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Orthogonal series approximation for boundary layers

Nevenka Adžić * **Z. Ovcin** †

Abstract

In this paper we shall consider a class of singularly perturbed problems described by the ordinary differential equation of second order with small parameter multiplying the highest derivative and the appropriate boundary conditions, which describes certain flow problems in fluid mechanics.

The solution of such problems displays boundary layers where the solution changes its values very rapidly.

The domain decomposition will be performed determining layer subintervals which are adapted to the possibility of spectral approximation. The division point for the boundary layer is determined using appropriate resemblance function, so that the length of the layer subinterval varies with the degree of the truncated orthogonal series.

The solution out of boundary layer is aproximated by the solution of the reduced problem, and the layer solutions is aproximated by truncated orthogonal series giving a smooth approximate solution upon the entire interval. The coefficients of the truncated series are evaluated using collocation technique.

The order-of-magnitude of the error is estimated using the principle of inverse monotonicity and the rate of convergence for spectral approximations.

Keywords: singular perturbation, pseudospectral approximation, Chebyshev polynomials, domain decomposition.

*Faculty of Engineering, University of Novi Sad, 21000 Novi Sad, Serbia, e-mail: nena@uns.ns.ac.yu

†Faculty of Engineering, University of Novi Sad, 21000 Novi Sad, Serbia

1 Introduction

We shall consider a self-adjoint boundary layer problem described by

$$L_\varepsilon y \equiv -\varepsilon^2 y''(x) + g(x)y(x) = h(x), \quad 0 \le x \le 1, \tag{1}$$

$$y(0) = A, \quad y(1) = B, \tag{2}$$

where function $g(x)$ satisfies the condition

$$g(x) \ge K^2 > 0, \quad K \in \mathbf{R}. \tag{3}$$

It is known that under the assumption (3) the self-adjoint problem has the unique solution $y(x) \in \mathbf{C}^2[0,1]$. In general, the solution displays two boundary layers of order $\mathcal{O}(\varepsilon)$. If we denote by $z(x)$ the solution of the reduced problem

$$g(x)z(x) = h(x), \quad 0 \le x \le 1,$$

then, if $z(0) \ne A$, the boundary layer occurs at $x = 0$, and if $z(1) \ne B$, we have the boundary layer at the endpoint $x = 1$.

The problem (1), (2) represents mathematical model of the large number of phenomena in sciences such as conduction and diffusion in fluid dynamics, theory of semiconductors and catalysis processes in chemistry and biology. It is of the great interest to describe the behavior of the exact solution of these problems, especially inside the layers.

Standard numerical methods give very poor results, so lately various special procedures, such as special grids, introduction of relaxation parameters and special discretizations, were developed.

Standard spectral approximation also fails when applied to singularly perturbed problems. In several papers (see e.g. [1]) the authors have developed the modification of standard spectral approximation, which assumes the division of the basic interval $[0,1]$ by the special procedure, in such a way that the length of the layer intervals is adapted to the application of truncated orthogonal series which is used to approximate the exact solution inside the layers. Out of the layers the reduced solution $z(x) = \frac{h(x)}{g(x)}$ was used to approximate the exact solution. This procedure has given very accurate results for small values of the perturbation parameter, even when low degree orthogonal polynomials were used.

The error of that approximation depends on the error at the division point, which can be easily estimated, and the error inside the layer which can't be estimated exactly. In this paper the numerical rate of convergence is used to enable us to estimate the accuracy of the spectral approximation of the layer solution.

In Section 2 the problem (1), (2) will be transformed in such a way that we can approximate layer solutions by the truncated Chebyshev series.

In Section 3 the procedure for the appropriate domain decomposition will be described.

In Section 4 the spectral approximation for the layer solution and the appropriate matrix equation will be constructed.

In Section 5 the error estimate for the layer solution will be discussed introducing the rate of exponential convergence for spectral approximation.

In Section 6 the rate of convergence for several well known numerical examples will be evaluated and the proposed error estimate compared to the exact error.

2 Transformation of the problem

Let us approximate the exact solution by

$$u(x) = \begin{cases} u_l(x) & 0 \le x < x_0 \\ z(x) & x_0 \le x \le 1 - x_1 \\ u_r(x) & 1 - x_1 < x \le 1, \end{cases} \tag{4}$$

where $u_l(x)$ and $u_r(x)$ are the left and right layer solutions. They are determined by the boundary value problems

$$L_\varepsilon u_l \equiv -\varepsilon^2 u_l''(x) + g(x) u_l(x) = h(x), \quad 0 \le x \le x_0, \tag{5}$$

$$u_l(0) = A, \quad u_l(x_0) = z(x_0) \tag{6}$$

and

$$L_\varepsilon u_r \equiv -\varepsilon^2 u_r''(x) + g(x) u_r(x) = h(x), \quad 1 - x_1 \le x \le 1, \tag{7}$$

$$u_r(1-x_1) = z(1-x_1), \quad u_r(1) = B. \tag{8}$$

Here $x_0, x_1 \in (0, \frac{1}{2})$ denote values that are going to be determined in the next section in such a way that layer solutions can be approximated in the best possible way by the truncated Chebyshev series.

3 Domain decomposition

The idea is to perform the domain decomposition, using the division points x_0 and $1-x_1$, in such a way that it provides the possibility of high accurate spectral approximation of the layer solutions $u_l(x)$ and $u_r(x)$.

Numerical examples show that a very small change of values x_0 and x_1 may cause that the error of the spectral approximation increases hundred times or even more.

The authors have developed a special procedure which enables to determine division points in terms of the degree of the truncated Chebyshev series which approximates layer solution. Here, we shall carry out this procedure for the left layer solution.

As the layer length is of order $\mathcal{O}(\varepsilon)$, we shall represent the division point x_0 in the form $x_0 = c\varepsilon$. The procedure bases on the introduction of the *resemblance function* for the layer solution $u_l(x)$.

Definition 1 *The resemblance function for the left layer solution of the problem (1),(2) upon the interval* $[0, x_0]$ *is the polynomial* $p_n(x)$ *of degree* $n \geq 2$*, such that*

1. $p_n(0) = A$ *and* $p_n(x_0) = z(0)$*,*
2. x_0 *is the stationary point for* $p_n(x)$*,*
3. $\text{sgn} p_n(x) > 0$ *if* $A > z(0)$ *and* $\text{sgn} p_n(x) < 0$ *if* $A < z(0)$*.*

Items 1.–3. provide that the resemblance function $p_n(x)$ upon the interval $[0, x_0]$ behaves very closely to the exact solution $y(x)$.

Lemma 1 *The polynomial*

$$p_n(x) = z(0) + (A - z(0)) \left(1 - \frac{x}{x_0}\right)^n, \quad n \geq 2, \tag{9}$$

represents the resemblance function for the left layer solution of the problem (1),(2) upon the interval $[0, x_0]$.

Proof: We have to verify the conditions from Definition 1.

1.
$$p_n(0) = z(0) + (A - z(0)) \left(1 - \frac{0}{x_0}\right)^n = A$$
and
$$p_n(x_0) = z(0) + (A - z(0)) \left(1 - \frac{x_0}{x_0}\right)^n = z(0).$$

2.
$$p_n'(x) = -\frac{n(A - z(0))}{x_0} \left(1 - \frac{x}{x_0}\right)^{n-1}.$$
When the left layer exists, $A - z(0) \neq 0$ and we have that $p_n'(x) = 0$ only for $x = x_0$, so x_0 is the stationary point.

3.
$$p_n''(x) = \frac{n(n-1)(A - z(0))}{x_0^2} \left(1 - \frac{x}{x_0}\right)^{n-2}.$$
For $x \in [0, x_0]$ it is obvious that
$$\mathrm{sgn} p_n''(x) = \mathrm{sgn}(A - z(0)).$$

In order to determine the division point x_0 we shall ask that the resemblance function has to satisfy the differential equation (5) at the layer point $x = 0$. This will give us

Lemma 2 *The value* c *that determines the division point* $x_0 = c\varepsilon$ *is given by*
$$c = \sqrt{\frac{n(n-1)}{g(0)}}. \tag{10}$$

Proof: If we introduce (9) into the differential equation (5), at the point $x = 0$ we obtain
$$-\varepsilon^2 \frac{n(n-1)}{c^2\varepsilon^2}(A - z(0)) + g(0)A = h(0).$$

With respect to $h(0) = g(0)z(0)$, the solution of the above equation for c, $c > 0$, will give us (10).

The existence of the square root in (10) is provided by the assumption (3).

4 Spectral approximation for the layer solution

In this section we shall carry out the construction of the spectral approximation for the left layer solution. The procedure for the right one is the same.

Once the division point x_0 is determined, we introduce the stretching variable

$$t = \frac{2x}{x_0} - 1,$$

which maps the layer subinterval $[0, x_0]$ into $[-1, 1]$.

Thus, transforming (5),(6) we come to the problem

$$L_l w \equiv -4w''(t) + c^2 G_l(t)w(t) = c^2 H_l(t), \quad -1 \leq t \leq 1, \tag{11}$$

$$w(-1) = A, \quad w(1) = z(x_0), \tag{12}$$

where we have used the notation

$$G_l(t) = g\left(\tfrac{x_0(t+1)}{2}\right), \quad H_l(t) = h\left(\tfrac{x_0(t+1)}{2}\right).$$

Now, the obtained problem can be solved by the use of standard spectral technique, approximating $w(t)$ by the truncated Chebyshev series

$$w_n(t) = {\sum_{k=0}^{n}}' a_k T_k(t). \tag{13}$$

The sign $'$ in $\sum'$, formula (13), means that we take $\frac{a_0}{2}$ instead of a_0 for $k = 0$.

The most common technique for evaluating the coefficients a_k, $k = 0, \ldots, n$, is collocation. Applying this method we come to the following theorem:

Theorem 1 *Let us define matrices*

$$\mathbf{L} = \{\ell_{ik}\}_{(n+1)\times(n+1)}, \quad \mathbf{H} = \{h_i\}_{(n+1)\times 1}$$

with

$$\ell_{0k} = (-1)^k, \quad \ell_{nk} = 1, \qquad k = 0, \ldots, n\ ,$$

$$\ell_{ik} = L_l T_k|_{t=t_i}, \quad h_i = c^2 H_l(t_i), \qquad i = 1, \ldots, n-1,$$

where $t_i = -\cos\frac{i\pi}{n}$ *are Gauss-Lobatto nodes and*

$$h_0 = A, \qquad h_n = z(x_0),$$

then the matrix $\mathbf{a} = \{a_k\}_{(n+1)\times 1}$*, where* a_k *are Chebyshev coeffients of the truncated series (13), is the solution of the matrix equation*

$$\mathbf{L} \cdot \mathbf{a} = \mathbf{H}. \tag{14}$$

Proof: If we introduce the truncated Chebyshev series $w_n(t)$ defined in (13) into the differential equation (11) and ask that it is satisfied at Gauss-Lobatto nodes, we come to

$$\sum_{k=0}^{n}{}' \left(-4T_k''(t_i) + c^2 G_l(t_i) T_k(t_i)\right) a_k = c^2 H_l(t_i), \quad i = 1, \ldots, n-1, \tag{15}$$

which represents the central $n-1$ rows of the matrix equation (14). Introducing $w_n(t)$ into boundary conditions (12), according to $T_k(\pm 1) = (\pm 1)^k$, we obtain two more equations

$$\sum_{k=0}^{n}{}'(-1)^k a_k = A, \quad \sum_{k=0}^{n}{}' a_k = z(x_0), \tag{16}$$

which represent the first and the last row of the matrix equation (14).

After solving matrix equation (14) for the unknown spectral coefficients a_k, $k = 0, \ldots, n$, we can represent the spectral approximation of the left layer solution upon subinterval $[0, x_0]$ as

$$v_n(x) = \sum_{k=0}^{n}{}' a_k T_k\left(\tfrac{2x}{x_0} - 1\right). \tag{17}$$

5 The error estimate and the rate of convergence

Using the proposed domain decomposition we can see that out of the boundary layers, i.e. for $x \in [x_0, 1 - x_1]$, the exact solution is approximated by the reduced solution $z(x)$. It is well known (see [5]) that the error of this approximation is given by

$$|y(x) - z(x)| < C(\varepsilon^2 + e^{-\frac{Kx}{\varepsilon}} + e^{-\frac{K(1-x)}{\varepsilon}}), \tag{18}$$

where C is a constant independent of K and ε, $K > 0$ is given by (3).

Inside the boundary layer, for $x \in [0, x_0]$, we have that

$$|y(x) - v_n(x)| \leq |y(x) - u_l(x)| + |u_l(x) - v_n(x)|.$$

The first term represents the error caused by the boundary condition at the point $x = x_0$ and it can be estimated by the use of the principle of inverse monotonicity. As

$$L_\varepsilon(\pm(y - u_l)) = 0,\ y(0) - u_l(0) = 0,\ \pm(y(x_0) - u_l(x_0)) \leq |y(x_0) - z(x_0)|,$$

we easily come to the conclusion that for all $x \in [0, x_0]$

$$\pm(y(x) - u_l(x)) \leq |y(x_0) - z(x_0)|.$$

According to (18), with respect to (10), this gives

$$|y(x) - u_l(x)| < C(\varepsilon^2 + e^{-\frac{K\sqrt{n(n-1)}}{\sqrt{g(0)}}}), \quad x \in [0, x_0]. \tag{19}$$

The second term

$$|u_l(x) - v_n(x)| = |w(t) - w_n(t)|$$

represents the error of the spectral approximation for the solution $u_l(x)$ of the problem (5),(6), i.e. (11),(12). This error is generated by

truncation error – defined as the error made by neglecting all spectral coefficients a_k, $k > n$, for the exact infinite Chebyshev series which represents $w(t)$, discretization error – which represents the difference between the first n terms of the exact infinite Chebyshev series and corresponding terms of the truncated Chebyshev series $w_n(t)$,

interpolation error – the error made by approximating $w(t)$ by truncated Chebyshev series whose coefficients are chosen to make the approximate agree with the solution $w(t)$ of the differential equation (11) at each of $n-1$ collocation points.

In order to estimate this error we shall apply two Rules-of-Thumb stated by Boyd in [3]

Assumption of equal errors: The discretization and interpolation errors are of the same order-of-magnitude as the truncation error.

Last coefficient error estimate: The truncation error is the same order-of-magnitude as the last coefficient retained in the truncation for series with geometric convergence.

The rate of convergence is determined using the exponential index of convergence

$$r = \lim_{n\to\infty} \frac{\log|\log(|a_n|)|}{\log n}. \tag{20}$$

The convergence is said to be geometric if $r = 1$.

6 Numerical examples

Example 1

We shall consider the boundary layer problem given in [4]

$$-\varepsilon^2 y''(x) + y(x) = -\cos^2 \pi x - 2(\varepsilon\pi)^2 \cos 2\pi x, \quad 0 \le x \le 1,$$

$$y(0) = 0, \quad y(1) = 0.$$

The reduced solution is $z(x) = -\cos^2 \pi x$, so we have boundary layers at both endpoints. The results are given only for the left layer, and for the right layer they are identical.

The layer subinterval, according to (10), is $[0, x_0] = [0, \sqrt{n(n-1)}\varepsilon]$.

Table 1: The numerical rate of exponential convergence

$\frac{\log\lvert\log(\lvert a_n\rvert)\rvert}{\log n}$	$n=8$	$n=16$	$n=32$	$n=64$
$\varepsilon=2^{-8}$	0.9958	0.7554	0.6044	0.5036
$\varepsilon=2^{-16}$	1.0561	0.9980	0.8526	0.7105
$\varepsilon=2^{-32}$	1.0561	0.9986	0.9763	0.8951
$\varepsilon=2^{-64}$	1.0561	0.9986	0.9763	0.9688

In Table 1 we give numerical rate of exponential convergence for several values of ε with n up to $n=64$, which shows almost geometric convergence of spectral approximation for ε sufficiently small.

Figure 1 presents the exact error $d(x)$ for $\varepsilon=2^{-16}$ and $n=8$, $d(x)$ is defined by $d(x) := \begin{cases} y(x)-v_n(x) & x\in[0,x_0) \\ y(x)-z(x) & x\in[x_0,2x_0] \end{cases}$. We can see that

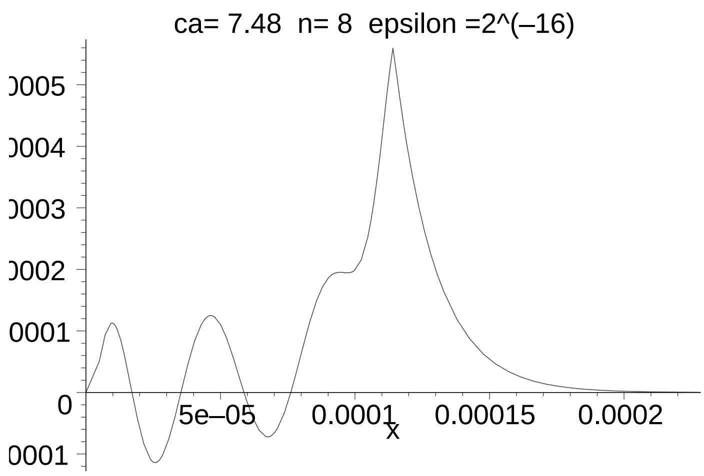

Figure 1: The exact error upon the interval $[0,2x_0]$ for $\varepsilon=2^{-16}$, $n=8$

the maximal error is achieved at the division point $x=x_0$, due to the difference between the value of the exact solution $y(x_0)$ and the value of the reduced solution $z(x_0)$ which we use for the boundary condition in the construction of the approximate solution.

In Table 2 we give maximal error $d(x)$ inside the layer subinterval (apart from the division point $x = x_0$) and the value of the last retained coefficient in the truncated Chebyshev series for several values of ε and n. It is easily seen that the order of magnitude of the maximal error is

Table 2: The maximal error inside the layer and the last coefficient

ε	$n = 8$		$n = 16$		$n = 32$		$n = 64$	
	$\|a_n\|$	$d(x)$	$\|a_n\|$	$d(x)$	$\|a_n\|$	$d(x)$	$\|a_n\|$	$d(x)$
2^{-8}	2.6e-04	3.6e-04	2.5e-07	3.0e-04	3.3e-13	3.0e-04	7.8e-25	3.0e-04
2^{-16}	2.6e-04	1.2e-04	2.5e-07	1.2e-07	3.3e-13	4.6e-09	7.8e-25	4.6e-09
2^{-32}	2.6e-04	1.2e-04	2.5e-07	1.2e-07	3.3e-13	1.6e-13	7.8e-25	1.1e-18
2^{-64}	2.6e-04	1.2e-04	2.5e-07	1.2e-07	3.3e-13	1.6e-13	7.8e-25	3.8e-25

the same as that of the last coefficient in the series.

Example 2

We shall consider the boundary layer problem given in [2]

$$-\varepsilon^2 y''(x) + \frac{1-\varepsilon}{(2-x)^2} y(x) = \frac{(1-\varepsilon)(x-1)}{(2-x)^2}, \; 0 \le x \le 1, \; y(0) = 0, \; y(1) = 0.$$

The reduced solution is $z(x) = x - 1$, so we have only the left boundary layer. The layer subinterval is $[0, x_0] = [0, 2\sqrt{n(n-1)}\varepsilon]$.

In Table 3 we give numerical rate of exponential convergence for several values of ε with n up to $n = 64$, which shows geometric convergence of spectral approximation. In Table 4 we give maximal errors inside the layer subinterval (apart from the division point $x = x_0$) and the values of the last retained coefficient in the truncated Chebyshev series for several values of ε and n.

Table 3: The numerical rate of exponential convergence

$\frac{\log\lvert\log(\lvert a_n\rvert)\rvert}{\log n}$	$n = 8$	$n = 16$	$n = 32$	$n = 64$
$\varepsilon = 2^{-8}$	1.0587	1.0029	0.9839	0.9832
$\varepsilon = 2^{-16}$	1.0561	0.9986	0.9763	0.9689
$\varepsilon = 2^{-32}$	1.0561	0.9986	0.9763	0.9369
$\varepsilon = 2^{-64}$	1.0561	0.9986	0.9763	0.9688

Table 4: The maximal error inside the layer and the value of the last coefficient

ε	$n = 8$		$n = 16$		$n = 32$		$n = 64$	
	$\lvert a_n\rvert$	$d(x)$	$\lvert a_n\rvert$	$d(x)$	$\lvert a_n\rvert$	$d(x)$	$\lvert a_n\rvert$	$d(x)$
2^{-8}	2.5e-04	1.2e-04	2.1e-07	9.9e-08	1.6e-13	7.2e-14	2.9e-26	1.2e-26
2^{-16}	2.6e-04	1.2e-04	2.5e-07	1.2e-07	3.3e-13	1.6e-13	7.7e-25	3.8e-25
2^{-32}	2.6e-04	1.2e-04	2.5e-07	1.2e-07	3.3e-13	1.6e-13	7.8e-25	4.2e-22
2^{-64}	2.6e-04	1.2e-04	2.5e-07	1.2e-07	3.3e-13	1.6e-13	7.8e-25	3.8e-25

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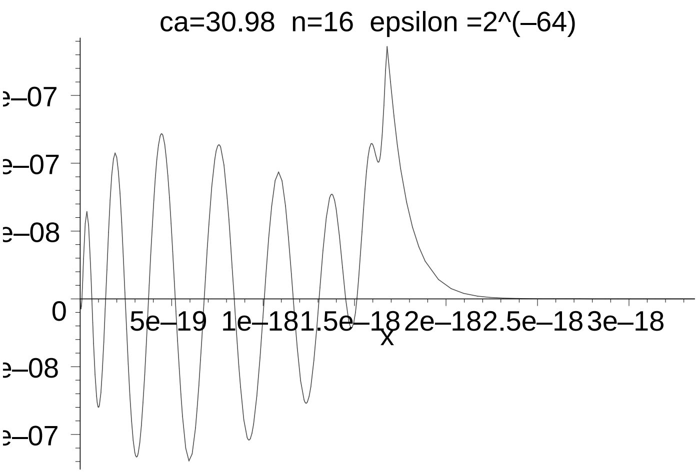

Figure 2: The exact error upon the interval $[0, 2x_0]$ for $\varepsilon = 2^{-64}$ and $n = 16$

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Aproksimacija graničnih slojeva ortogonalnim Čebisevljevim redom

UDK 517.955

Posmatra se jedna klasa singularno poremećenih problema opisanih običnom diferencijalnom jednačinom drugog reda sa malim parametrom uz drugi izvod i odgovarajućim konturnim uslovima, koji opisuju probleme protoka u mehanici fluida. Reenja ovakvih problema ispoljavaju granične slojeve u kojima rešenje naglo menja svoje vrednosti. Izvrena je dekompozicija domena u kojoj se odredjuju slojni podintervali koji su prilagodjeni primeni spektralne aproksimacije. Deobena tačka za slojni interval odredjuje se pomoću funkcije sličnosti tako da dužina slojnog intervala zavisi od stepena konačnog ortogonalnog reda. Rešenje van sloja aproksimirano je rešenjem redukovanog problema, a rešenje unutar sloja konačnim ortogonalnim redom što daje glatku aproksimaciju nad celim intervalom. Koeficijenti konačnog reda izračunati su primenom kolokacione tehnike. Red veličine greške je ocenjen na osnovu principa inverzne monotonije i reda konvergencije spektralne aproksimacije.