_3 \neq 0,\ q_4 \neq 0$ is shown in Fig.2. Appropriate springs are placed in the joints of the introduced kinematic chain depending on the type of joint (cylindrical for the prismatic joint, and spiral for the revolute one) which are used for modelling elastic properties of the beam ***I***. Stiffnesses of the introduced springs are such that the potential energy of this system of springs is equal to the potential energy of the elastic beam

***I*** determined by the equation (22).

The relative joint displacements $q_i$, i=2,...,7 are determined by the following relation

$$(q_2 \dots q_7)^T = (\{\vec{u}_E^{(E)}\}^T \ \{\vec{\delta}^{(E)}\}^T)^T. \tag{23}$$

The local coordinate systems $C_i x_i y_i z_i$, i=1,7 are rigidly connected to the bodies ***1,...,7***, respectively, where $C_i$ is the centre of mass of the $i$th body. Without reducing generality, it is assumed that in the referent configuration of the considered system of bodies $q_i = 0$, i=1,...,7 holds for the generalised coordinates. In further presentation, denotations ( $^{(i)}$), { $^{(i)}$} and [ $^{(i)}$] will be used for the row matrix, the column matrix and the square matrix whose elements are formed in relation to the local coordinate system of the $i$th body, respectively.

The matrices of transformation of the coordinates $[A_{i,j}]$, i,j=0,1,...,7 from the local coordinate system connected to the body ***j*** into the local coordinate system connected to the body ***i***, where the index 0 denotes the referent coordinate system $Oxyz$, have, according to [5], the form

$$[A_{i,j}] = [I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad for \quad \mathrm{i = j}, \tag{24}$$

$$[A_{i-1,i}] = [I] + \bar{\chi}_i(1 - \cos q_i)[e_i^{d(i)}]^2 + \bar{\chi}_i(\sin q_i)[e_i^{d(i)}], \tag{25}$$

$$[A_{i,j}] = [A_{i,i+1}][A_{i+1,j}] \ \text{ for } \ \mathrm{i + 1 < j}. \tag{26}$$

The right-hand side of the relation (25) represents the matrix of transformation in the Rodrigue's form and $[e_i^{d(i)}]$ denotes the dual object which corresponds to the vector $\vec{e}_i$ and it represents a skew-symmetric matrix formed by projections of this vector on the axes of the local coordinate system $C_i x_i y_i z_i$. The symbol $\chi_i$ denotes the identifier of a joint type: if the $i$th joint is revolute, then $\chi_i = 0$, and if it is prismatic, then $\chi_i = 1$, and then $\bar{\chi}_i = 1 - \chi_i$ holds.

It should be emphasized that the matrix of transformation of the coordinates is orthogonal and symmetrical.

In the referent configuration of the considered system of bodies the coordinates of the unit vectors of the axes of joints in relation to the referent coordinate system are:

$$\left.\begin{array}{lll} \{\vec{e}_2^{(0)}\} = (1\ \ 0\ \ 0)^T\,, & \{\vec{e}_3^{(0)}\} = (0\ \ 1\ \ 0)^T, & \{\vec{e}_4^{(0)}\} = (0\ \ 0\ \ 1)^T \\ \{\vec{e}_5^{(0)}\} = (1\ \ 0\ \ 0)^T\,, & \{\vec{e}_6^{(0)}\} = (0\ \ 1\ \ 0)^T\,, & \{\vec{e}_7^{(0)}\} = (0\ \ 0\ \ 1)^T \end{array}\right\}. \tag{27}$$

The unit vector $\vec{e}_1$ constantly has the direction of the y-axis of the referent coordinate system, and the axes of joint, which are determined by the vectors $\vec{e}_5$ , $\vec{e}_6$ and $\vec{e}_7$, intersect at the point $E'$. As the bodies 2,...,6 are fictive, in further consideration it is taken that

$$\vec{\rho}_{ii} = \vec{0}\,, \quad \vec{\rho}_i = \vec{0}\,, \quad i = 2, ..., 6 \tag{28}$$

as well as $O_5 \equiv O_6 \equiv O_7 \equiv E'$.

Stiffnesses of the introduced springs in the joint in Fig.2, on the basis of (22), are

$$\left.\begin{array}{lll} c_2 = \frac{AE}{L}\,, & c_3 = \frac{12EI_z}{L^3}\,, & c_4 = \frac{12EI_y}{L^3} \\ c_5 = \frac{GJ}{L}\,, & c_6 = \frac{EI_y}{L}\,, & c_7 = \frac{EI_z}{L} \end{array}\right\}. \tag{29}$$

## 2.3 Differential equations of motion

Taking into account the presented description of the kinematic chain from Section 2.2, which is used for modelling an elastic beam, the mass and the tensor of inertia of the bodies 1,...,6 are $m_i = 0$, $[J_{C_i}^{(i)}] = [0]$, i=1,...,6. Hence, the kinetic energy of the considered system of bodies is

$$T = \frac{1}{2}\sum_{\alpha=1}^{7}\sum_{\beta=1}^{7} a_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta, \tag{30}$$

where the coefficients of metric tensor according to [5,6] are defined by

$$a_{\alpha\beta} = m_7(\vec{T}_{\alpha(7)}^{(\alpha)})\{\vec{T}_{\beta(7)}^{(\alpha)}\} + \bar{\chi}_\alpha\bar{\chi}_\beta(\vec{e}_\alpha^{(7)})[J_{C_7}^{(7)}]\{\vec{e}_\beta^{(7)}\} \tag{31}$$

where

$$\vec{T}_{\alpha(7)} = \frac{\partial \vec{r}_{C_7}}{\partial q_\alpha} = \bar{\chi}_\alpha \vec{e}_\alpha \times \vec{R}_{\alpha(7)} + \chi_\alpha \vec{e}_\alpha, \tag{32}$$

$$\vec{R}_{\alpha(7)} = \sum_{k=\alpha}^{7} (\chi_k q_k \vec{e}_k + \vec{\rho}_{kk}) + \vec{\rho}_7, \tag{33}$$

The potential energy $\Pi$ of the system is

$$\Pi = \Pi_G + \Pi_c, \tag{34}$$

where

$$\Pi_G = -m_7 \vec{g} \cdot \vec{r}_{C_7}, \quad \vec{g} - \text{the acceleration of gravity}, \tag{35}$$

is a part of the potential energy arising due to the gravity force, and

$$\Pi_c = \frac{1}{2} \sum_{i=1}^{7} c_i q_i^2 \,, \quad c_1 = 0 \tag{36}$$

is a part of the potential energy arising from elastic forces of springs in the joints, when it is assumed that the springs are undeformed for $q_i = 0$, i=2,...,7.

The generalized force which corresponds to the generalized coordinate $q_i$, i=1,...,7 is

$$Q_i = -\frac{\partial \Pi}{\partial q_i} + Q_i^p, \tag{37}$$

where $Q_i^p$ represents the component of the generalized forces which arises from internal driving forces in the $i-$th joint and it is determined by the following relation

$$Q_i^p = (\bar{\chi}_i \vec{M}_i + \chi_i \vec{P}_i) \cdot \vec{e}_i. \tag{38}$$

In (38), $\vec{M}_i$ denotes the moment of a internal driving force couple, and $\vec{P}_i$ denotes the internal driving force of the $i-$th joint. In our case $\vec{P}_i = \vec{0}$, i=1,...,7; $\vec{M}_1 = M\vec{e}_1$; $\vec{M}_i = \vec{0}$, i=2,...,7.

The Lagrange's equations of the second kind for the considered system

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i, \quad i = 1, ..., 7 \tag{39}$$

written in the covariant form are

$$\begin{aligned} \sum_{i=1}^{7} a_{ki}\ddot{q}_i + \sum_{i=1}^{7}\sum_{j=1}^{7} \Gamma_{ij,k}\dot{q}_i\dot{q}_j &= -c_k q_k + m_7(0 \quad -g \quad 0)\{\vec{T}^{(0)}_{k(7)}\} + Q^p_k, \\ k &= 1, ..., 7 \end{aligned} \tag{40}$$

where $\Gamma_{ij,k}$ is the Christoffel symbol of the first kind defined by

$$\Gamma_{ij,k} = \frac{1}{2}\left[\frac{\partial a_{jk}}{\partial q_i} + \frac{\partial a_{ik}}{\partial q_j} - \frac{\partial a_{ij}}{\partial q_k}\right].$$

# 3 Conclusion

This paper has presented the procedure of modelling of a light elastic beam in the case of small elastic deformations which is based on presentation of the light elastic beam in the form of a system of rigid bodies which are interconnected by joints having appropriate springs placed in them (spiral or cylindrical). Stiffnesses of these springs have been determined on the basis of the expression for potential energy of the elastic beam expressed as a function of components of the vector of elastic displacement and the vector of elastic rotation calculated for the elastic centre of the beam. Also, the components of these vectors have been taken as relative joint displacements of the introduced system of rigid bodies. The modelling process has been illustrated on the example of an elastic beam rotating about a fixed vertical axis, with a rigid body whose mass is much larger than the beam mass fixed to its free end. Considerations in this paper can be easily expanded to the case of kinematic chains formed by light elastic beams.

In contrast to the existing methods, the presented procedure of modelling of an elastic beam does not require discretization of the beam. The considerations presented in this paper represent a contribution to realization of ideas about expanding the application of well developed methods of dynamics of the system of rigid bodies to the analysis of the system with elastic bodies.

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## Modeliranje lakog elastičnog štapa sistemom krutih tela

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U radu je pokazano da se laki elastični štap, u slučaju malih elastičnih deformacija, može modelirati kinematičkim lancem bez grananja formiranim od krutih tela koja su medjusobno povezana pasivnim rotacionim ili prizmatičnim zglobovima sa odgovarajućim oprugama u njima. Uvedenim oprugama modeliraju se elastična svojstva štapa. Potencijalna energija elastičnog štapa izražena je kao funkcija od komponenti vektora elastičnog pomeranja i vektora elastičnog obrtanja sračunatih za elastični centar štapa čime je postignuto da matrica krutosti štapa bude dijagonalna. Pošto je potencijalna energija uvedenog sistema tela sa oprugama izražena u funkciji relativnih zglobnih pomeranja, dobijena je dijagonalna matrica krutosti. Osim toga, ove dve matrice krutosti su jednake. Proces modeliranja demonstrira se na primeru elastičnog štapa koji se obrće oko nepokretne ose, a za čiji je slobodni kraj fiksirano kruto telo čija je masa mnogo veća od mase štapa. Za ovaj mehanički sistem formirane su diferencijalne jednačine kretanja u simboličkoj formi. Opisana tehnika modeliranja ima za cilj da se proširi upotreba dobro razvijene metodologije dinamike sistema krutih tela na analizu sistema sa elastičnim telima.