

# Basic general concepts in the network analysis

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## Abstract

This survey is concerned oneself with the study of those types of material networks which can be met both in civil engineering and also in electrotechnics, in mechanics, or in hydrotechnics, and of which behavior lead to linear problems, solvable by means of Finite Element Method and adequate algorithms.

Here, it is presented a unitary theory of networks met in the domains mentioned above and this one is illustrated with examples for the structural networks in civil engineering, electric circuits, and water supply networks, but also planar or spatial mechanisms can be comprised in this theory.

The attention is focused to make evident the essential properties and concepts in the network analysis, which differentiate the networks under force from other types of material networks. To such a network a planar, connected, and directed or undirected graph is associated, and with some vector fields on the vertex set this graph is endowed.

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# 1 Introduction

The Finite Element Method (FEM) is a mathematical method of analysis of the behaviour of a material structure under the action of some external (or/and internal) factors based on the structure formal decomposition in structural elements which can be individually studied by using known methods.

So, a material structure consists of a finite number of objects interconnected one to another. If one chooses a finite number of material particles on the boundaries of structural elements, called *nodal points* (or, simply, *nodes*), we also ask for the structure to be connected, i.e. there exists at least one chain of structural elements that links any pair of nodal points. But, generally, the structural elements can have two, three, or more nodal points. Let us denote by  $\Gamma (\neq \emptyset)$  the set of nodes of a structure  $\Omega$ , and by  $\{\Omega^e\}$  the set of its structural elements. Assume  $|\Gamma| = n$  and  $|\{\Omega^e\}| = m$ , where  $|A|$  denotes the number of elements of the set inside.

In FEM one defines a system of so-called *variables* of the problem, and also a system of *external* (or/and *internal*) *factors* (called, as "forces") loading the structure; these are introduced with the help of two scalar or vector fields defined on  $\Gamma$ .

At the beginning, any problem solved by FEM needs two important phases:

**I. Geometrical modelling** of the structure. This stage, called "discretization", must be made accurately both with respect to the choosing of an enough number of elements, and also having in view their physical and geometrical properties.

By a *discretization* of a structure  $\Omega$  we will understand the choosing of a pair  $(\{\Omega^e\}, \Gamma)$ , consisting in a system  $\{\Omega^e\}_m$ , ( $e \in \{1, 2, \dots, m\} \doteq \overline{1, m}$ ), of  $m$  structural elements, and in a set  $\Gamma$  of  $n$  nodes on the boundaries of elements, which satisfy the following conditions:

$$(D_1). \cup_{e \in \overline{1, m}} \Omega^e = \Omega,$$

$$(D_2). (\forall) e, e' \in \overline{1, m}, e \neq e', \Gamma^{ee'} := \Omega^e \cap_{\Gamma} \Omega^{e'} \subset \Gamma^*,$$

where  $\cap_{\Gamma}$  is an intersection restricted to the node set  $\Gamma$  and  $\Gamma^*$  is a submanifold of the Euclidean affine space  $E^3$  of dimension strictly

less than each of the two dimensions of considered manifolds, that is  $\dim \Gamma^* < \min\{\dim \Omega^e, \dim \Omega^{e'}\}$ .

So, the set of common nodes  $\Gamma^{ee'}$  is either the empty set, or is entirely enclosed in  $\Gamma^*$ , which can be a curve or a surface of  $E^3$ . Certainly, because  $\Gamma$  is finite,  $\Gamma^{ee'}$  is of dimension zero. In the case when  $\Gamma^{ee'} = \emptyset$  the elements  $\Omega^e$  and  $\Omega^{e'}$  are not directly interconnected, that is  $\Omega^e$  and  $\Omega^{e'}$  do not have common nodes.

A discretization having  $m$  structural elements and  $n$  nodes will be denoted by  $(\{\Omega^e\}, \Gamma)_{m,n}$ .

The elements are characterized from the physical point of view with respect to the material properties, and from the geometrical point of view with respect to scheme of their disposition in the structure and with respect to the dimension of the geometric varieties (manifolds) which with these elements are assimilated. This last aspect is summarized in the following table of correspondences:

<i>Structural elements</i>	<i>Geometric manifolds</i>
- straight rods (or curved bars)	→ 1 - dimensional: line segments (or arcs of curves)
- flat plates (or curved shells)	→ 2 - dimensional: plane domains (or surface portions)
- solids	→ 3 - dimensional: spatial domains

The particles of a structure are assimilated with the points of the corresponding manifold; the points are considered to be 0-dimensional manifolds in the Euclidean affine space  $E^3$ .

In the following parts of the paper only a particular case of structures, namely the networks, will be considered.

## 2 The networks and their graphs

A *material network* (or, simply, a *network*) is a material structure whose each individual component does connect exactly two nodes, and two different components which are interconnected at a node do have not any other common intersection. Shortly, we can say that networks are 2 - nodal element material structure. Any network must have at least three nodes and two structural elements.

Among the most important 2-nodal material structures, appearing in the technical problems, we enumerate the following loaded networks:

- (a) structural networks with structural elements: beams, rods, arcs,...
- (b) electric circuits with structural elements: resistors, capacitors,...
- (c) pipe networks with structural elements: pipelines,...
- (d) spatial mechanisms with structural elements: truss elements, links,...

and others, whose equilibrium equations are linear or almost linear in the sense that we may approximate them by linear equations; also we can consider the case of linear differential or differential-algebraic equations (see [15]).

But anyway, we can observe the networks under consideration usually provide one-dimensional problems in the analysis by FEM; therefore their structural elements can be simply assimilated with straight line segments. Thus, to these networks we can associate some graphs.

So, to every material network  $\Omega$  we can associate a finite, planar, directed/undirected, connected and simple (without multiple arcs/edges) graph  $G$ , with the help of a *transfer mapping*  $\phi : \Omega \rightarrow G$ , suitable defined for each case.

Looking for general (similar) properties of several types of material networks, in order to make use of specific mathematical methods of analysis, we observe that two kinds of networks are frequently considered, namely, loaded networks and transportation networks.

In order to represents a loaded network or a transportation network,  $G$  will be chosen as a specific graph, either of the form (S, E), or of the form (S, A), where S denotes the set of vertices (corresponding to the nodes of  $\Omega$ ) and E or A are the sets of edges or of arcs, respectively, corresponding to the structural elements.

- (i) If a network is represented by a directed graph of which arc set is endowed with a scalar field we will say that it is a *network with flow* and will be denoted by  $\mathbf{G}_\Phi$ . Usually, this is the case of transportation networks.
- (ii) If a network is represented by a graph of which vertex set is endowed with a vector field we will say that it is a *network under force* and will be denoted by  $\mathbf{G}_F$ . Usually, this is the case of loaded networks.

Notations  $G_\Phi$  and  $G_F$  designate the graphs associated to the considered above networks. Sometimes, the notations  $G_\Phi \doteq \vec{G}$  and  $G_F \doteq G$  or  $G_F \doteq \vec{G}$  for

the graphs are preferable, to put in evidence that the graph  $\vec{G}$  is directed while  $G$  is not.

The most adequate mathematical methods used in connection with these types of networks for optimization of a functional are simplex procedure and finite element method, respectively. Among the networks enumerated above, (b) and (c) are of the first type, while (a), (b), (c) and (d) can all be comprised in the second class. The graphs associated to some networks of different types on one and the same material structure  $\Omega$  (such as the cases (b) and (c)) will be different. Moreover, we may associate to a network either an directed graph or an undirected one, with respect to our convenience for the solving a specific technical problem.

Let us assume that a material structure  $\Omega$  is a network and consider a discretization  $(\{\Omega^e\}, \Gamma)_{m,n}$  of  $\Omega$  having  $m$  structural elements and  $n$  nodes. For the nodes of the network and the vertices of the associated graph will be used the same notations, because to the material points (in  $\Omega$ ) or to their geometrical images by  $\phi$  (in  $G$ , or in  $\vec{G}$ ) we can associate the same local (sometimes, global) coordinates. Thus, to the node set  $\Gamma$  it corresponds in the graph  $G$  (or in  $\vec{G}$ ) the set  $S = \{x_i\}_n$  of vertices and to the set of structural elements  $\{\Omega^e\}$  it corresponds either the set  $E = \{a^e\}_m$  of edges of  $G$ , or the set  $A = \{\vec{a}^e\}_m$  of arcs of  $\vec{G}$ ; in  $E$  (or in  $A$ ),  $a^e$  (or  $\vec{a}^e$ ) is the edge (or, the arc) that represents the element  $\Omega^e$  for any  $e \in \{1, 2, \dots, m\}$ . When an edge  $a^e$  connects two vertices  $x_i$  and  $x_j$ , ( $i, j \in \{1, 2, \dots, n\}$ ), that is the points  $x_i$  and  $x_j$  are nodes of a structural element  $\Omega^e$ , will be used the following notation  $a^e = [x_i, x_j]$ ; a pair of vertices  $(x_i, x_j) \in S \times S$  are said to be *adjacent* if there exists an edge  $a^e$  connecting them. Similar notions and notations can be considered for the arcs.

For a network  $\Omega$  the set  $\Gamma^{ee'}$  either is  $\emptyset$ , or contains only one node; so, the edges  $a^e$  and  $a^{e'}$  either are not incident, or are incident at a vertex  $x_i$  for an  $i \in \{1, 2, \dots, n\}$ , respectively; similar situation for the arcs. The set of all edges (or arcs) incident with a given vertex,  $x_i$ , will be denoted by  $Star[i]$ ; this one is a subset of  $E$  or of  $A$ . Analogously, the set of all structural elements which have a common node,  $x_i$ , will be denoted by  $[x_i]$ . So, if  $\Gamma^{ee'} = \{x_i\}$ , we can write  $\Omega^e, \Omega^{e'} \in [x_i]$ , and this fact is represented in the graph by the relations  $a^e, a^{e'} \in Star[i]$ . Obviously,  $[x_i]$  can contains more than two structural elements.

If it is necessary to be used a graph of type  $G$  when we analyze a network  $\Omega$ , we can associate to the graph two matrices:

1).  $I(G) := [p_i^e]$ , where  $p_i^e = \begin{cases} 1, & a^e \in Star[i] \\ 0, & a^e \notin Star[i] \end{cases}$ , called *matrix of incidence*, and

2).  $A(G) := [q_{ij}]$ , where  $q_{ij} = \begin{cases} 1, & [x_i, x_j] \in E \\ 0, & [x_i, x_j] \notin E \end{cases}$ , called *matrix of adjacency*,

which belong to the linear spaces  $M(m, n; |)$  and  $M^s(n, n; |)$ , respectively.

These matrices indicate the interconnection of the structural elements by positions occupied of the number 1, or of the numbers +1 and -1, in the case of  $I(\vec{G})$ , when indicates the incidence of an arc towards interior and exterior at a vertex, respectively.

Also, sometimes it is useful to put in evidence a preferential enumeration of structural elements of a network of which matrix of adjacency  $A(G)$  is known.

Thus, for a fixed index "i" ( $\in \overline{1, n}$ ) the structural element of the network,  $\Omega^e$ , represented in the graph  $G$  by the edge  $[x_i, x_j]$  (with  $j > i$ , because  $A(G)$  is symmetric), will be labeled as  $a^e$  by an index "e" ( $\in \overline{1, m}$ ) in an increasing order of those indices "j" that correspond to the nonzero entries of the matrix. In other words, the sequence  $\{a^1, a^2, \dots, a^m\}$  will be an ordering in the union

$$\cup_{i \in \overline{1, n}} \{Star[i] - (\cup_{i' < i} Star[i'])\},$$

where  $Star[i]$  denotes the set of all edges  $[x_i, x_j] \in E$  incident at the vertex labeled by "i". The natural number  $|Star[i]| \doteq \deg(i)$  is called *degree* of  $x_i$ ; we have  $1 \leq \deg(i) \leq m$ .

In such a manner, with the help of the matrix of adjacency, one can obtain a good correspondence  $e \leftrightarrow (i, j)$  between the elements of  $E$  and the subsets of  $S \times S$ .

The networks with flow  $\mathbf{G}_\Phi$  are not proper to be analyzed by FEM. So we pass to present general bases and properties of the second type of networks considered above, those of networks under force,  $\mathbf{G}_F$ .

### 3 The networks under force and associated FEM

Let  $\Omega$  be a 2-nodal material structure, i.e. a material network. In order to define on  $\Omega$  a "network under force structure",  $\mathbf{G}_F$ , four fields will be used: two of them will introduce the structure and other two will connect it with an appropriate FEM.

A *network under force* is a material network of which graph  $G = (S, E)$  is endowed with:

- (i). a vector field on the vertex set, defined by the function

$$F : S \rightarrow |^p, (p = 1, 2, 3, \dots),$$

of which values

$$F(x_i) = \mathbf{F}_i, (i \in \{1, 2, \dots, n\}),$$

called *nodal forces*, load the nodes, and

- (ii). a scalar field on the edge set

$$k : E \rightarrow |^*_+, (|^*_+ = (0, +\infty)),$$

of which values

$$k(a^e) = r^e, (e \in \{1, 2, \dots, m\}, r^e > 0),$$

called *perviousness*, give an influence about the network state under the action of  $\mathbf{F}$ .

The fields  $\mathbf{F}$  and  $k$  are independent one from another, however the presence of  $k$  is put in evidence only by the action of  $\mathbf{F}$ . Particularly,  $\mathbf{F}$  can also be a scalar field, when  $p = 1$ . The field  $\mathbf{F}$  is defined, while the field  $k$  by the material nature and by the geometrical shape of network is determined, whenever  $\Omega$  "works", i.e.  $\mathbf{F}$  acts.

Thus a network under force will be denoted, as was shown before, by  $G_F$ , and the pair  $(\mathbf{F}, k)$  is its determinant couple;  $\mathbf{F}$  is said to be the *external field* and  $k$  is the *structural field* of the network. The previous pencil of fields designate the "action" of  $\mathbf{F}$  and the "influence" of  $k$  about the working network.

As a result of the  $(\mathbf{F}, k)$  presence on  $G_F$  the answer consists in other two assignments induced on the network:

- (I). The first is a vector field on the node set

$$X : S \rightarrow |^p, (p = 1, 2, 3, \dots),$$

of which values

$$X(x_i) = \mathbf{u}_i, (i \in \{1, 2, \dots, n\}, n = |S|),$$

are called *nodal vectors* with components the *nodal parameters*:  $u_i^1, \dots, u_i^p$ .

We will call  $\mathbf{X}$  the *nodal state field*, because each of the nodal vectors, given by a  $1 \times p$  matrix  $[u_i^1 \dots u_i^p] = \mathbf{u}_i^T$ , describes the network state at the node  $x_i$ , for  $i \in \overline{1, n}$ . This field can be extended to the whole network as follows.

The number of nodal parameters on the whole network is equal to  $n' = n \cdot p$ , where  $p$  denotes the number of parameters per node.

The state of the network at an arbitrary point  $x$  can be approximate by a matrix  $[u^1 \dots u^p] = \bar{\mathbf{u}}^T$ , where  $\bar{\mathbf{u}}$  is the image-vector of  $x$  by the *extended state field*  $\bar{\mathbf{X}} : G_F \rightarrow |^p$ , defined such that  $\bar{\mathbf{X}}|_S = \mathbf{X}$  and the  $p$  components of the column-vector  $\bar{\mathbf{X}}(x) = \bar{\mathbf{u}}$ , called *parameters of state*, by means of some interpolation functions are expressed. For instance, the interpolation function expressing one of the state parameters, say  $u^1$ , by a polynomial of the form

$$u^1 = a_1 + a_2 s + \dots + a_{2p} s^{2p-1}, \dots$$

is given. Then, another can be its derivative, etc., depending on the network type that we have in view. In any case the  $u^1$  is the most important state parameter for almost all problems, so, we will call it the *basic approximation function* at  $x$ .

We observe that in the case of one-dimensional problems only one variable is needed. This is a real number  $s$  in the closed interval  $[s_i, s_j]$ , of which ends are the local coordinates of the nodes  $x_i$  and  $x_j$ ; to each value of  $s$  it corresponds a position of the point  $x$  on  $a^e = [x_i, x_j]$ , considered as a "linear element". Thus we can consider  $n' = 2p$ , because  $x$  belongs to a "local structure" with two nodes.

The coefficients  $a_1, a_2, \dots, a_{2p}$  depend on the nodal parameters defined by  $\mathbf{X}$ .

Indeed, replacing the nodal coordinates into  $u^1$  we obtain a system of  $2p$  linear equations for each pair of indices  $(i, j)$

$$u_i^1 = \sum_{k=1}^{2p} a_k s_i^{k-1}, \dots, u_j^1 = \sum_{k=1}^{2p} a_k s_j^{k-1}, \dots \quad (1)$$

where the terms in the left hand sides of the  $2p$  equations are the nodal parameters. This system can be solved with respect to  $a_k$ , ( $k \in \{1, 2, \dots, 2p\}$ ). Now we replace the coefficients so obtained into the polynomial, which can be reordered with respect to the nodal parameters as

$$u^1 = \Phi^1(s) \cdot \mathbf{u}^e, \dots \quad (2)$$

where  $\Phi^1(s), \dots$ , called *shape functions*, are  $1 \times 2p$  matrices of nodal coordinates, while the column vector  $u^e$ , associated to  $a^e$ , is the same for all shape functions.

(II). The second assignment induced on  $G_F$  by  $(\mathbf{F}, k)$  is a bipartite scalar field on the edge set,

$$\tilde{F} : E \rightarrow |\times|,$$

of which values on the edges

$$\tilde{F}([x_i, x_j]) = (F_{ij}, F_{ji}),$$

called *couple-field of internal forces*, be the following conditions are related :

$$(a^*). F_{ij} \neq 0 \Leftrightarrow (i, j) \mapsto [x_i, x_j] = a^e (\in E),$$

$$(b^*). F_{ij} + F_{ji} = 0, \text{ for all } i, j \in \{1, \dots, n\}, i \neq j.$$

This answer consists in a system of  $2m$  "internal forces" appearing as couples in the structural elements of  $G_F$ . These represent the action of some nodes upon others, and may be assumed of "mechanical" nature according to  $(b^*)$ , which is a mechanical axiom (see [22]).

Because to each pair  $(i, j)$  of unordered indices, chosen such that  $[x_i, x_j] \in E$ , it corresponds a unique  $e \in \overline{1, m}$ , the absolute values of the images by  $\tilde{F}$  are equal and we can denote by  $I^e = |F_{ij}| = |F_{ji}|$  the common value; this is called *intensity* of the internal forces induced by  $(\mathbf{F}, k)$  on the structural element "e".

Finally, we remark that when it is necessary to use a directed graph  $\vec{G}$ , the edge set  $E$  will be replaced by the arc set  $A$  all over in the previous stages. Such an example can be met in the Section 8.

## 4 One dimensional problems and stiffness matrices

To each element "e" of the network one associates the *elemental stiffness matrix*

$$K^e = \int_{\Omega^e} B^T \cdot D \cdot B d\mu,$$

where  $\Omega^e$  is the domain occupied in  $E^3$  by element and  $d\mu$  is its measure.

But, if the element can be assimilated with an one dimensional manifold and represented in the graph  $G$  by the edge  $a^e$ , its stiffness matrix becomes

$$K^e = \int_{s_i}^{s_j} B^T \cdot D \cdot B ds, \quad (3)$$

where the limits of the integral are the local coordinates of the nodes  $x_i$  and  $x_j$ .

Also,  $s \in [s_i, s_j]$  is the coordinate of an arbitrary point  $x \in a^e$ ,  $B^T$  is the  $1 \times 2p$  matrix of derivatives (sometimes, of second order) of the shape function  $\Phi^1(s) \doteq \Phi(s)$ ,  $B = (B^T)^T$ , and  $D$  is the characteristic constant expressing the local material property. The interval length of integration,  $s_j - s_i = L (= L^e)$ , also plays a role in the element geometry, being another factor that defines the perviousness  $r^e$  of the element.

Thus, we can conclude that the values of the field  $k$  on the edges 'e' are constants of the form  $r^e = r^e(D, 1/L)$ ,  $e \in \overline{1, m}$ , because  $L$  behaves oneself "like a resistance" relatively to the action of  $\mathbf{F}$ .

When the elemental geometry is not important (as, for instance, in the case of electric circuits) one can takes  $L = 1$ , and the integral will be extended to  $[0,1]$ .

The case of networks under force with only one parameter per node is not only interesting but also it is very frequent. This is the case that offer some possibilities of the FEM generalization by emphasizing of common properties for a lot of networks, such as the electric circuits, or some beam structures, or others, which lead to one dimensional problems.

Thus, if  $p=1$ ,  $\mathbf{X}$  becomes a scalar field and nodal vectors  $\mathbf{u}_i$  will have only one coordinate, as such the upper indices will be omitted.

At an arbitrary point  $x \in [x_i, x_j]$  the extended state field  $\bar{X} : G_F \rightarrow |$  has the value  $\bar{X}(x) = \bar{u}$  with only one state parameter, defined by a single polynomial of the form

$$u = a_1 + a_2s.$$

The previous system of linear equation, (1), becomes

$$\begin{bmatrix} u_i \\ u_j \end{bmatrix} = \begin{bmatrix} 1 & s_i \\ 1 & s_j \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Solving this system, one obtains the solution

$$a_1 = \frac{1}{s_j - s_i}(s_j u_i - s_i u_j), \quad a_2 = \frac{1}{s_j - s_i}(-u_i + u_j).$$

Using these coefficients it results the following form of interpolation function

$$u = \frac{1}{s_j - s_i}(s_j - s) u_i - \frac{1}{s_j - s_i}(s_i - s) u_j, \quad (4)$$

which also can be written under the matrix form

$$u = \Phi(s) \cdot u^e, \quad \text{with } \Phi(s) = \begin{bmatrix} \frac{1}{L}s_j - \frac{s}{L} & -\frac{1}{L}s_i + \frac{s}{L} \end{bmatrix}, \quad u^e = \begin{bmatrix} u_i \\ u_j \end{bmatrix}, \quad (5)$$

where  $L = s_j - s_i$ . From here one obtains  $B = [-\frac{1}{L} \quad \frac{1}{L}]$  and, using the formula of  $K^e$ , the elemental stiffness matrix is derived

$$K^e = r^e \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad (r^e = \frac{D}{L}) \quad (6)$$

First we observe that, although the element  $a^e$ , ( $e \in \overline{1, m}$ ), is an arbitrary one in  $G_F$ , it appears a typical matrix,  $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ , one and the same for all the  $m$  elements of E. On the other hand, when we effectively write any one of the matrices  $K^e$ , ( $e \in \overline{1, m}$ ), it is necessary only to multiply this constant matrix by  $r^e$ , of the form given before; in the case here exposed this factor is  $r^e = \frac{D}{L}$ , being the value of the field  $k$  on the structural element  $a^e$ . As we already seen, sometimes, the number  $L$  do not differ from an element to

another being a (nonzero) constant with respect to  $E$ , thus it can be reduced even to the value 1. At the same time,  $D$  is a constant characteristic for each element of the structure, which also can have a common value for a lot of elements. In the case of complex structures, with elements of different types, or made of different materials,  $D$  can be even a square matrix of constants; this is the case of structures consisting in beams, rods, for which  $D$  is the so-called ‘matrix of elastic constants’, of size  $6 \times 6$ , etc.(see [12]).

In order to realize the stipulated synthesis the following Kirckhoff’s law, known from the theory of electric circuits (see [15]):

” *The sum of currents traversing a cutset of the network equals zero*”, to the network under force will be extended. In our case this is expressed by the formula

$$F_i = \sum_{j \in J} F_{ij}, \quad (7)$$

where  $J \subset \{1, 2, \dots, n\}$  is a family of indices chosen such that the following condition  $\{[x_i, x_j] \mid j \in J\} = \text{Star}[i]$  holds. This expresses the fact that:

“ *The sum of internal forces incident at a node is equal to the nodal force which load that node*”.

So, if  $F_i$  is an external force applied to the network at a node ‘i’,  $F_{ij}$  appears as ‘a part’ of  $F_i$  that acts along to the isolated element  $[x_i, x_j] = a^e$ , at the node ‘i’.

Thus, the well known (in FEM) condition of element equilibrium

$$K^e \cdot \mathbf{u} = \mathbf{q},$$

where  $\mathbf{u}(= \mathbf{u}^e)$  is the matrix of unknowns (with entries: nodal parameters),  $K^e \cdot \mathbf{u}$  is the vector of a system of ‘elastic forces’ that has to equilibrate the system of forces applied to the nodes represented by the vector  $\mathbf{q}(= \mathbf{q}^e)$ , in this case becomes

$$[k_{\alpha\beta}^e] \cdot \begin{bmatrix} u_i \\ u_j \end{bmatrix} = \begin{bmatrix} F_{ij} \\ F_{ji} \end{bmatrix}, \quad (\alpha, \beta \in \{i, j\}). \quad (8)$$

In such a manner to every element  $a^e$  of  $G_F$  one can associate a symmetric, but singular matrix, of type  $2 \times 2$ ,

$$[k_{\alpha\beta}^e] = r^e \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad (9)$$

called *elemental stiffness matrix*, which can be adapted for entire class of one-parameter networks by choosing appropriate values for  $r^e$ .

We remark that the previous matrix equation cannot be solved with respect to the unknowns  $(u_i, u_j)$ ; more exactly, the linear system of the specified unknowns is not uniquely determined, according to the condition (b\*) in (II).

Extending the stiffness matrix to the set of all nodes (or of whole network) but having in attention only the considered above element 'e', we locate the nodal connection  $(i, j)$  into the  $n \times n$  matrix  $K_{ij}^e$  corresponding to a network  $G_F$  with  $n$  nodes. So, we first define the *location matrix* of an element 'e' by

$$K_{ij}^e = [\bar{k}_{\alpha\beta}^e], \tag{10}$$

where

$$\bar{k}_{\alpha\beta}^e = \begin{cases} k_{\alpha\beta}^e, & \{\alpha\beta\} \subset \{i, j\} \\ 0, & \text{otherwise} \end{cases}, \tag{10'}$$

that is the matrix that locates in the structure the element  $a^e$  has the form

$$K_{ij}^e = \begin{bmatrix} 0 & \vdots & 0 & \vdots & 0 \\ \cdots & k_{ii}^e & \cdots & k_{ij}^e & \cdots \\ 0 & \vdots & 0 & \vdots & 0 \\ \cdots & k_{ji}^e & \cdots & k_{jj}^e & \cdots \\ 0 & \vdots & 0 & \vdots & 0 \end{bmatrix}, \quad (k_{ij}^e = k_{ji}^e). \tag{11}$$

It results that for each  $e \in \overline{1, m}$  the diagonal of the elemental stiffness matrix is a part of the diagonal of this location matrix corresponding to the chosen element.

To construct the *aggregate rigidity matrix*,  $K$ , associated to the given network means to arrange the corresponding stiffness matrices of the elements in the position of the global array and to add these matrices. Thus we have

$$K = \sum_{e=1}^m K_{ij}^e, \quad (i, j) \in \{1, \dots, n\} \times \{1, \dots, n\}, \tag{12}$$

and summation is made for those pairs  $(i, j)$  for which in the matrix of incidence,  $I(G)$ , for a chosen  $e \in \{1, \dots, m\}$ ,  $\delta_i^e = \delta_j^e = 1$  appear.

The aggregate rigidity matrix  $K$  is symmetric. Sometimes it is also a banded matrix (the nonzero entries form a band along the main diagonal), but this fact is not true in general. In this sense we can formulate the following result:

*The aggregate rigidity matrix of a network under force is a simple banded matrix if and only if the network graph is a simple elementary chain.*

Examples to illustrate this assertion can be seen below.

## 5 Energy functional. The equilibrium equation and associated subspaces.

The energy functional of an isolated element  $[x_i, x_j] = a^e$  (the index 'e' will be omitted) by the following expression is defined

$$E(a^e) = \frac{1}{2} \mathbf{u}^T \cdot K \cdot \mathbf{u} - \mathbf{u}^T \cdot \mathbf{q}, \quad (13)$$

where  $\mathbf{u}(= \mathbf{u}^e)$  and  $\mathbf{q}(= \mathbf{q}^e)$  are column matrices with  $2p$  entries which are the coordinates of the vectors  $\mathbf{u}_i, \mathbf{u}_j$  and  $\mathbf{F}_i, \mathbf{F}_j$ , respectively, corresponding to the extremities  $x_i, x_j$  of the structural element  $a^e$ , and  $K(= K^e)$  is the stiffness matrix of the same element.

The well known condition of the minimum energy, that is  $\delta E(a^e) = 0$ , leads to

$$K \cdot \mathbf{u} = \mathbf{q}, \quad (14)$$

the equilibrium equation of an element (or of the structure, when  $K$  is the aggregate rigidity matrix). The vector  $\mathbf{q}$  contains the nodal forces due both to the external actions (loads) and to the internal ones (stresses, etc.). This vector can be associated to each element  $a^e$  by the values of the field  $\mathbf{F}$  on the nodes  $x_i$  and  $x_j$ , or to a network by the values of  $\mathbf{F}$  on  $S$ .

About the sizes of the matrices in the equilibrium equation we have to say that when  $K$  is the aggregate rigidity matrix this is of type  $n' \times n'$ , where  $n' = n \cdot p$ , ( $n = |S|$ ,  $p$  is the number of parameters per node), while  $\mathbf{u}$  and  $\mathbf{q}$  are both of type  $n' \times 1$ .

If the equilibrium equation is referred to the whole network or to a chain of it, the previous vectors will belong to some direct sums of subspaces, that is

$$\mathbf{u} \in \bigoplus_{i=1}^n U_i \doteq U, \quad \mathbf{q} \in \bigoplus_{i=1}^n V_i \doteq V, \quad (15)$$

where  $U$  and  $V$  are  $n \cdot p$  - dimensional real vector spaces of nodal parameters and of nodal forces, respectively, while  $U_i$  and  $V_i$ , ( $i \in \{1, \dots, n\}$ ), are subspaces of  $|^{n \cdot p}$  isomorphic with the real vector space  $|^p$ :

$$U_i, V_i \leq |^{n \cdot p}, \quad \dim |U_i = \dim |V_i = p. \quad (16)$$

In these relations  $n$  denotes the number of nodes of the network or of the chain.

So, the considered above vectors can be obtained with the help of relations

$$\mathbf{u} = \sum_{i=1}^n \mathbf{u}_i, \quad \mathbf{q} = \sum_{i=1}^n \mathbf{F}_i \quad (17)$$

where the terms in the sums are the images of the  $n$  nodes of  $S_1 \subseteq S$  by one of the vector fields

$$X : S_1 \rightarrow |^p, \quad X(x_i) = \mathbf{u}_i, \quad (i \in \{1, 2, \dots, n\}),$$

and

$$F : S_1 \rightarrow |^p, \quad F(x_i) = \mathbf{F}_i, \quad (i \in \{1, 2, \dots, n\}),$$

respectively, where  $S_1$  denotes the set of nodes of the chain or of the network.

For instance, a typical term for a component vector of  $\mathbf{u}$  is of the form

$$\mathbf{u}_i = \underbrace{[0, \dots, 0]}_{(i-1)p}, u_i^1, \dots, u_i^p, \underbrace{[0, \dots, 0]}_{(n-i)p}]^T,$$

for  $i=1, 2, \dots, n$ , ( $n = |S_1|$ ). Similar expressions can have the terms of  $\mathbf{q}$ .

For the one-dimensional problems it is enough to take  $p = 1$  everywhere above. The aggregate rigidity matrix  $K$  can be computed as was shown before, in the Section 4.

In the sequel the previous concepts on the network theory will be illustrated.

## 6 Structural networks

This type of networks, of which structural elements are beams or rods, is not only well known and used in the civil engineering but it is typical for the general case above mentioned and, so, it is easy to be adapted to the scheme given before. However, some correspondences will be put in evidence, but for a structural network consisting in a discretized cable  $\Omega$ , transformed in a chain of piecewise smooth arcs interconnected at nodes and structured as a network under force  $\mathbf{G}_F$ .

Let then  $\Omega = AB$  be an elastic (extensible) cable of length  $L$ , in equilibrium state under gravitational forces, with ends A and B, of which supports at the same level are placed. The Young's modulus  $E$  and the cross section area  $A$  complete the physical and geometrical properties of the cable.

An orthonormal frame  $R^o = \{O; \mathbf{i}, \mathbf{j}, \mathbf{k}\}$  of the Euclidean affine space  $E^3$  is chosen such that  $O \equiv A$ ,  $\mathbf{v} = \overrightarrow{AB}$  is a vector collinear with  $\mathbf{i}$ , the plane of the cable is  $(Oxz)$ , and the orientation of the  $(Oz)$  axis is indicated by the sense of gravitational force.

Now we consider a discretization of  $\Omega$  consisting in a system  $\{\Omega^e\}_n$  of arcs  $\Omega^e = M_{i-1}M_i$ , ( $e = i \in \overline{1, n}$ ,  $M_0 \equiv A$ ,  $M_n \equiv B$ ), as structural elements, and in a set  $\Gamma = \{A, M_i\}_{n+1}$  of nodes. Thus we can associate to  $\Omega$  a graph  $G = (S, E)$  of which vertex set  $S$  consists in the network nodes, that is  $\phi|_{\Gamma}(M_i) = M_i$ , and the edge set  $E$  is the collection  $\{\overline{M_{i-1}M_i}\}_n$  of straight line segments, the images of the arcs by the transfer mapping  $\phi$ , i.e.  $\phi(\Omega^e) = \overline{M_{i-1}M_i}$ . Let us denote by  $M'_i = pr_{(Ox)}M_i$ , ( $i \in \overline{1, n}$ ), the projections of nodes on the  $(Ox)$  axis, and let be  $d_i = |pr_{\mathbf{v}}\overrightarrow{OM}_i - pr_{\mathbf{v}}\overrightarrow{OM}_{i-1}|$ , ( $pr_{\mathbf{v}}\overrightarrow{OM}_i > 0$ ). If the coordinates of the nodes are  $M_i(x_i, 0, z_i)$ , then  $d_i = x_i - x_{i-1}$  and the cable span is given by  $d = \sum_{i=1}^n d_i = \|\mathbf{v}\|$ .

Assume, initially, the cable is in equilibrium under the only one distributed force on it,  $\mathbf{g}$ , that is under its weight as a permanent load.

Now we transform  $\Omega$  in a network under force,  $\mathbf{G}_F$ , by attaching of a variable load  $\mathbf{p}$  consisting in snow weight, or/and in the weight of a pipe line; suppose this last one is bound to the cable at nodes  $M_i \mapsto P_i$ , where  $P_i$  are the positions of  $M_i$  in the new equilibrium state after applying of  $\mathbf{p}$ , such that the projections on  $(Ox)$  to be conserved, (see Figure 1). So,  $\mathbf{G}_F = \bigcup_{i=1}^n P_{i-1}P_i$ .

The network  $\mathbf{G}_F$ , in the loaded cable case, by the following pencil of fields  $(\mathbf{F}, k)$  on the associated graph  $\hat{G} = (\hat{S}, \hat{E})$ ,  $\hat{S} = \{P_i\}$ ,  $\hat{E} = \{\overline{P_{i-1}P_i}\}_n$ , can be defined:

- (i).  $\mathbf{F} : \hat{S} \rightarrow |\mathbb{P}$  is a vector field on the vertex set of which values, the

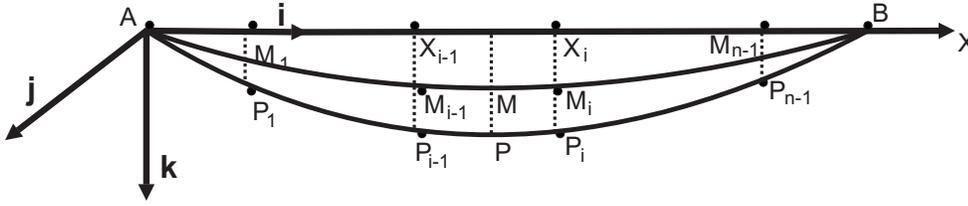


Figure 1: Discretized cable with additional load

nodal forces  $F(P_i) = (g_i + p_i)d_i$ , ( $i \in \overline{1, n}$ ), by the concentrated loads are given, and

(ii).  $k : \hat{E} \rightarrow |^*_+$  is a scalar field on the edge set of which values, the perviousness  $k(\overline{P_{i-1}P_i}) = r^e$ , ( $e = i \in \overline{1, n}$ ,  $r^e > 0$ ), by the material and geometrical nature of the cable are determined, such as the stretching rigidity and the length of projections of the structural elements on the level line (AB). These are constants of the form  $r^e = r^e(D, 1/L)$  and, when  $\mathbf{p}$  acts vertically, we may consider  $D = EA$  and we replace the element length  $L^e = s_i - s_{i-1}$  by the length  $d_i$  of its projection on (Ox) axis, such that we can take

$$r^e = \frac{EA}{d_i}, \quad (e = i = 1, \dots, n). \quad (18)$$

Here also must be mentioned that  $p = 2$ , when  $\mathbf{p}$  acts vertically, and  $p = 3$ , otherwise.

The answer of  $\mathbf{G}_F$ , as a consequence of the  $(\mathbf{F}, k)$  presence, consists in:

(I). a vector field  $X : \hat{S} \rightarrow |^p$ , of which values at nodes, the nodal vectors  $X(P_i) = z(x_i) + w(x_i)$ , by the sum of deflections before and after applying of the load  $\mathbf{p}$  is defined, and

(II). a bipartite scalar field  $\tilde{F} : E \rightarrow | \times |$ , of which values on the edges  $\tilde{F}(\overline{P_{i-1}P_i}) = (N_{i-1}^+, N_i^-)$ , by the axial tensile forces is given.

In (II) we used the notations  $N_i^- = -\|N_{\mathbf{q}}^-(x_i)\|$  and  $N_i^+ = +\|N_{\mathbf{q}}^+(x_i)\|$ , where  $\mathbf{q} = \mathbf{g} + \mathbf{p}$  is the total load,  $N_{\mathbf{q}}(x) = N_{\mathbf{q}}^-(x) + N_{\mathbf{q}}^+(x)$  is the axial tensile force acting tangentially to the smooth arc of  $\mathbf{G}_F$  at a point P, of abscise  $x$ , decomposed at each point  $P_i, (i \in \overline{1, n}$ , where the curve is not necessarily differentiable), in the two components  $N_{\mathbf{q}}^-(x_i) \in \lim_{P \rightarrow P_i} T_P(P_{i-1}P_i)$  and  $N_{\mathbf{q}}^+(x_i) \in \lim_{P \rightarrow P_i} T_P(P_iP_{i+1})$ . Moreover,  $|N_i^-| = |N_i^+|$  always when  $\mathbf{q}$  is a

vertical load.

It follows that both the conditions (a) and (b) of  $\tilde{F}$  are satisfied.

The meaning of these assertions can be found in our paper [8].

Thus the algorithm to obtain the aggregate stiffness matrix for the whole structure  $G_F$  can be applied as was described in 4.

Using (6) and (18) we have the following elemental stiffness matrix

$$K^e = EA \cdot \begin{bmatrix} d_i^{-1} & -d_i^{-1} \\ -d_i^{-1} & d_i^{-1} \end{bmatrix} = [k_{\alpha\beta}^e], \quad (\alpha, \beta = i - 1, i).$$

If we denote  $d_i^{-1} \doteq a_i$ , ( $i \in \overline{1, n}$ ), one obtains the aggregate rigidity matrix

$$K = EA \cdot \begin{bmatrix} a_1 & -a_1 & 0 & \cdots & 0 & 0 \\ -a_1 & a_1 + a_2 & -a_2 & \cdots & 0 & 0 \\ 0 & -a_2 & a_2 + a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} + a_n & -a_n \\ 0 & 0 & 0 & \cdots & -a_n & a_n \end{bmatrix}.$$

We observe  $K$  is a band matrix because the graph  $G_F$  is a chain. It can be used to compute the vertical displacements  $\mathbf{w}(x_i)$  after applying the load  $\mathbf{p}$ .

### Numerical example

A practical example for such a cable both with a numerical approach and an analytical solution was studied in our paper [8]. Let us consider an elastic cable of length  $L = 100$  m, with supports placed at the same level, and endowed with a supplementary load  $\mathbf{p}$  of  $60$  daN/m<sup>2</sup>; the Young's modulus of the cable is  $E = 1.2 \cdot 10^4$  and the cross section area is  $A = 31.462 \text{ cm}^2$ . Using a C program, called "CABLE 1", for a discretization containing then arcs of equal length, the aggregate stiffness matrix  $K$  and the vector of nodal displacements were computed. For instance, by solving (14), this last one was obtained as

$$X(P_i) = [0 \ 0.14 \ 1.12 \ 2.60 \ 4.17 \ 7.03 \ 3.98 \ 2.47 \ 1.12 \ 0.12 \ 0.01]^T.$$

## 7 Electric circuits as networks under force

We consider an electric circuit consisting of a voltage source, some operational voltage amplifiers, linear resistors, and linear capacitors. Thus, it appears as a network with structural elements: resistors and capacitors, and so we shall associate to it two kinds of stiffness matrices. The model which we propose in this case is the following:

As nodal forces loading some nodes of the circuit we consider the currents applied at those nodes due the voltage source or the amplifiers, in accordance with the mathematical modelling made in the case of modified nodal voltage analysis, and as internal forces appearing in the structural elements, the currents traversing the branches. The scalar field of perviousness either by the conductivity or by capacity of the structural elements will be represented. Finally, the nodal vectors (here with only one component) will be the voltages at nodes, which represent just the nodal parameters. So, this model corresponds to a network under force  $\mathbf{G}_F$  with only one parameter per node, ( $p = 1$ ).

In such a manner we defined the four considered above functions:

$$F : S \rightarrow |^P, \quad F(x_i) = I_i (\geq 0), \quad (i \in \overline{1, n}; p = 1),$$

$$k : E \rightarrow |^*_+, \quad (k_1)k(a^e) = G^e (= 1/R_e), \quad or$$

$$(k_2)k(a^e) = C^e, \quad (e \in \overline{1, m}),$$

$$X : S \rightarrow |^P, \quad X(x_i) = U_i, \quad (i \in \overline{1, n}; p = 1), \quad and$$

$$\tilde{F} : E \rightarrow | \times |, \quad \tilde{F}(a^e) = (I_{ij}, I_{ji}), \quad (a^e = [x_i, x_j]),$$

where  $G^e, R_e, C^e$  denote the conductance, the resistance, and the capacitance, respectively, of the structural element  $a^e$ , ( $e \in \overline{1, m}$ ), the nodal values  $U_i$  are the voltages at the nodes  $x_i$ , ( $i \in \overline{1, n}$ ), and  $I_i, I_{ij}$  are the currents loading the nodes  $x_i$ , or traversing the structural elements  $[x_i, x_j]$ , respectively.

In order to separate the set  $E$  of structural elements (resistors and capacitors) one by one we choose the vertex set  $S$  of the graph  $G = (S, E)$  associated to  $\mathbf{G}_F$  such that to comprise each of them quite between two nodes of  $S$ . Thus

the values  $(k_1)$  and  $(k_2)$  in the list given above are exclusively one another. The voltage source and the operational amplifiers do not be placed at nodes. However, these define the field  $\mathbf{F}$  of external forces loading some nodes,  $k$  is given by network nature,  $\mathbf{X}$  and  $\tilde{F}$  are induced fields on the working network  $\mathbf{G}_F$ .

If the element  $a^e = [x_i, x_j](\in E)$  is a linear resistor, according to the Ohm's law, we have

$$I_{ij} = G^e(U_i - U_j),$$

and, if it is a linear capacitor, also we have

$$I_{ij} = C^e(\dot{U}_i - \dot{U}_j),$$

for  $i, j \in \overline{1, n}$  and  $e \in \overline{1, m}$ , where  $\dot{U} = dU/dt$  ( $t$  denotes time).

We now observe that in both cases the corresponding linear or differential equations associated to these relations can be written as matrix equations of the form (8) by using some specific stiffness matrices:

$$\begin{bmatrix} I_{ij} \\ I_{ji} \end{bmatrix} = [k_{\alpha\beta}^e] \cdot \begin{bmatrix} U_i \\ U_j \end{bmatrix}, \quad \begin{bmatrix} I_{ij} \\ I_{ji} \end{bmatrix} = [h_{\alpha\beta}^e] \cdot \begin{bmatrix} \dot{U}_i \\ \dot{U}_j \end{bmatrix}, \quad (\alpha, \beta = i, j), \quad (19)$$

where  $[k_{\alpha\beta}^e]$  and  $[h_{\alpha\beta}^e]$  are the stiffness matrices of the form (9) with  $r^e = G^e$  or  $r^e = C^e$ , as perviousness of the element 'e', with respect to the fact that this one is a linear resistor or a linear capacitor, respectively.

Now we define the location matrix of each element 'e', separately for the electric circuit elements: resistors or capacitors, according to (10), and we obtain

$$K_{ij}^e = [\bar{k}_{\alpha\beta}^e] \quad \text{and} \quad H_{ij}^e = [\bar{h}_{\alpha\beta}^e]. \quad (20)$$

The aggregate stiffness matrices of the network will be

$$K = \sum_{e=1}^m K_{ij}^e \quad \text{and} \quad H = \sum_{e=1}^m H_{ij}^e, \quad (21)$$

where the sums contain nonzero terms only for those  $e \in \overline{1, m}$  which correspond to the resistors and capacitors, respectively, as structural elements.

The matrices  $K$  and  $H$  can be used to write the network equilibrium equations, which in the case of electric circuits on the Kirchhoff's laws are based.

Writing out the Kirchhoff's equations for all cut sets isolating the nodes we obtain the following system of implicit ODEs that gives the equilibrium of  $G_F$

$$H \dot{\mathbf{u}} + K \mathbf{u} = \mathbf{q}, \quad (22)$$

where  $K$  and  $H$  are the primary and secondary stiffness matrices, respectively,  $\mathbf{u}$  is the column-vector of nodal values of voltages, and  $\mathbf{q}$  is the column-vector of currents loading the nodes of the network.

If  $H$  is a regular matrix we can obtain the normal form of matrix equation (22)

$$\dot{\mathbf{u}} = A \mathbf{u} + B, \quad (23)$$

where  $A = -H^{-1} \cdot K$ ,  $B = H^{-1} \cdot \mathbf{q}$ , and the system can be solved in an usual way.

But there are some cases when  $H$  is singular. Then the system of ODEs is of index  $> 0$ . Such a system also can be solved by splitting it up in a semi-implicit form containing both differential equations and algebraic equations. The method was given by Kampowsky, Rentrop and Schmidt ([15, pp.32-39]). They also made a very important index-classification of implicit ODEs.

### Example.

In order to illustrate the previous stiffness method of circuit analysis and to put in evidence the differences between this one and that given in [15] for a modified nodal voltage analysis we choose to present the same circuit as that in the mentioned above paper. It consists of an initial voltage source with an initial voltage signal  $U_{in}$ , two linear resistors with resistances  $R_1$  and  $R_2$ , two linear capacitors with capacitances  $C^1$  and  $C^2$ , and two operational amplifiers generating the currents  $I_1$  and  $I_2$  that load two nodes of the circuit, as in Figure 2.

In Figure 3 one illustrates the field of currents (as external forces) that loads the node set  $S = \{x_1, \dots, x_5\}$ , the scalar field of perviousness (capacitances and conductances) on the edge set  $E = \{a^1, \dots, a^4\}$ , ( $a^1 = [x_1, x_2]$ ,  $a^2 = [x_1, x_5]$ ,  $a^3 = [x_2, x_3]$ ,  $a^4 = [x_3, x_4]$ ), the field of voltages (as nodal

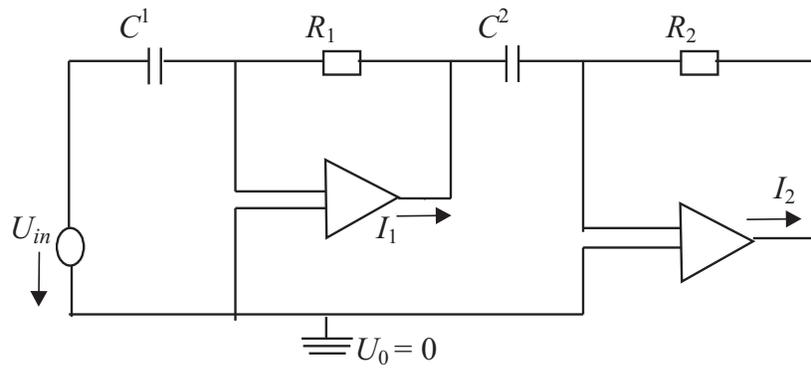


Figure 2: The network  $G_F$  (Circuit with a double operational amplifier)

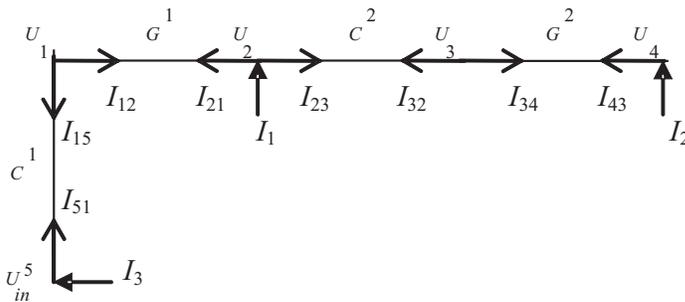


Figure 3: The graph  $G$  associated to the network

parameters) on the node set, and the induced system of currents (as internal forces) on the edge set of the graph. For example, if we have in view the elements :

$a^1 = [x_1, x_2]$ , then  $r(a^1) = G^1 (= 1/R_1)$  and the equilibrium equation is

$$G^1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} I_{12} \\ I_{21} \end{bmatrix},$$

or

$a^2 = [x_1, x_5]$ , then  $r(a^2) = C^1$  and the equilibrium equation is

$$C^1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \dot{U}_1 \\ \dot{U}_{in} \end{bmatrix} = \begin{bmatrix} I_{15} \\ I_{51} \end{bmatrix}, \text{ etc.}$$

The correspondent location matrices are

$$K_{12}^1 = \begin{bmatrix} G^1 & -G^1 & 0 & 0 & 0 \\ -G^1 & G^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and } H_{15}^2 = \begin{bmatrix} C^1 & 0 & 0 & 0 & -C^1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -C^1 & 0 & 0 & 0 & C^1 \end{bmatrix}.$$

respectively.

We observe we have two of a type of such nonzero location matrices and using the formulas (20), (10') and (21) we can obtain the two aggregate stiffness matrices associated to the network

$$K = \begin{bmatrix} G^1 & -G^1 & 0 & 0 & 0 \\ -G^1 & G^1 & 0 & 0 & 0 \\ 0 & 0 & G^2 & -G^2 & 0 \\ 0 & 0 & -G^2 & G^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} C^1 & 0 & 0 & 0 & -C^1 \\ 0 & C^2 & -C^2 & 0 & 0 \\ 0 & -C^2 & C^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -C^1 & 0 & 0 & 0 & C^1 \end{bmatrix}.$$

Now the equilibrium equation of the whole network can be written under the form (22), where the following vectors will be used:  $u = [U_1 U_2 U_3 U_4 U_{in}]^T$ ,  $\dot{u} = [\dot{U}_1 \dot{U}_2 \dot{U}_3 \dot{U}_4 \dot{U}_{in}]^T$ , and  $q = [0 I_1 0 I_2 I_3]^T$  with  $I_3 = C^1(\dot{U}_{in} - \dot{U}_1)$ .

## 8 Pipe line networks for water supply

In their book [13] the authors suggested the possibility to make an analogy between a water supply network and an electric circuit, by putting in correspondence the currents traversing an RLC circuit with water flowing in a pipe and the voltage drop with difference in pressure at the two ends of the pipe. However, this analogy cannot be completely made even for some simple types of circuits and water supply networks by using only the enumerated above two pairs of correspondent terms and by writing out their equilibrium equations provided by Kirchhoff's laws. In order to obtain a new improvement of a mathematical model it is necessary to analyze more technical details.

Let us consider a ring network of water supply,  $\Omega$ , of which mathematical models with the help either of a directed planar graph  $\vec{G} = (S, A)$ , or of an undirected topological graph  $G = (S, E)$  can be introduced;  $S = \{x_i\}_n$  is the set of vertices that corresponds to the network nodes and to the set of structural elements (pipes) $\{\Omega^e\}$  it corresponds the set  $A = \{\vec{a}^e\}_m$  of arcs of  $\vec{G}$ , (Figure 4), or the set  $E = \{a^e\}$  of edges of  $G$ . Thus we can organize  $\Omega$  either as a network with flow  $\mathbf{G}_\Phi$ , or as a network under force  $\mathbf{G}_F$ . To apply FEM we consider only this last case, but the graph which will be used is  $\vec{G}$  for which the water yield senses give the orientation of the arcs. So, we endow this graph with the fields indicated in the Section 3 for a network of  $\mathbf{G}_F$  type, as follows.

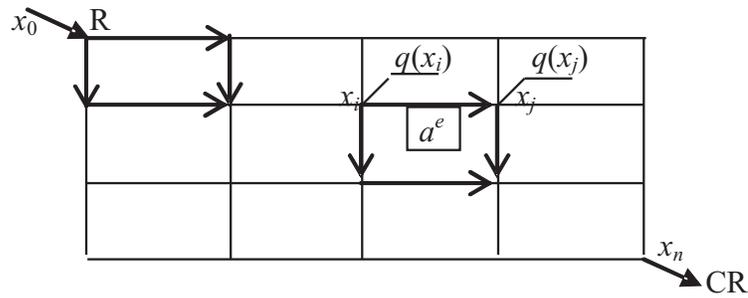


Figure 4: The graph  $\vec{G}$  associated to the ring network

In this section the notation  $\vec{a}^e = (x_i, x_j)$  means that there exist two nodes  $x_i$  and  $x_j$  connected by the arc  $\vec{a}^e \in A$ .

For every pipe of the ring water network, represented in the graph  $\vec{G}$  by an arc  $\vec{a}^e$ , we denote by  $Q_t(\vec{a}^e)$  the transitory yield and by  $Q_c(\vec{a}^e)$  the consumption yield. Then the mean flow rate  $\varphi$  on  $G_F$  will be given by its values on the arcs

$$\varphi(\vec{a}^e) = Q_t(\vec{a}^e) + q(\vec{a}^e),$$

where  $q(\vec{a}^e) = Q_c(\vec{a}^e)/2$  is the mean consumption yield on the pipe (because, if there exists a lot of consumers along the considered above pipe one consider the arithmetic average of the individual consummations). Also the following condition is needed:  $Q_t(\vec{a}^e) + Q_c(\vec{a}^e) \leq c(\vec{a}^e)$ , where  $c(\vec{a}^e)$  denotes the capacity of  $\vec{a}^e$ ; the case of equality means the arc is saturate.

Now we define the field  $\mathbf{F}$  on S by

$$\mathbf{F}(x_i) := \sum_{\vec{a}^e \in \omega_x^-} Q_t(\vec{a}^e) + \sigma(x_i), \quad (24)$$

where  $\omega_x^-, (x = x_i)$ , denotes the subset of arcs of A incident towards interior with the vertex  $x$ , and  $\sigma(x_i)$  denotes the water surplus (excess) at the same node provided occasionally by some reservoir; for a current node  $x_i \in S$  we have  $\sigma(x_i) \geq 0$ . Finally, we observe that in this case we can take  $p = 1$  and the Kirchhoff's "currents law" (KCL) can be applied as:

$$\mathbf{F}(x_i) = \sum_{\vec{a}^e \in \omega_x^+} \varphi(\vec{a}^e) + q(x_i), \quad (25)$$

where  $\omega_x^+, (x = x_i)$ , is the set of all arcs of A incident towards exterior with the vertex  $x$ , and  $q(x_i)$  denotes the water consumption at the same node (Figure 5).

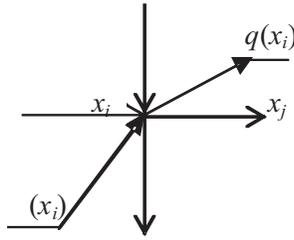


Figure 5: Water consumption at an isolated node

The scalar field values on A,  $k(a^e) \doteq r^e, (e \in \overline{1, m})$ , in this case are given by

$$r^e = k_o \cdot \frac{D^n}{L}, \quad (26)$$

where  $k_o$  is a specific constant,  $L$  and  $D$  are the length and diameter of the pipe represented by the arc  $\vec{a}^e$ , respectively, and  $n$  is a number in the closed real interval  $[1.85, 2.0]$ , (according to [11]).

The two assignments appearing in the network  $\mathbf{G}_F$ , as consequence of the presence of the pencil  $(\mathbf{F}, k)$ , are the following:

The first is a vector field on S,  $\mathbf{X} : S \rightarrow |\mathbb{P}$ , of which values,  $\mathbf{u}_i$ , the nodal vectors, in the case of water supply network are defined by the pressures at

nodes  $X(x_i) := h_i$ , ( $i \in \{1, 2, \dots, n\}$ ); the difference  $h^e = h_i - h_j$  is the *head loss* along the arc  $\vec{a}^e = (x_i, x_j)$ . Again we observe that  $p = 1$ , as well as for electric circuits. We put  $u = [h_1 \ h_2 \ \dots \ h_n]^T$  for the vector of nodal parameters.

The second is a bipartite scalar field on the arc set  $A$ ,  $\tilde{F} : A \rightarrow |\times|$ , given by  $\tilde{F}(\vec{a}^e) = (F_{ij}, F_{ji})$ , ( $\vec{a}^e = (x_i, x_j)$ ), where the internal forces are defined as

$$F_{ij} := [\varphi(\vec{a}^e)]^2 \quad \text{if } \vec{a}^e = (x_i, x_j) \quad \text{and} \quad \vec{a}^e \in \omega_x^+, (x = x_i), \quad \text{and}$$

$$F_{ji} := -[\varphi(\vec{a}^e)]^2 \quad \text{if } \vec{a}^e = (x_i, x_j) \quad \text{and} \quad \vec{a}^e \in \omega_x^-, (x = x_i),$$

conventionally, for the opposite sense of the arc representing water flowing.

The equilibrium equation of each ring and, finally, of the whole network, requires that the sum of all head losses vanishes. This is equivalent in the electric circuit theory with the well known Kirchhoff's 'voltage law' (KVL). Symbolically, it can be written as ([11]):  $\sum h^e = 0$ , with  $h^e = sQ^n$ ,  $s$  being the *resistance modulus* of a pipe 'e' and  $Q$  is the mean flow on it. Naturally, this is true for an ideal water supply network. Usually, a correction of the form  $(Q \pm \Delta q)$  on each pipe, when  $\sum h^e = \Delta h (> 0)$ , is needed.

In accordance with the previous considerations on each arc  $\vec{a}^e \in \vec{G}$  we have

$$[\varphi(\vec{a}^e)]^2 = r^e(h_i - h_j), \quad \text{if } \vec{a}^e = (x_i, x_j), \quad (27)$$

where  $r^e$  denotes the perviousness, given here by the value (26).

In order to establish the equilibrium equation of the whole network first will be written this equation for a current pipe 'e' under the matrix form

$$[k_{\alpha\beta}^e] \cdot \begin{bmatrix} h_i \\ h_j \end{bmatrix} = \begin{bmatrix} (Q^e)^2 \\ -(Q^e)^2 \end{bmatrix}, \quad (\alpha, \beta \in \{i, j\}), \quad (28)$$

where  $Q^e = \varphi(\vec{a}^e)$  is the flow rate of the corresponding arc  $\vec{a}^e$  and  $[k_{\alpha\beta}^e]$  is its stiffness matrix (of the form (9) with  $r^e$  given by (26)).

Now, using (10) and (10'), we define the location matrix  $K_{\alpha\beta}^e$  for the structural element 'e', and, with the help of (12), we can obtain the aggregate rigidity matrix  $K$ .

Finally, the asked equilibrium equation (14),  $K \cdot u = q$ , can be written by using the vector  $q = [q(x_i) - \sigma(x_i)]^T$  of nodal forces.

## References

- [1] Abramov, N.N.: Teoria i metodica rasciota sistem podaci i raspredelenia vodi; I.L.S., Moskva, 1972.
- [2] Arnold, V.I.: Les méthodes mathématiques de la Mécanique classique; Ed. Mir, Moskou, 1976.
- [3] Berge, C.: Théorie des graphes et ses applications; Dunod, Paris, 1967.
- [4] Boja, N., Filimon, E.: *Sur les graphes conjugués utilisés aux réseaux annulaires d'alimentation en eau*; Bul.St.Teh., I.P."Traian Vuia", Timisoara, Ser. Constructii, Tom.22(36), **2**, (1977), 393-397.
- [5] Boja, N.: *On the aggregate rigidity matrices in the network analysis*; Anal.t.Univ."Ovidius", Constana, Seria Matem.,Vol 3(**1**), (1995), 29-37.
- [6] Boja, N.: *Mathematical modelling of some electric circuits as networks under force*; Italian J.of Pure and Appl. Math., Udine, Nr.2, (1997), 151-158.
- [7] Boja, N.: *One-dimensional problems in the network under force analysis*; Bull.for Appl.and Comp.Math., Budapest, BAM-1303, (LXXXI-A), (1997), 103-110.
- [8] Boja, N., Binzar, T.: *Stiffness matrices of discretized cables, I*; Bull.for Appl.and Comp.Math., Budapest, BAM-1463, (LXXXV-A), (1998), 169-178.
- [9] Cook, D.R.: Concepts and applications of finite element analysis; Wiley, New York, 1974.
- [10] Folic, R., Kovacevic, D.: *Numerical modelling in Civil Engineering Structural Analysis*; Proceed. of the 10<sup>th</sup> Symp.of Math. and its Appl., Timisoara, (2003), 391-400.
- [11] Gheniev, N.N., Abramov, N.N.,Pavlov, V.I.: Water supply (in russian); Moskva, 1953.
- [12] Green, A.E., Zerna,W.: Theoretical Elasticity; Clarendon Press, Oxford, 1954.
- [13] Hirsch, M., Smale, S.: Differential Equations, Dynamical Systems and Linear Algebra; Pure and Appl. Math., Columbia Univ., New York, 1976.

- [14] Jeter, M.N.: *Mathematical Programming – An Introduction in Optimization*; Pure and Appl. Math., **102**, Dekker, New York, 1986.
- [15] Kampowsky, W., Rentrop, P., Schmidt, W.: *Classification and numerical simulation of electric circuits*; Surv.Math.for Ind., vol.2, **1**, Springer-Verlag, Wien, New York, (1992), 23-65.
- [16] Matsumoto, T.: *On the dynamics of electrical networks*; J. Diff. Equations, vol. 21, **1**, (1976), 179-196.
- [17] Parton, V.Z., Perlin, P.I.: *Mathematical Methods of the Theory of Elasticity*, Vol.I, Mir.Publ., Moskow, 1984.
- [18] Poceski, A.: *Mixed Finite Element Method*; Springer – Verlag, New York, 1992.
- [19] Pop, N.: *An intuitive way for teaching Finite Element Method*; Proceed. of 8<sup>th</sup> SEFI European Seminar on Math. and Eng. Educ., Praga, (1995), 174-178.
- [20] Smale, S.: *On the mathematical foundations of electrical circuit theory*; J. Diff. Geom., (1972), 193-210.
- [21] Takens, F.: *Geometric aspects of nonlinear R..L.C. networks*; Lect. Notes in Math., **468**, Springer - Verlag, Berlin-Heidelberg-New York, (1974), 305-332.
- [22] Truesdell, C.: *A first course in rational continuum mechanics*; J. Hopkins Univ., Baltimore, Maryland, 1972.
- [23] Zienkiewicz, O.C.: *La méthode des éléments finis*; Mc.Grow, Paris,1979.

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## **Osnovni opšti koncepti analize mreža**

UDK 517.9

Ovaj pregledni rad se bavi studijom onih tipova materijalnih mreža koje se sreću u građevinarstvu, elektrotehnici, mehanici i hidrotehnici, a njihovo ponašanje vodi linearnim problemima rešivim pomoću metode konačnih elemenata adekvatnim algoritmima.

Prikazana je unitarna teorija mreža koj se, zatim, ilustruje primerima strukturnih mreža u građevinarstvu, električnim kolima, mrežama vodovodnim kao i ravanskim i prostornim mehanizmima.

Pažnja se koncentriše na pokazivanje esencijalnih osobina i konceptata analize mreža razdvajajući mrežu pod dejstvom sile od ostalih vrsta materijalnih mreža. Takvoj mreži je pridružen ravanski, povezan, usmeren ili neusmeren graf. Ona je opremljena takodje nekim vektorskim poljima na skupu diskontinualnih tačaka.