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Approximate solution for SPP with discontinuous source term

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Abstract

In this paper we shall consider self-adjoint singularly perturbed problem described by the ordinary differential equation of second order with small parameter multiplying the highest derivative and discontinuous source term, and the appropriate boundary conditions, which describes steady state of certain flow problems. The solution displays both boundary layers and an interior layer.

The domain decomposition will be performed determining layer subintervals which are adapted to polynomial approximation. The division points for the interior layer are determined by the procedure similar to the one for boundary layers using appropriate resemblance function.

The solution out of boundary layer is approximated by the solution of the reduced problem, and the layer solutions is approximated by truncated orthogonal series giving a smooth approximate solution upon the entire interval. The coefficients of the truncated series are evaluated using pseudospectral technique.

The rate of convergence is examined and the order-of-magnitude of the error is given, using the principle of inverse monotonicity and the behavior of the pseudospectral approximations.

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Numerical example is included and it shows the high accuracy of the presented method.

Keywords: *singular perturbation, pseudospectral approximation, discontinuous source term, domain decomposition*

1 The exact solution of the problem

In this paper we shall consider a self-adjoint singularly perturbed reaction-diffusion boundary value problem in one dimension with a discontinuous source term. We want to find $u_\varepsilon \in C^1(0,1) \cap C^2(I_d),\ I_d = (0,d) \cup (d,1)$, such that

$$Lu \equiv -\varepsilon^2 u_\varepsilon'' + g(x)u_\varepsilon = f(x),\ x \in I_d,\ u_\varepsilon(0) = a,\ u_\varepsilon(1) = b, \qquad (1)$$

where $f(x)$ has a single discontinuity at the point $x = d$, i.e.

$$f(d^-) \neq f(d^+), \qquad d \in (0,1)$$

and $f(x)$ is sufficiently smooth on I_d. The function $g(x)$ is sufficiently smooth on $[0,1]$ and satisfies the condition

$$g(x) \geq \alpha > 0, \qquad x \in [0,1].$$

P.A. Farrell, J.J. Miller, E. O' Riordan and G. I. Shishkin have shown in [1] that the differential operator L which occurs in (1) satisfies minimum principle on $[0,1]$, i.e. if for an arbitrary function $w(x) \in C[0,1] \cap C^2(I_d)$ the following conditions hold

$$w(0) \geq 0, \quad w(1) \geq 0$$

$$Lw(x) \geq 0 \quad \forall x \in I_d$$

$$w(d^-) = w(d^+), \quad w'(d^-) \geq w'(d^+),$$

then $w(x) \geq 0$ for all $x \in [0,1]$.

They have also shown that the boundary value problem (1) has the unique solution $u_\varepsilon \in C^1(0,1) \cap C^2(I_d)$ given by the expression which contains two constants that require further determination

$$u_\varepsilon(x) = \begin{cases} y_1(x) + (a - y_1(0))\phi_1(x) + A\phi_2(x), & x \in (0,d) \\ y_2(x) + (b - y_2(1))\phi_2(x) + B\phi_1(x), & x \in (d,1) \end{cases} \tag{2}$$

where $y_1(x)$ and $y_2(x)$ are particular solutions of the differential equations

$$-\varepsilon^2 y_1'' + g(x)y_1 = f(x), \quad x \in (0,d) \tag{3}$$

$$-\varepsilon^2 y_2'' + g(x)y_2 = f(x), \quad x \in (d,1), \tag{4}$$

functions $\phi_1(x)$ and $\phi_2(x)$ are the solutions of the boundary value problems

$$-\varepsilon^2 \phi_1'' + g(x)\phi_1 = 0, \quad x \in (0,1), \quad \phi_1(0) = 1, \phi_1(1) = 0 \tag{5}$$

$$-\varepsilon^2 \phi_2'' + g(x)\phi_2 = 0, \quad x \in (0,1), \quad \phi_2(0) = 0, \phi_2(1) = 1 \tag{6}$$

and constants A and B are chosen in such a way that $u_\varepsilon \in C^1(0,1)$.

Following this idea we can construct the explicit form of the exact solution, which is stated in the following theorem

Theorem 1 *Let*

$$D = \begin{vmatrix} \phi_1(d) & \phi_2(d) \\ \phi_1'(d) & \phi_2'(d) \end{vmatrix}, \tag{7}$$

$$E_i = \begin{vmatrix} \phi_i(d) & y_2(d^+) - y_1(d^-) \\ \phi_i'(d) & y_2'(d^+) - y_1'(d^-) \end{vmatrix}, \quad F_i = \frac{E_i}{D}, \quad i = 1,2 \tag{8}$$

where $\phi_1(x)$ and $\phi_2(x)$ are solutions of the boundary value problems (5), (6) and $y_i(x)$, $i = 1,2$ are particular solutions of the differential equations (3), (4). The exact solution of the problem (1) can be represented as

$$u_\varepsilon(x) = \begin{cases} y_1(x) + (a - y_1(0))\phi_1(x) + (F_1 - y_2(1) + b)\,\phi_2(x) & x \in (0,d) \\ y_2(x) + (b - y_2(1))\phi_2(x) + (F_2 - y_1(0) + a)\,\phi_1(x) & x \in (d,1) \end{cases} \tag{9}$$

and $u_\varepsilon(x) \in C^1(0,1) \cap C^2(I_d)$.

Proof: Applying the principle of inverse monotonicity to the problems (5) and (6) we can see that

$$0 < \phi_i(x) < 1\,,\ x \in (0,1), \quad i = 1,2$$

which means that these functions don't have internal extreme. According to the boundary conditions we conclude that

$$\phi_1'(x) < 0 \quad \text{and} \quad \phi_2'(x) > 0\,, \quad x \in (0,1)$$

which guaranties that $D \neq 0$ and implies the existence of constants F_1 and F_2.

In the next step we shall prove that the solution (9) satisfies the differential equation in (1).

Let $x \in (0,d)$. As

$$u_\varepsilon''(x) = y_1''(x) + (a - y_1(0))\phi_1''(x) + (F_1 - y_2(1) + b)\phi_2''(x)$$

we can see that

$$\begin{aligned}
Lu_\varepsilon \ &\equiv\ -\varepsilon^2 u_\varepsilon''(x) + g(x)u_\varepsilon(x) = \\
&=\ -\varepsilon^2 y_1''(x) - \varepsilon^2(a - y_1(0))\phi_1''(x) - \varepsilon^2(F_1 - y_2(1) + b)\phi_2''(x) + \\
&\quad\ +g(x)y_1(x) + g(x)(a - y_1(0))\phi_1(x) + g(x)(F_1 - y_2(1) + b)\phi_2(x) = \\
&=\ (F_1 - y_2(1) + b)(-\varepsilon^2\phi_2''(x) + g(x)\phi_2(x)) + \\
&\quad\ +(a - y_1(0))(-\varepsilon^2\phi_1''(x) + g(x)\phi_1(x)) - \varepsilon^2 y_1''(x) + g(x)y_1(x).
\end{aligned}$$

Using (5),(6) and (3) we can see that

$$Lu_\varepsilon = f(x), \qquad x \in (0,d).$$

In the same way we can prove that

$$Lu_\varepsilon = f(x), \qquad x \in (d,1).$$

In the third step we have to verify that the proposed solution $u_\varepsilon(x)$ satisfies the boundary conditions in (1).

As $\phi_1(0) = 1$, $\phi_1(1) = 0$, $\phi_2(0) = 0$ and $\phi_2(1) = 1$ we can see that

$$u_\varepsilon(0) = y_1(0) + (a - y_1(0))\phi_1(0) + (F_1 - y_2(1) - b)\,\phi_2(0) = a$$

and

$$u_\varepsilon(1) = y_2(1) + (b - y_2(1))\phi_2(1) + (F_2 - y_1(0) + a)\,\phi_1(1) = b.$$

The last step is to prove that $u_\varepsilon(x)$ and its first derivative are continuous at the point $x = d$.

From (9) we can see that

$$\begin{aligned} &u_\varepsilon(d^-) - u_\varepsilon(d^+) = \\ &= y_1(d^-) + (a - y_1(0))\phi_1(d) + F_1\phi_2(d) - y_2(1)\phi_2(d) + b\phi_2(d) - \\ &- y_2(d^+) - (b - y_2(1))\phi_2(d) - F_2\phi_1(d) + y_1(0)\phi_1(d) - a\phi_1(d) = \\ &= y_1(d^-) + \frac{E_1}{D}\phi_2(d) - y_2(d^+) - \frac{E_2}{D}\phi_1(d) = \\ &= y_1(d^-) - y_2(d^+) + \frac{E_1\phi_2(d) - E_2\phi_1(d)}{D}. \end{aligned}$$

As

$$\begin{aligned} E_1\phi_2(d) &= \phi_1(d)\phi_2(d)y_2'(d^+) - \phi_1(d)\phi_2(d)y_1'(d^-) - \\ &\quad - \phi_1'(d)\phi_2(d)y_2(d^+) + \phi_1'(d)\phi_2(d)y_1(d^-) \end{aligned}$$

and

$$\begin{aligned} E_2\phi_1(d) &= \phi_1(d)\phi_2(d)y_2'(d^+) - \phi_1(d)\phi_2(d)y_1'(d^-) - \\ &\quad - \phi_1(d)\phi_2'(d)y_2(d^+) + \phi_1(d)\phi_2'(d)y_1(d^-) \end{aligned}$$

we obtain that

$$\begin{aligned} E_1\phi_2(d) - E_2\phi_1(d) &= \big(y_2(d^+) - y_1(d^-)\big)\left(\phi_1(d)\phi_2'(d) - \phi_1'(d)\phi_2(d)\right) = \\ &= \big(y_2(d^+) - y_1(d^-)\big)\cdot D. \end{aligned}$$

Thus,

$$u_\varepsilon(d^-) - u_\varepsilon(d^+) = 0$$

which means that the function $u_\varepsilon(x)$ is continuous for all $x \in (0,1)$.

Starting from

$$u_\varepsilon'(x) = \begin{cases} y_1'(x) + (a - y_1(0))\phi_1'(x) + (F_1 - y_2(1) + b)\,\phi_2'(x) & x \in (0,d) \\ y_2'(x) + (b - y_2(1))\phi_2'(x) + (F_2 - y_1(0) + a)\,\phi_1'(x) & x \in (d,1) \end{cases}$$

we can see that

$$u'_\varepsilon(d^-) - u'_\varepsilon(d^+) = y'_1(d^-) - y'_2(d^+) + \frac{E_1\phi'_2(d) - E_2\phi'_1(d)}{D} \ .$$

As

$$E_1\phi'_2(d) - E_2\phi'_1(d) = \big(y'_2(d^+) - y'_1(d^-)\big) \cdot D$$

we obtain that

$$u'_\varepsilon(d^-) - u'_\varepsilon(d^+) = 0$$

which means that the first derivative of the function $u_\varepsilon(x)$ is continuous for all $x \in (0,1)$.

This proves that the exact solution $u_\varepsilon(x)$, given by (9), belongs to the space $C^1(0,1) \cap C^2(I_d)$.

2 The asympthotic solution

In practice it might be quite difficult to determine particular solutions $y_1(x)$ and $y_2(x)$ of the problems (3) and (4), so the exact solution of the given problem (1) can't be evaluated. Instead of the exact solution it might be practical to deal with the asympthotic solution. The asympthotic solution for the problem with homogenious boundary conditions was constructed by authors in [4]. In this section we will adopt that result to the problem (1).

Let $z(x)$ represent the solution of the reduced problem

$$g(x)z(x) = f(x), \quad x \in I_d. \tag{10}$$

The jump of function $f(x)$ at the point $x = d$ implies that the solution of the reduced problem (10) also has a discontinuity at $x = d$, so we shall represent it as

$$z(x) = \begin{cases} z_l(x), & x \in (0,d) \\ z_r(x), & x \in (d,1) \end{cases} \tag{11}$$

where

$$g(x)z_l(x) = f(x), \quad x \in (0,d) \tag{12}$$

$$g(x)z_r(x) = f(x), \quad x \in (d,1) \ . \tag{13}$$

Theorem 2 *Let*

$$D_i = \begin{vmatrix} \phi_i(d) & z_r(d^+) - z_l(d^-) \\ \phi_i'(d) & z_r'(d^+) - z_l'(d^-) \end{vmatrix} , \quad M_i = \frac{D_i}{D} , \quad i = 1,2 \tag{14}$$

where D is defined by (7) and the functions $\phi_1(x)$ and $\phi_2(x)$ are solutions of the boundary value problems (5),(6). The function $u_a(x) \in C^1(0,1) \cap C^2(I_d)$ defined by

$$u_a(x) = \begin{cases} z_l(x) - z_l(0)\phi_1(x) + (M_1 - z_r(1))\,\phi_2(x) & x \in (0,d) \\ z_r(x) - z_r(1)\phi_2(x) + (M_2 - z_l(0))\,\phi_1(x) & x \in (d,1) \end{cases} \tag{15}$$

represents the asympthotic solution to the problem (1), i.e.

$$|u_\varepsilon(x) - u_a(x)| \le C\varepsilon^2 , \quad x \in I_d , \tag{16}$$

where $C = \sup_{x \in I_d} |z''(x)|$.

Proof: In order to prove that the asympthotic solution $u_a(x)$, given by (15), belongs to the space $C^1(0,1) \cap C^2(I_d)$ we have to prove that $u_a(x)$ and its first derivative are continuous at the point $x = d$.

From (15) we can see that

$$\begin{aligned} u_a(d^-) - u_a(d^+) &= z_l(d^-) - z_l(0)\phi_1(d) + M_1\phi_2(d) - z_r(1)\phi_2(d) - \\ &\quad - z_r(d^+) + z_r(1)\phi_2(d) - M_2\phi_1(d) + z_l(0)\phi_1(d) = \\ &= z_l(d^-) + \frac{D_1}{D}\phi_2(d) - z_r(d^+) - \frac{D_2}{D}\phi_1(d) = \\ &= z_l(d^-) - z_r(d^+) + \frac{D_1\phi_2(d) - D_2\phi_1(d)}{D}. \end{aligned}$$

As

$$\begin{aligned} D_1\phi_2(d) &= \phi_1(d)\phi_2(d)z_r'(d^+) - \phi_1(d)\phi_2(d)z_l'(d^-) - \\ &\quad - \phi_1'(d)\phi_2(d)z_r(d^+) + \phi_1'(d)\phi_2(d)z_l(d^-) \end{aligned}$$

and

$$\begin{aligned} D_2\phi_1(d) &= \phi_1(d)\phi_2(d)z_r'(d^+) - \phi_1(d)\phi_2(d)z_l'(d^-) - \\ &\quad - \phi_1(d)\phi_2'(d)z_r(d^+) + \phi_1(d)\phi_2'(d)z_l(d^-) \end{aligned}$$

we obtain that

$$\begin{aligned} D_1\phi_2(d) - D_2\phi_1(d) &= \left(z_r(d^+) - z_l(d^-)\right)\left(\phi_1(d)\phi_2'(d) - \phi_1'(d)\phi_2(d)\right) = \\ &= \left(z_r(d^+) - z_l(d^-)\right) \cdot D. \end{aligned}$$

Thus,

$$u_a(d^-) - u_a(d^+) = 0$$

which proves that the function $u_a(x)$ is continuous for all $x \in (0,1)$.

Starting from

$$u_a'(x) = \begin{cases} z_l'(x) - z_l(0)\phi_1'(x) + (M_1 - z_r(1))\,\phi_2'(x) & x \in [0,d) \\ z_r'(x) - z_r(1)\phi_2'(x) + (M_2 - z_l(0))\,\phi_1'(x) & x \in (d,1] \end{cases}$$

we can see that

$$u_a'(d^-) - u_a'(d^+) = z_l'(d^-) - z_r'(d^+) + \frac{D_1\phi_2'(d) - D_2\phi_1'(d)}{D}\ .$$

As

$$D_1\phi_2'(d) - D_2\phi_1'(d) = \left(z_r'(d^+) - z_l'(d^-)\right) \cdot D$$

we obtain that

$$u_a'(d^-) - u_a'(d^+) = 0,$$

which proves that the first derivative of the function $u_a(x)$ is continuous for all $x \in (0,1)$.

Further we have to prove that $u_a(x)$ is the asymthotic solution of the problem (1).

Let $x \in (0,d)$. As

$$u_a''(x) = z_l''(x) - z_l(0)\phi_1'' + (M_1 - z_r(1))\phi_2''$$

we can see that

$$\begin{aligned} Lu_a &\equiv -\varepsilon^2 u_a''(x) + g(x)u_a(x) = \\ &= -\varepsilon^2 z_l''(x) + \varepsilon^2 z_l(0)\phi_1'' - \varepsilon^2(M_1 - z_r(1))\phi_2'' + \\ &\quad + g(x)z_l(x) - g(x)z_l(0)\phi_1 + g(x)(M_1 - z_r(1))\phi_2 = \\ &= (M_1 - z_r(1))(-\varepsilon^2\phi_2'' + g(x)\phi_2) - z_l(0)(-\varepsilon^2\phi_1'' + g(x)\phi_1) \\ &\quad + g(x)z_l(x) - \varepsilon^2 z_l''(x). \end{aligned}$$

Using (5), (6) and (12) we can see that

$$Lu_a = f(x) - \varepsilon^2 z_l''(x), \qquad x \in (0, d).$$

In the same way we can prove that

$$Lu_a = f(x) - \varepsilon^2 z_r''(x), \qquad x \in (d, 1).$$

Thus,

$$Lu_a \equiv -\varepsilon^2 u_a + g(x)u_a = f(x) - \varepsilon^2 z''(x), \qquad x \in I_d. \tag{17}$$

Starting from $Lu_\varepsilon = f(x)$, using (17), we can see that

$$L(\pm(u_\varepsilon - u_a)) = \pm\varepsilon^2 z''(x) \le \varepsilon^2 |z''(x)|, \quad x \in I_d.$$

As the operator L satisfies minimum principle on $[0, 1]$ we conclude that

$$|u_\varepsilon(x) - u_a(x)| \le C\varepsilon^2, \quad x \in I_d, \quad \text{where } C = \sup_{x \in I_d} |z''(x)|.$$

3 Piecewise approximation of the solution

We can see that

$$z(d^-) = z_l(d^-) = \frac{f(d^-)}{g(d)} \quad \text{and} \quad z(d^+) = z_r(d^+) = \frac{f(d^+)}{g(d)} \ ,$$

which means that the solution $u_\varepsilon(x)$, besides boundary layers at $x = 0$ and $x = 1$, also has an interior layer at $x = d$. It is well known that the length of these layers is of order $\mathcal{O}(\varepsilon)$.

When we want to apply pseudospectral approximation to the solution of the problem (1), the first step is to divide the interval $[0, 1]$ into six subintervals using division points $x_0 = c_0\varepsilon$, $x_l = d - c_l\varepsilon$, $x = d$, $x_r = d + c_r\varepsilon$ and $x_1 = 1 - c_1\varepsilon$.

Upon subintervals $[x_0, x_l]$ and $[x_r, x_1]$ the exact solution is approximated by the reduced solution $z(x)$, and upon the other four subintervals the approximate solution is represented as the sum of the reduced solution and appropriate layer solution. This means that the approximate solution can be represented as

$$u(x)=\begin{cases} z_l(x)+u_0(x) & x\in[0,c_0\varepsilon] \\ z_l(x) & x\in(c_0\varepsilon,d-c_l\varepsilon) \\ z_l(x)+u_l(x) & x\in[d-c_l\varepsilon,d) \\ z_r(x)+u_r(x) & x\in(d,d+c_r\varepsilon] \\ z_r(x) & x\in(d+c_r\varepsilon,1-c_1\varepsilon) \\ z_r(x)+u_1(x) & x\in[1-c_1\varepsilon,1] \end{cases} \tag{18}$$

where functions $u_0(x)$, $u_l(x)$, $u_r(x)$ and $u_1(x)$ represent layer solutions which satisfy

$$-\varepsilon^2 u_0''(x)+g(x)u_0(x)=\varepsilon^2 z_l''(x),\ x\in[0,c_0\varepsilon], \tag{19}$$

$$u_0(0)=a-z_l(0),\quad u_0(c_0\varepsilon)=0$$

$$-\varepsilon^2 u_l''(x)+g(x)u_l(x)=\varepsilon^2 z_l''(x),\ x\in[d-c_l\varepsilon,d) \tag{20}$$

$$-\varepsilon^2 u_r''(x)+g(x)u_r(x)=\varepsilon^2 z_r''(x),\ x\in(d,d+c_r\varepsilon] \tag{21}$$

$$u_l(d-c_l\varepsilon)=0,\quad u_r(d+c_r\varepsilon)=0 \tag{22}$$

$$z_l(d^-)+u_l(d^-)=z_r(d^+)+u_r(d^+), \tag{23}$$

$$z_l'(d^-)+u_l'(d^-)=z_r'(d^+)+u_r'(d^+), \tag{24}$$

$$-\varepsilon^2 u_1''(x)+g(x)u_1(x)=\varepsilon^2 z_r''(x),\ x\in[1-c_1\varepsilon,1], \tag{25}$$

$$u_1(1-c_1\varepsilon)=0,\quad u_1(1)=b-z_r(1).$$

The second step is to represent layer solutions in the form of truncated Chebyshev series.

In some of their earlier papers concerning standard self-adjoint SPP (see e.g. [3]), the authors have shown that the accuracy of the spectral approximation vitally depends on the choice of the division points $x_0 = c_0\varepsilon$ and $x_1 = 1-c_1\varepsilon$. The optimal choice was derived by the use of so-called resemblance function, determining numbers c_0 and c_1 in terms of degree n of the appropriate truncated Chebyshev series. It was proved that when ε is small enough

$$c_0=\sqrt{\frac{n(n-1)}{g(0)}}\quad\text{and}\quad c_1=\sqrt{\frac{n(n-1)}{g(1)}}.$$

It is necessary to perform the same procedure to evaluate the interior division points $x_l = d - c_l\varepsilon$ and $x_r = d + c_r\varepsilon$.

Definition 1 *The resemblance function for the point $x_r = d + c_r\varepsilon$ is a polynomial $q(x)$ of degree n such that*

a) $q(x_r) = 0$ is the minimum for $q(x)$ if $z_r(d^+) < z_l(d^-)$, and maximum if $z_r(d^+) > z_l(d^-)$

b) $q(x)$ is concave if $z_r(d^+) < z_l(d^-)$, and convex if $z_r(d^+) > z_l(d^-)$ for all $x \in (d, d + c_r\varepsilon)$

c) $q(d) = \frac{z_l(d^-) - z_r(d^+)}{2} = z^*$.

Verifying the conditions from Definition 1 it can be easily proved that the following lemma holds:

Lemma 1 *Polynomial*

$$q(x) = z^* \left(\frac{d + c_r\varepsilon - x}{c_r\varepsilon} \right)^n$$

is the resemblance function for the point $x_r = d + c_r\varepsilon$.

The division point is evaluated from the request that resemblance function has to satisfy the appropriate differential equation at the layer point.

Lemma 2 *For sufficiently small ε the number c_r which determines the division point $x_r = d + c_r\varepsilon$ is given by*

$$c_r = \sqrt{\frac{n(n-1)}{g(d)}}.$$

Proof: We introduce $q(x)$ into the differential equation (21) and ask that it is satisfied at the layer point $x = d$, which gives us

$$n(n-1) \cdot z^* - c_r^2 g(d) \cdot z^* = -c_r^2 \varepsilon^2 z_r''(d).$$

The positive solution of the above equation is

$$c_r = \sqrt{\frac{n(n-1)z^*}{g(d)z^* - \varepsilon^2 z_r''(d)}}.$$

If ε is sufficiently small, we can neglect the term $\varepsilon^2 z_r''(d)$, so we come to

$$c_r = \sqrt{\frac{n(n-1)}{g(d)}}.$$

Using the same procedure for the division point $x_l = d - c_l\varepsilon$ we obtain that

$$c_l = \sqrt{\frac{n(n-1)}{g(d)}}. \tag{26}$$

4 Spectral approximation

The procedure for boundary layer solutions $u_0(x)$ and $u_1(x)$ of the problems (19) and (25) was constructed in some earlier authors' papers (see e.g. [2]). The construction required that the boundary layer subintervals $[0, c_0\varepsilon]$ and $[1 - c_1\varepsilon, 1]$ were transformed into $[-1, 1]$, and after that the layer solutions were approximated by

$$\bar{u}_0(x) = \sum_{k=0}^{n}{}' \xi_k \, T_k\left(\tfrac{2x}{c_0\varepsilon} - 1\right) \tag{27}$$

and

$$\bar{u}_1(x) = \sum_{k=0}^{n}{}' \eta_k \, T_k\left(\tfrac{2(x-1)}{c_1\varepsilon} + 1\right). \tag{28}$$

The prime sign in $\sum'$ means that we divide coefficient for $k = 0$ by 2. For example, in formula (27), we take $\frac{\xi_0}{2}$ instead of ξ_0.

The coefficients ξ_k and η_k, $k = 0, \ldots, n$ were evaluated from the request that (27) and (28) satisfy the boundary conditions and the differential equations in (19) and (25) at Gauss-Lobatto collocation points.

The same technique must be used for interior layer solutions $u_l(x)$ and $u_r(x)$.

We introduce two stretching variables given by

$$x = \varphi(t) = \frac{c_l \varepsilon}{2}(t-1) + d, \tag{29}$$

(it transforms the interior layer subinterval $[d - c_l\varepsilon, d]$ into $[-1, 1]$) and

$$x = \psi(t) = \frac{c_r \varepsilon}{2}(t+1) + d \tag{30}$$

(which transforms the interior layer subinterval $[d, d + c_r\varepsilon]$ into $[-1, 1]$). After that we approximate the interior layer solutions that satisfy (20)-(24) by truncated Chebyshev series of degree n

$$\bar{u}_l(x) = \bar{u}_l\left(\tfrac{c_l\varepsilon}{2}(t-1) + d\right) = w_l(t) = {\sum_{k=0}^{n}}' \beta_k T_k(t) \tag{31}$$

and

$$\bar{u}_r(x) = \bar{u}_r\left(\tfrac{c_r\varepsilon}{2}(t+1) + d\right) = w_r(t) = {\sum_{k=0}^{n}}' \gamma_k T_k(t). \tag{32}$$

Theorem 3 *If we define matrices*

$$\mathbf{A} = \{a_{ik}\}_{(2n+2)\times(2n+2)}, \quad \mathbf{B} = \{b_i\}_{(2n+2)\times 1}$$

with

$$a_{0,k} = (-1)^k, \quad k = 0, \dots, n\ , \qquad a_{0,k} = 0, \quad k = n+1, \dots, 2n+1\ ,$$

$$a_{i,k} = -\tfrac{4}{c_l^2} T_k''(t_i) + g\left(\varphi(t_i)\right) T_k(t_i),\ k = 0, \dots, n, \quad a_{i,k} = 0,\ k = n+1, \dots, 2n+1,$$
$$i = 1, \dots, n-1$$

$$a_{n,k} = 1, \quad k = 0, \dots, n, \qquad a_{n.k} = (-1)^{k+1}, \quad k = n+1, \dots, 2n+1$$

$$a_{n+1,k} = k^2, \quad k = 0, \dots, n, \qquad a_{n.k} = (-1)^k (k-n-1)^2, \quad k = n+1, \dots, 2n+1$$

$$a_{n+i+1,k} = 0, \quad k = 0, \dots, n,$$

$$a_{n+i+1,k} = -\tfrac{4}{c_r^2} T_{k-n-1}''(t_i) + g\left(\psi(t_i)\right) T_{k-n-1}(t_i),\ k = n+1, \dots, 2n+1,$$
$$i = 1, \dots, n-1$$

$$a_{2n+1,k} = 0, \quad k = 0, \dots, n\ , \qquad a_{2n+1,k} = 1, \quad k = n+1, \dots, 2n+1\ ,$$

$$b_0 = 0, \quad b_i = \varepsilon^2 z_l''(\varphi(t_i)),\ i = 1, \dots, n-1, \quad b_n = z_r(d^+) - z_l(d^-)$$

$$b_{n+1} = z_r'(d^+) - z_l'(d^-), \quad b_{n+1+i} = \varepsilon^2 z_r''(\psi(t_i)),\ i = 1, \dots, n-1, \quad b_{2n+1} = 0$$

where $t_i = -\cos\frac{i\pi}{n}$ *are Gauss-Lobatto nodes, then the spectral coefficients* β_k *and* γ_k *represent the elements of the matrix* $\mathbf{X} = \{x_k\}_{(2n+2)\times 1}$,

$$x_k = \beta_k,\ k = 0, \dots n, \quad x_k = \gamma_{k-n-1},\ k = n+1, \dots, 2n+1$$

which is the solution of the matrix equation

$$\mathbf{A} \cdot \mathbf{X} = \mathbf{B}. \tag{33}$$

Proof: Introducing (31) and (32) into equations (22) and (23), with respect to $T_k(\pm 1) = (\pm 1)^k$, we obtain the first and the last row of the matrix equation

$$\sum_{k=0}^{n}{}'(-1)^k \beta_k = 0\ , \quad \sum_{k=0}^{n}{}'\gamma_k = 0$$

and the $n+1$-st row

$$\sum_{k=0}^{n}{}' \left(\beta_k - (-1)^k \gamma_k\right) = z_r(d^+) - z_l(d^-)\ .$$

Introducing (31) and (32) into (24), with respect to $T_k'(\pm 1) = (\pm 1)^k k^2$, we obtain the $(n+2)$-nd row of the matrix equation

$$\sum_{k=0}^{n}{}' k^2 \left(\beta_k - (-1)^{k+1} \gamma_k\right) = z_r'(d^+) - z_l'(d^-).$$

Introducing (31) and (32) into (20) and (21), using Gauss-Lobatto collocation points $t_i = \cos\frac{i\pi}{n}$, $i = 1, \dots, n-1$, we obtain the other $2n-2$ rows of the matrix equations

$$\sum_{k=0}^{n}{}' \left(-\tfrac{4}{c_l^2} T_k''(t_i) + a\left(\varphi(t_i)\right) T_k(t_i)\right) \beta_k = \varepsilon^2 z_l''\left(\varphi(t_i)\right),\ i = 1, \dots, n-1$$

and

$$\sum_{k=0}^{n}{}' \left(-\tfrac{4}{c_r^2} T_k''(t_i) + a\left(\psi(t_i)\right) T_k(t_i)\right) \gamma_k = \varepsilon^2 z_r''\left(\psi(t_i)\right),\ i = 1, \dots, n-1\ .$$

5 The error estimate and the rate of convergence

Using the proposed domain decomposition we can see that out of the boundary layers, i.e. for $x \in [x_0, x_l]$ and $x \in [x_r, x_1]$, the exact solution is approximated by the reduced solution $z(x)$. It is well known (see [1] and [5]) that the error of this approximation is given by

$$|u_\varepsilon(x) - z_l(x)| < C(\varepsilon^2 + e^{-\frac{Kx}{\varepsilon}} + e^{-\frac{K(d-x)}{\varepsilon}}) \tag{34}$$

and

$$|u_\varepsilon(x) - z_r(x)| < C(\varepsilon^2 + e^{-\frac{K(x-d)}{\varepsilon}} + e^{-\frac{K(1-x)}{\varepsilon}}) \tag{35}$$

where C is a constant independent of K and ε and $K > 0$ satisfies

$$g(x) \geq K^2 > 0, \qquad x \in [0, 1].$$

Inside the interior layer, for example for $x \in [x_l, d)$, we have that

$$|u_\varepsilon(x) - z_l(x) - \bar{u}_l(x)| \leq |u_\varepsilon(x) - z_l(x) - u_l(x)| + |u_l(x) - \bar{u}_l(x)|.$$

The first term represents the error caused by the boundary condition at the point $x = x_l$ and it can be estimated by the use of the minimum principle

$$|u_\varepsilon(x) - z_l(x) - u_l(x)| \leq |u_\varepsilon(x_l) - z(x_l)|.$$

According to (34), with respect to (26), this gives

$$|u_\varepsilon(x) - z_l(x) - u_l(x)| < C(\varepsilon^2 + e^{-\frac{K\sqrt{n(n-1)}}{\sqrt{g(d)}}}), \quad x \in [x_l, d). \tag{36}$$

The second term

$$|u_l(x) - \bar{u}_l(x)|$$

represents the error of the spectral approximation for the solution $u_l(x)$ of the problem (5),(6), i.e. (11),(12). This error can be estimated by:

Last coefficient error estimate: The error is the same order-of-magnitude as the last coefficient retained in the truncation for series with geometric convergence.

The convergence is geometric if the exponential index of convergence, defined by

$$r = \lim_{n\to\infty} \frac{\log|\log(|\beta_n|)|}{\log n}, \tag{37}$$

is equal to 1.

6 Numerical example

We have taken numerical example given in [1] to test our results

$$-\varepsilon^2 u_\varepsilon'' + u_\varepsilon = \begin{cases} 0.7 & x \in [0, 0.5) \\ -0.6 & x \in (0.5, 1] \end{cases}$$

$$u_\varepsilon(0) = 0 \ , \quad u_\varepsilon(1) = 0.$$

We have the interior layer at the point $x = 0.5$.

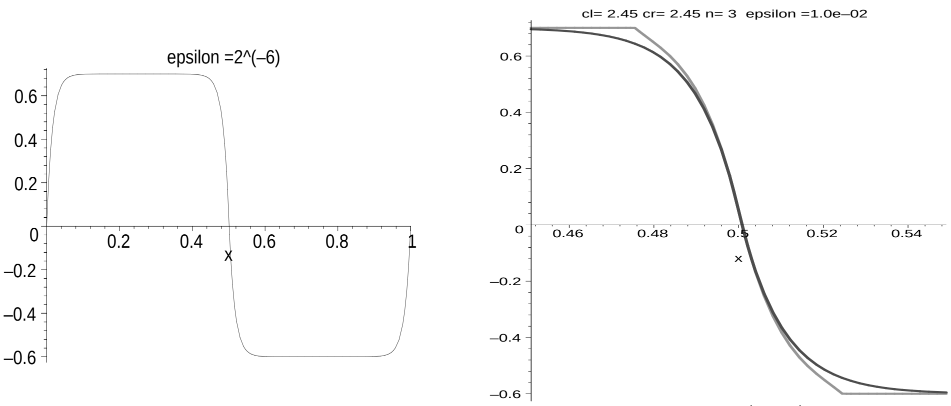

Figure 1: Asympthotic solution on the whole interval(left) and exact together with the approximate solution on middle subinterval (right)

For the construction of the asympthotic solution we used reduced solutions

$$z_l(x) = 0.7 \qquad \text{and} \qquad z_r(x) = -0.6$$

and functions

$$\phi_1(x) = \frac{e^{-\frac{x}{\varepsilon}} - e^{\frac{x-2}{\varepsilon}}}{1 - e^{-\frac{2}{\varepsilon}}}$$

and

$$\phi_2(x) = \frac{e^{\frac{x-1}{\varepsilon}} - e^{-\frac{x+1}{\varepsilon}}}{1 - e^{-\frac{2}{\varepsilon}}}.$$

For the construction of the pseudospectral approximation we used division points $x_l = 0.5 - \sqrt{n(n-1)}\varepsilon$ and $x_r = 0.5 + \sqrt{n(n-1)}\varepsilon$.

Figure 1 represents the graph of the asympthotic solution and the approximate solution together with the exact one constructed by the proposed procedure upon the interval $[0.5-2\sqrt{n(n-1)}\varepsilon, 0.5+2\sqrt{n(n-1)}\varepsilon]$, which includes interior layer. Quite modest values $\varepsilon = 10^{-2}$ and $n = 3$ were chosen in purpose to distinguish the approximate solution from the exact one.

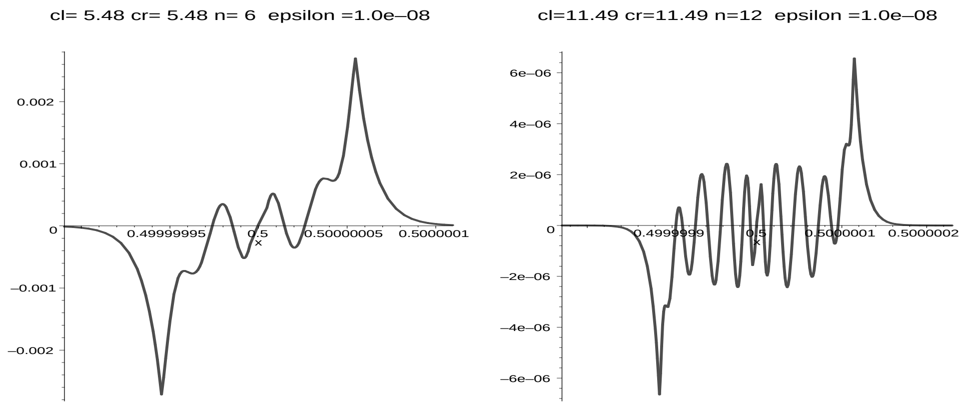

Figure 2: The exact error

Table 1: The numerical rate of exponential convergence

$\frac{\log\lvert\log(\lvert a_n\rvert)\rvert}{\log n}$	$n = 4$	$n = 8$	$n = 16$	$n = 32$
$\varepsilon = 2^{-2}$	0.3292	-.4482	0.2510	0.5534
$\varepsilon = 2^{-4}$	1.1975	0.8835	0.4875	-.0186
$\varepsilon = 2^{-8}$	1.2090	1.0834	1.0082	0.9806
$\varepsilon = 2^{-16}$	1.2090	1.0834	1.0082	0.9806

Figure 2 represents the difference between the exact solution and the approximate one upon the interval $[0.5-2\sqrt{n(n-1)}\varepsilon, 0.5+2\sqrt{n(n-1)}\varepsilon]$ for $\varepsilon = 10^{-8}$ when $n = 6$ and $n = 12$. It shows a high accuracy of the presented method. The maximal error is achieved at the division points x_l and x_r due to the error caused by the boundary conditions (22).

In Table 1 we give numerical rate of exponential convergence for several values of ε with n up to $n = 64$, which shows geometric convergence of spectral approximation.

Table 2: The maximal error inside the layer and the last coefficient

ε	$n=4$		$n=8$		$n=16$		$n=32$	
	$\lvert\beta_n\rvert$	$d(x)$	$\lvert\beta_n\rvert$	$d(x)$	$\lvert\beta_n\rvert$	$d(x)$	$\lvert\beta_n\rvert$	$d(x)$
2^{-2}	5.9e-03	2.1e-01	1.7e-04	6.7e-01	1.6e-07	7.4e+00	2.1e-13	9.0e+02
2^{-4}	5.9e-03	5.2e-03	1.7e-04	1.9e-03	1.6e-07	2.1e-02	2.1e-13	2.6e+00
2^{-8}	5.9e-03	4.8e-03	1.7e-04	7.4e-05	1.6e-07	7.8e-08	2.1e-13	1.0e-13
2^{-16}	5.9e-03	4.8e-03	1.7e-04	7.4e-05	1.6e-07	7.8e-08	2.1e-13	1.0e-13

In Table 2 we give the exact error $d(x)$ and the last retained coefficient β_n of the approximate solution. It shows that the error is the same order of magnitude as the last retained Chebyshev coefficient.

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Približno rešenje za SPP sa prekidnim slobodnim članom

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Posmatra se samoadjungovani singularno perturbovani problem opisan običnom diferencijalnom jednačinom drugog reda sa malim parametrom uz najviši izvod i prekidnim slobodnim članom uz dodatak odgovarajućih konturnih uslova, koji opisuje stabilno stanje odredjenih problema protoka. Rešenje ima kako granične, tako i unutrašnji sloj.

Izvršena je dekompozicija domena u kojoj su slojni intervali prilagodjeni polinomnoj aproksimaciji. Deobene tačke za unutršnji sloj odredjene su procedurom analognom onoj za granične slojeve uz korišćenje odgovarajuće funkcije sličnosti.

Rešenje van slojeva aproksimirano je rešenjem redukovanog problema, a slojna rešenja konačnim ortogonalnim redovima, što daje glatku aproksimaciju nad čitavim intervalom. Koeficijenti konačnih redova izračunati su pseudo-spektralnom tehnikom.

Ispitana je brzina konvergencije i dat je red veličine greške korišćenjem principa inverzne monotonije i ponašanjem pseudo-spektralne aproksimacije.

Uključen je numerički primer koji pokazuje visoku tačnost predloženog metoda.