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On the lateral vibrations of an elastic rod with varying compressive force

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Abstract

We study lateral vibration of a simply supported axially compressed elastic rod with rotary inertia. The axial force is assumed to be a function of time. Stability of the solution is examined. The conditions are examined under which the temporal evolution of the system is a slowly varying and regularly slowly varying function. Also some properties of the generalized solution to the problem are examined. Our results on stability boundary obtained by dynamic, method agree with the results obtained by static (Euler) method.

Keywords: *elastic rod, lateral vibration*

1 Formulation of the problem

Consider a rod BC of length l simply supported at both ends. The support at the end B is unmovable while at the end C the rod has movable joint. The axis of the rod is initially straight. At the end C the rod is loaded by a compressive force F that is a function of time t and whose action line coincides with the rod axis in the undeformed (initial

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state). Let $\bar{x} - B - \bar{y}$ be a rectangular Cartesian coordinate system with axis $\bar{x}$ oriented along the rod axis in the initial state (see figure 1).

In what follows we consider only the plane motion (in the plane $\bar{x} - B - \bar{y}$) so that (see [1]) the use D'Alembert's principle for an element of length dS leads to

$$\frac{\partial H}{\partial S} = -q_x, \qquad \frac{\partial V}{\partial S} = -q_y, \qquad \frac{\partial M}{\partial S} = -V\frac{\partial x}{\partial S} + H\frac{\partial y}{\partial S} - m, \quad (1.1)$$

where $S \in [0, l]$ is the arc–length of the rod axis, x and y are coordinates of an arbitrary point on the rod axis in the deformed state along the $\bar{x}$ and $\bar{y}$ axis, respectively, H and V are components of the contact force (representing the influence of the part of the rod $[0, S)$ on the part $[S, l])$ along the $\bar{x}$ and $\bar{y}$ axis respectively, and q_x, q_y and m are the intensities of the distributed forces per unit length of the rod axis along the $\bar{x}$ and $\bar{y}$ axis and the intensity of the distributed couple, respectively.

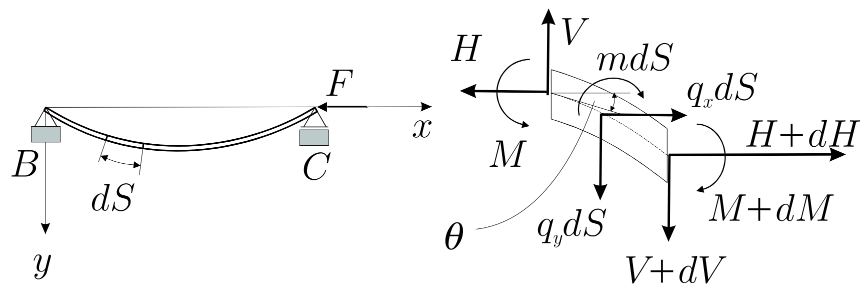

Figure 1: Coordinate system and load configuration

In figure 1 we showed positive directions of cross–sectional quantities. Also in our case the only distributed loads are inertial loads, so that

$$q_x = -\rho\frac{\partial^2 x}{\partial t^2}; \qquad q_y = -\rho\frac{\partial^2 y}{\partial t^2}; \qquad m = -J\frac{\partial^2 \theta}{\partial t^2}, \qquad (1.2)$$

where ρ is the (line) density of the rod, J is its moment of inertia of a part of the rod of unith length and t is the time. In what follows we shall retain the term $J\dfrac{\partial^2 \theta}{\partial t^2}$ representing the rotary inertia of the rod cross–section. By using (1.1) and (1.2) we obtain

$$\begin{aligned} \frac{\partial H}{\partial S} &= \rho\frac{\partial^2 x}{\partial t^2}, \qquad \frac{\partial V}{\partial S} = \rho\frac{\partial^2 y}{\partial t^2}, \\ \frac{\partial M}{\partial S} &= -V\cos\theta + H\sin\theta + J\frac{\partial^2\theta}{\partial t^2}, \\ \frac{\partial x}{\partial S} &= \cos\theta, \qquad \frac{\partial y}{\partial S} = \sin\theta, \qquad \frac{1}{r} = \frac{d\theta}{dS}, \end{aligned} \tag{1.3}$$

where θ is the angle between the tangent to the rod axis and $\bar{x}$ axis and r is the radius of curvature of the rod axis.

To the system (1.3) we adjoin the constitutive equation connecting the curvature of the rod axis $(1/r)$ with the bending moment M. We take the classical Bernoulli–Euler rod theory for which

$$M = EI\frac{1}{r}, \tag{1.4}$$

where $EI = const.$ is the bending rigidity of the rod. The boundary conditions corresponding to the rod shown in figure 1, are

$$\begin{aligned} H(l,t) &= -F(t), \qquad M(0,t) = 0, \qquad M(l,t) = 0, \\ x(0,t) &= 0; \qquad y(0,t) = 0; \qquad y(l,t) = 0. \end{aligned} \tag{1.5}$$

The trivial solution to system (1.3), (1.5) in which the axis of the rod remains straight, reads

$$\begin{aligned} &H^0 = -F, \qquad V^0 = 0, \qquad M^0 = 0, \\ &x^0(S,t) = S, \qquad y^0(S,t) = 0, \qquad \theta^0(S,t) = 0. \end{aligned} \tag{1.6}$$

Let $H = H^0 + \Delta H, ...\theta = \theta^0 + \Delta\theta$. By substituting this in (1.3), (1.5) neglecting the higher order terms in the perturbations $\Delta H, ...\Delta\theta$, we obtain

$$EI\frac{\partial^4 \Delta y}{\partial S^4} = -\rho\frac{\partial^2 \Delta y}{\partial t^2} - F(t)\frac{\partial^2 \Delta y}{\partial S^2} + J\frac{\partial^4 \Delta y}{\partial^2 t \partial^2 S}. \tag{1.7}$$

where we used $\dfrac{\partial^2 \Delta\theta}{\partial t^2} = \dfrac{\partial^2}{\partial t^2}\left(\dfrac{\partial \Delta y}{\partial S}\right)$ and $\dfrac{1}{\Delta r} = \dfrac{\partial^2 \Delta y}{\partial S^2}$ (see $(1.3)_5$). The boundary conditions corresponding to (1.7) are

$$\begin{aligned} &\frac{\partial^2 \Delta y}{\partial S^2}(0,t) = 0, \qquad &&\frac{\partial^2 \Delta y}{\partial S^2}(l,t) = 0, \\ &\Delta y(0,t) = 0, &&\Delta y(l,t) = 0. \end{aligned} \tag{1.8}$$

To system (1.7), (1.8) certain initial conditions must be prescribed and we shall do it later. By introducing the dimensionless quantities

$$\begin{aligned} &u = \frac{\Delta y}{l}, \quad m = \frac{Ml}{E_0 I}, \quad \tau = t\sqrt{\frac{EI}{\rho l^4}}, \\ &\lambda = \frac{Fl^2}{EI}, \quad \xi = \frac{S}{l}, \qquad \alpha = \frac{J}{\rho l^2}, \end{aligned} \tag{1.9}$$

from (1.7), (1.8) we obtain

$$\frac{\partial^4 u}{\partial \xi^4} + \lambda \frac{\partial^2 u}{\partial \xi^2} - \alpha \frac{\partial^4 u}{\partial \xi^2 \partial \tau^2} + \frac{\partial^2 u}{\partial \tau^2} = 0, \quad 0 < \xi < 1, \ \ \tau > 0, \tag{1.10}$$

and

$$\frac{\partial^2 u}{\partial \xi^2}(0,t) = 0, \qquad \frac{\partial^2 u}{\partial \xi^2}(1,t) = 0, \qquad u(0,t) = 0, \qquad u(1,t) = 0. \tag{1.11}$$

2 Solutions to equations (1.10) & (1.11)

2.1 Some mathematical tools

2.1.1 Karamata's class of functions

In studying equation (1) we use Karamata's class (cf. [5], [7]) of regularly varying functions and a natural extension of them, the class of rapidly varying functions (cf. [6]). We cite theirs definitions.

Definition 1 *A positive measurable function ρ defined on some neighbourhood $[a,\infty)$ of infinity is called **regularly varying** at infinity of index α if for each $\lambda > 0$ and some $\alpha \in \mathbb{R}$*

$$\lim_{x\to\infty} \rho(\lambda x)/\rho(x) = \lambda^{\alpha}.$$

*If $\alpha = 0$, then ρ is called **slowly varying**. We denote it by an L.*

It is well-known that the function L is slowly varying at infinity if and only if it may be written in the form

$$L(x) = c(x)\exp\{\int_a^x (\varepsilon(t)/t)dt, \quad x \geq a$$

for some $a > 0$, where ε and c are measurable and for $x \to \infty$, $\varepsilon(x) \to 0$ and $c(x) \to c \in (0,\infty)$. If $c(x) \equiv c$, L is called *normalized.* A regularly varying function ρ is of the form $\rho(x) = x^{\alpha}L(x)$, $\alpha \in \mathbb{R}$.

Definition 2 *A positive measurable function ρ defined on $[a,\infty)$ is called **rapidly varying at infinity of index** ∞ if for $x \to \infty$*

$$\rho(\lambda x)/\rho(x) = \begin{cases} \infty, & \text{for } \lambda > 1 \\ 0, & \text{for } 0 < \lambda < 1 \end{cases}$$

*and is called **rapidly varying at infinity of index** $-\infty$ if for $x \to \infty$*

$$\rho(\lambda x)/\rho(x) = \begin{cases} 0, & \text{for } \lambda > 1 \\ \infty, & \text{for } 0 < \lambda < 1. \end{cases}$$

*Together they are called **rapidly varying at infinity**.*

2.1.2 The space of distributions

Let Ω denote an open subset of $\mathbb{R}^2$. The *support* of a function φ defined on Ω is the closure in Ω of the set $\{x \in \Omega; \varphi(x) \neq 0\}$. The space $\mathcal{D}(\Omega)$ is the space $\{\varphi \in \mathcal{C}^{\infty}(\mathbb{R}^2);\ \mathrm{supp}\varphi \subset \Omega\}$. A sequence $\{\varphi_j\} \subset \mathcal{D}(\Omega)$ converges in $\mathcal{D}(\Omega)$ to zero if and only if there exists the compact set $K \subset \Omega$:

1. supp$\varphi_j \subset K$, $j \in \mathbb{N}$;

2. for every $\alpha \in (N \cup \{0\})^2 \equiv \mathbb{N}_0^2$, $\varphi_j^\alpha \to 0$ uniformly on K.

$\mathcal{D}'(\Omega)$ is the space of all continuous linear functionals on $\mathcal{D}(\Omega)$. It is called *the space of distributions* on Ω. Every locally integrable function f on Ω defines the *regular distribution* denoted by $[f]$.

Every distribution has all derivatives and $D^\alpha D^\beta = D^\beta D^\alpha$. If a function f has $f^{(\alpha)} \in L_{loc}(\Omega)$, then $D^\alpha[f] = [f^{(\alpha)}]$. A locally integrable function on Ω has $g \in \mathcal{D}'(\Omega)$ as *α-derivative in the sense of distributions* means that the regular distribution $[f]$ has derivative equal g, $D^\alpha[f] = g$.

The differentiation of distributions is a linear and continuous mapping $\mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$. If $\{f_j\} \subset \mathcal{D}'(\Omega)$ is a sequence of solutions to a partial differential equation and if this sequence converges in $\mathcal{D}'(\Omega)$, then the limit of this sequence is also a solution to this equation.

If $\{F_j\}$ is a sequence of continuous functions on Ω and if it converges uniform on every compact set $K \subset \Omega$ to zero, then $\{[F_j]\}$ converges to zero in $\mathcal{D}'(\Omega)$, as well.

2.2 The existence and the character of found solutions

We consider the equation

$$\Big(\frac{\partial^4}{\partial \xi^4} + \lambda(t)\frac{\partial^2}{\partial \xi^2} - \alpha\frac{\partial^4}{\partial \xi^2 \partial t^2} + \frac{\partial^2}{\partial t^2}\Big)u(\xi,t) = 0,\ 0 < \xi < 1,\ t > 0, \quad (2.1)$$

with the boundary conditions:

$$u(0,t) = u(1,t) = u_\xi^{(2)}(0,t) = u_\xi^{(2)}(1,t) = 0, \ \ t > 0, \quad (2.2)$$

and with $\alpha > 0$.

Let us suppose that $u_k(\xi,t) = A_k \sin k\pi\xi \cdot T_k(t)$, $k \in \mathbb{N}$, where A_k are arbitrary constants, $A_k \neq 0$, $k \in \mathbb{N}$. Then $u_k(\xi,y)$ satisfies boundary conditions (2.2) for any T_k and every $k \in \mathbb{N}$. Moreover, in order that $u\,(\xi,t)$ satisfies (2.1) $T_k, k \in \mathbb{N}$, has to satisfy the equation:

$$T_k^{(2)}(t)(\alpha(k\pi)^2 + 1) + (k\pi)^2((k\pi)^2 - \lambda(t))T_k(t) = 0, \ \ t > 0. \quad (2.3)$$

Since $\alpha > 0$, equation (2.3) can be written in the form

$$T_k^{(2)}(t) - q_k(t)T_k(t) = 0, \quad t > 0, \tag{2.4}$$

where

$$q_k(t) = \frac{(k\pi)^2}{\alpha(k\pi)^2 + 1}(\lambda(t) - (k\pi)^2), \quad t > 0. \tag{2.5}$$

Equation (2.4) is of particular importance for applied mathematics and this accounts for the vast amount of research connected with it. We cite only three books, we use in this paper. These are [2], [3] and [4].

Definition 3 *A nontrivial solution T to (2.4) is said to be oscillatory if there exists a sequence $\{t_n\}, t_n \to \infty$, such that $T(t_n) = 0$ for each $n \in \mathbb{N}$.*

Theorem 1. *Suppose that $\lambda \in \mathcal{C}((0,\infty))$ and moreover for a $k = k_0 \in \mathbb{N}$*

$$\begin{cases} \lambda(t) \geq (k_0\pi)^2, & t \geq t_1 > 0, \\ \lambda(t) - (k_0\pi)^2 \neq 0 & t \geq t_2 > 0. \end{cases} \tag{2.6}$$

Then equation (2.3) has for all $1 \leq k \leq k_0$ a positive, convex and decreasing solution $T_{1k}(t)$ on (t_0, ∞) for some $t_0 \geq t_1$ (which depends on k). This solution is:

1) Slowly varying at ∞, $T_{1k} = L_{1k}(t)$, if and only if for each $\mu > 1$

$$x \int_x^{\mu x} q_k(t)\, dt \to 0, \quad x \to \infty.$$

2) Regularly varying at ∞ of index $\alpha_k, T_{1k} = t^{\alpha_{1k}} L_{1k}(t)$ if and only if for each $\mu > 1$

$$x \int_x^{\mu x} q_k(t)\, dt \to \frac{c_k(\mu - 1)}{\mu} > 0, \quad x \to \infty.$$

where $\alpha_{1k} = \frac{1}{2}\left(1 - \sqrt{1 + 4c_k}\right)$.

3) Rapidly varying at ∞ of index $-\infty$ if and only if for each $\mu > 1$

$$x \int_{x}^{\mu x} q_k(t)\, dt \to \infty, \quad x \to \infty.$$

A corresponding linearly independent solution T_{2k} is:

In case 1),$T_{2k}(t) = tL_{2k}(t)$; L_{1k} and L_{2k} are normalized and such that

$$L_{2k}(t) \sim (L_{1k}(t))^{-1}, \quad t \to \infty.$$

In case 2), $T_{2k}(t)$ is an increasing and regularly varying at ∞, $T_{2k}(t) = t^{\alpha_{2k}} L_{2k}(t)$, where $\alpha_{2k} = \frac{1}{2}\left(1 + \sqrt{1 + 4c_k}\right)$. The functions $L_{ik}, i = 1, 2,$ are normalized and

$$L_{2k}(t) \sim \{(1 - 2\alpha_{1k}) L_{1k}(t)\}^{-1}, \quad t \to \infty.$$

In case 3), $T_{2k}(t)$ is an increasing and regularly varying.

It can be given a shorter version of Theorem 1 with the same supposition on $\lambda(t)$ and with $1 \le k \le k_0$:

Corollary 1. *All solutions $T(t) = K_1 T_1(t) + K_2 T_2(t)$, where K_1 and K_2 are constants, to equation* (2.3) *are*:

1) In case $x \int_{x}^{\mu x} q_k(t)dt \to 0$, $x \to \infty$ for each $\mu > 1$, either slowly varying or regularly varying of index 1.

2) In case $x \int_{x}^{\mu x} q_k(t)dt \to \dfrac{c_k(\mu - 1)}{\mu} > 0$, $x \to \infty$ for each $\mu > 1$, regularly varying of index α_{1k} or α_{2k} (α_{ik}, $i = 1, 2$, are given in Theorem 1).

3) In case $x \int_{x}^{\mu x} q_k(t)dt \to \infty$, $x \to \infty$ for each $\mu > 1$, rapidly varying.

Proof of Theorem 1. We consider equation (2.3) in the form (2.4) with q_k given by (2.5). Since $\lambda \in \mathcal{C}((0, \infty))$, $q_k \in \mathcal{C}((0, \infty))$,as well. Consequently, (2.3) has a fundamental system of two solutions. To prove that for a $t_0 > 0$ there exists a positive decreasing solution to (2.3) we use the following Lemma:

Lemma 1. *([4, p. 10]) Let for some $a > 0, q \in \mathcal{C}((0, \infty)), q(x) \ge 0$ and $q(x) \ne 0, x \ge x_0 > 0$. Then the equation $y^{(2)}(x) - q(x) y(x) = 0$ has positive decreasing solutions on (x_0, ∞) for some $x_0 > a$.*

Suppose that $\lambda(t)$ satisfies condition (2.6) for a $k = k_0$,then $q_k(t)$ given by (2.5) satisfies conditions for q in Lemma 1 for every $1 \leq k \leq k_0$. Therefore (2.4) has a positive decreasing solution on (t_0, ∞) for some $t_0 > 0$ and $1 \leq k \leq k_0$.

The assertions of Theorem 1 denoted by 1), 2) and 3) follow from Theorem 1, Theorem 2 and Theorem 3 in [4], respectively.

In the next theorem the conditions are easier to verify than in Theorem 1. Consequently such theorem is more suitable for use.

Theorem 2. *([4]) Let the suppositions on $\lambda(t)$ be the same as in Theorem 1. If $\tau^2 q_k(t) \to c_k, t \to \infty, 1 \leq k \leq k_0$, then all decreasing solutions of (2.4) are slowly or rapidly or regularly varying functions with index $\alpha_k = \left(1 - \sqrt{1 + 4c_k}\right)$ in the later case, according as $c_k = 0, c_k = \infty, c_k \in (0, \infty)$.*

The proof is obvious and follows from the proof of Theorem 1.

If we omit condition (2.6), then the following theorem is valid:

Theorem 3. *Let $\lambda \in \mathcal{C}([t_0, \infty))$ for some $t_0 > 0$. If $L_{ik}, i = 1, 2$, denote two normalized slowly varying functions for $k = k_0 \in \mathbb{N}$, then there exist two linearly independent solutions T_{1k_0}, T_{2k_0} to (2.3):*

1) Of the form $T_{1k_0}(t) = L_{1k_0}(t), T_{2k_0} = tL_{2k_0}(t)$ if and only if $x \int_0^x q_{k_0}(t)\, dt \to 0, x \to \infty$ for a $k_0 \in \mathbb{N}$. Moreover one has

$$L_{2k_0}(t) \sim (L_{1k_0}(t))^{-1}, t \to \infty.$$

2) Of the form $T_{1k_0}(t) = t^{\alpha_{ik_0}} L_{1k_0}(t), i = 1, 2$, if and only if $x \int_0^x (-q_{k_0}(t))\, dt \to C_{k_0}, x \to \infty, -\infty < C_{k_0} < \frac{1}{4}, C_{k_0} \neq 0$, where $\alpha_{ik_0}, i = 1, 2$, are two roots of the equation $\alpha^2 - \alpha + C_{k_0} = 0, \alpha_{1k_0} < \alpha_{2k_0}$. Moreover

$$L_{2k_0}(t) \sim \{(1 - 2\alpha_{1k_0}) L_{1k_0}(t)\}^{-1}, t \to \infty.$$

Proof. It follows from Theorem 1.10 and Theorem 1.11 in [4].

Remark. In case 2), when $0 < C_{k_0} < \frac{1}{4}$,the both solutions are always unbounded. If $-\infty < C_{k_0} < 0$, then $\alpha_{1k_0} < 0$ and $\alpha_{2k_0} > 0$. Consequently $T_{1k_0}(t) \to 0, t \to \infty$ and $T_{2k_0}(t)$is unbounded.

In case 1) at least one of the solutions is unbounded.

A natural question arises: What we can assert about the solutions to (2.3) if in Theorem 3 the number $C_k \in (1/4, \infty)$. By Hille's oscillation criterion (cf.[8]) we known that

$$\lim_{x\to\infty} \sup\, x \int_x^\infty (-q_k(t))dt > \frac{1}{4}$$

is a sufficient condition for (2.3) to have the both linearly independent oscillatory solutions.

Let us remark that in [9] one can find the behavior of zeros of oscillatory solutions to equation of the form (2.4).

There is an immense literature dealing with the question of oscillatory solutions to equations of the form (2.4). Such an one, applied to equation (2.3) is the following:

Theorem 4. *Let $\lambda(t)$ be a twice continuously differentiable function and $\lambda(t) < (k\pi)^2, t \geq t_0$, for a $k \geq k_0$. Moreover, suppose that*

$$\int_{t_0}^\infty \left| q_{k_0}^{-3/2}(\tau)\, q_{k_0}^{(2)}(\tau) \right| d\tau < \infty. \tag{2.7}$$

Then the equation (2.3) for $k = k_0$ has a fundamental system of solutions satisfying

$$\begin{aligned} T(t) &\sim [-q_k(t)]^{-1/4} \exp\left\{ \pm i \int_{t_0}^t [-q_k(\tau)]^{1/2} d\tau \right\}, \\ T^{(1)}(t) &\sim \pm [-q_k(t)]^{-1/4} \exp\left\{ \pm i \int_{t_0}^t [-q_k(\tau)]^{1/2} d\tau \right\}, \quad t \to \infty. \end{aligned}$$

This is a consequence of Theorem 14 in [3, chapter IV].

A more precise theorem is a consequence of Theorem 15 in [3, chapter IV].

Theorem 5. *Let $\lambda(t)$ be a real continuous function for $t \geq t_0 > 0$, and let the integrals*

$$\int_t^\infty (1+q_k(\tau))\,d\tau,$$
$$g_1(t)=\int_t^\infty (1+q_k(\tau))\cos 2\tau d\tau,$$
$$g_2(t)=\int_t^\infty (1+q_k(\tau))\sin 2\tau d\tau, \quad \textit{for a } k=k_0\in\mathbb{N},$$

exist and suppose

$$\int_{t_0}^{\infty} |1+q_k(\tau)|\,|g_j(\tau)|\,d\tau<\infty, \qquad j=1,2, \ \ k=k_0.$$

Then equation (2.3) has a fundamental system of solutions T_1, T_2 satisfying:

$$\begin{aligned} T_1(t) &= \cos t+o(1), \quad T_2(t)=\sin t+o(1), \\ T_1^{(1)}(t) &= -\sin t+o(1), \quad T_2^{(1)}(t)=\cot s+o(1), \end{aligned}$$

for $t\to\infty$.

Comments. It is an interesting question: How much the number $\alpha>0$ influences on the asymptotic behavior of solutions to (2.3). It is easily seen that the number α does not figure in (2.6) which is the basic supposition on $\lambda(t)$ of Theorem 1. The asymptotic behavior of solutions depends in Theorem 1 only on

$$\lim_{x\to\infty} x\int_x^{\mu x} q_k(t)dt \quad \text{for each } \mu>1.$$

The number α can have an influence on this limit only in the case 2) when $C_k\neq 0$. This influence reflects on α_1 and α_2 but only on theirs numerical values and can not change the sign.

The same conclusion is with Theorem 2 and Theorem 3 case 1). But in Theorem 3 case 2) if for an $\alpha=\alpha_0$, C_{k_0} is less than $\frac{1}{4}$, then the both solutions are unbounded. If we can find $0<\alpha<\alpha_0$ such that $\dfrac{\alpha_0(k\pi)^2+1}{\alpha(k\pi)^2+1}C_k>\dfrac{1}{4}$, then the solutions to (2.3) with this value of α are oscillatory solutions. Similarly changing α we can obtain non–oscillatory unbounded solutions instead of oscillatory solutions.

In theorems 4 and 5 the existence of the given solutions does not depend on $\alpha > 0$.

If in equation (2.3) λ is a constant, then it is easily to find solutions of this equation.

An example for $\lambda(t)$ which satisfies conditions of Theorem 4 is $\lambda(t) = \frac{t+1}{t+2} = 1 - \frac{1}{t+2}$. This function is twice continuously differentiable:

$$\lambda^{(1)}(t) = \frac{1}{(t+2)^2} > \lambda^{(2)}(t) = -\frac{2}{(t+2)^3},$$

for $t \geq 0$ it is monotone increasing because of $\lambda^{(1)}(t) > 0,\ t \geq 0$, and $\lim_{t\to\infty} \lambda(t) = 1$. Consequently $\lambda(t) < \pi^2,\ t \geq 0$. Now,

$$-q_1(t) = \frac{\pi^2}{\alpha\pi^2 + 1}(\pi^2 - \lambda(t)) = A(\pi^2 - \lambda\,(t))).$$

Thus $|q_1^{(2)}(t)| = A\frac{2}{(t+2)^3}$ and integral (2.7) exists. The assertion of Theorem 4 for $\lambda(t) = \frac{t+1}{t+2}$ is valid.

2.3 Generalized solutions

In Subsection 2.3 we found a sequence of solutions $u_k(\xi, y) = A_k \sin k\pi\xi\ T_k(t)$, $k \in \mathbb{N}$, to (2.1), (2.2), which are of a special form. The characteristic of such solution is that the initial condition is $u_k(\xi, 0) = A_k T_k(0) \sin k\pi\xi$, which is a very narrow class of functions. It is easily seen that any finite sum $\Sigma u_k(\xi, t)$ produce a new solution to (2.1), (2.2). In the following we will construct generalized solutions to (2.1), (2.2) by using some series, but preserving the basic properties of classical solutions, natural for the mechanical applications.

Naturally, the interesting solutions to (2.1), (2.2) are those which are bounded. Therefore we chose to prove the following theorem:

Theorem 6. *Let*

1) $\lambda\,(t) \in \mathcal{C}^\infty\,((0,\infty))\,,$

2) $\lim_{t\to 0^+} \lambda\,(t) = 0,$

3) $\lambda^{(1)} \in L^1([0,\infty))$,

4) there exists $k_0 \in \mathbb{N}$ *such that* $\sup\limits_{t \geq 0} |\lambda(t)| = \Lambda < (k_0\pi)^2$.

Then for every $k \geq k_0, k \in \mathbb{N}$, *equation (2.4) has a unique solution* $T_k(t)$ *satisfying the same initial conditions* $T_k(0) = T_0, T^{(1)}(0) = T_0^1, k \geq k_0$. *The elements of the sequence* $\{T_k(t)\}_{k \geq k_0}$ *are bounded on* $[0,\infty)$, *uniformly in* $k \geq k_0$. *Moreover*

$$u(\xi,t) = \sum_{k=k_0}^{\infty} A_k (\sin k\pi\xi) T_k(t), \tag{3.1}$$

where $|A_k| \leq \frac{N}{k^{\beta+\gamma}}, \gamma > 0, 1 \leq \beta \leq 5$ *represents function in* ξ *with values in* $\mathcal{D}'((0,\infty))$ *which is a generalized solution to (2.1),(2.2).*

Remark. We have first to explain the meaning of the sentence "... which is a generalized solution to (2.1), (2.2)."

For every $\xi \in (0,1)$, $u(\xi,t)$ defines a regular distribution; $u(\xi,t)$ is a function in ξ with values in $\mathcal{D}'((0,\infty))$ which has the $\beta - 1$ continuous partial derivatives in ξ and other derivatives are in the sense of distributions (cf. 2.1.2).

Proof. Because of suppositions 1) and 2) we can continuously extend $\lambda(t)$ to $(-\varepsilon,\infty)$, $\varepsilon > 0$ such that $\lambda(t) = 0$, $t \in (-\varepsilon, 0]$. Then for every $k \in \mathbb{N}$ there exists a solution T_k to (2.4) with the initial condition $T_k(0) = T_0$ and $T_k^{(1)}(0) = T_0^1$.

If in (2.4) we introduce q_k given by (2.5) and multiply so obtained equation by $T_k^{(1)}(t)$, after integrtion between 0 and t, we obtain for every $k \in \mathbb{N}$, $k \geq k_0$, $t \geq 0$:

$$\begin{aligned}\frac{1}{2}(T_k^{(1)}(t))^2 &+ \frac{(k\pi)^4}{\alpha(k\pi)^2+1}\frac{1}{2}T_k^2(t) - \\ &- \frac{(k\pi)^2}{\alpha(k\pi)^2+1}\int_0^t \lambda(\tau)T_k^{(1)}(\tau)T_k(\tau)d\tau = B,\end{aligned} \tag{3.2}$$

where B is a constant.

From (3.2) it follows that

$$B = \frac{1}{2}\Big(T_0^1\Big)^2 + \frac{(k\pi)^4}{\alpha(k\pi)^2+1}\frac{1}{2}\Big(T_0\Big)^2. \tag{3.3}$$

Integrating by parts the integral in (3.2), this gives

$$\frac{1}{2}\Big(T_k^{(1)}(t)\Big)^2 \quad + \frac{(k\pi)^2}{\alpha(k\pi)^2+1}\,\frac{(k\pi)^2-\lambda(t)}{2}\,T_k^2(t)+$$

$$+\frac{1}{2}\,\frac{(k\pi)^2}{\alpha(k\pi)^2+1}\int\limits_0^t \lambda^{(1)}(\tau)T_k^2(\tau)d\tau = B.$$

Whence

$$\frac{(k\pi)^2}{\alpha(k\pi)^2+1}((k\pi)^2-\Lambda)\frac{(T_k(t))^2}{2} \le$$

$$\le B + \int\limits_0^t \frac{|\lambda'(\tau)|}{(k\pi)^2-\Lambda}\,\frac{(k\pi)^2}{\alpha(k\pi)^2+1}\,\frac{(k\pi)^2-\Lambda}{2}(T_k(\tau))^2 d\tau. \tag{3.4}$$

Now we can use

Lemma 2. *(cf. Lemma 1 in [2, p. 107]). Let* $u,v\ge 0,\ c_1>0$ *and* u *satisfy the inequality*

$$u(t) \le c_1 \int\limits_0^t u\,(\tau)\,v\,(\tau)\,d\tau, \quad t\ge 0.$$

Then

$$u\,(t) \le c_1 \exp\left(\int\limits_0^t v\,(\tau)\,d\tau\right), \quad t\ge 0.$$

Applying this Lemma to (3.4) we have the inequality

$$\frac{1}{2}\,\frac{(k\pi)^2}{\alpha(k\pi)^2+1}((k\pi)^2-\Lambda)(T_k(t))^2 \le B\exp\Big(\frac{1}{(k_0\pi)^2-\Lambda}\int\limits_0^\infty |\lambda^{(1)}(\tau)|d\tau\Big).$$

Now it is easily seen that there exists a constant M such that $|T_k(t)|\le M,\ t\ge 0,\ k\ge k_0$.

It remains only to prove that $u(\xi, t)$, given by (3.1) is a generalized solution to (2.1), (2.2). The finite sum

$$u_K(\xi, t) = \sum_{k=k_0}^{K} A_k \sin k\pi\xi T_k(t)$$

is also a solution to (2.1), (2.2), we have only to prove that $u_K(\xi, t)$ converges in $\mathcal{C}((0,1) \times (0,\infty))$, when $k \to \infty$.

Since

$$|A_k \sin k\pi\xi T_k(t)| \leq |A_k|M \leq \frac{NM}{k^{\beta+\gamma}},\ (\xi, t) \in (0,1) \times (0,\infty)), 1 \leq \beta \leq 5,$$

for every $k \geq k_0$, the sequence $\{u_K(\xi, t)\}_{K \geq k_0}$ converges in $\mathcal{C}((0,\infty) \times (0,1))$ and consequently in $\mathcal{D}'((0,1) \times (0,\infty))$. Thus $[u(\xi, t)]$ is a solution to ((2.1), (2.2) (cf. 2.1.2).

It is also

$$|(A_k \sin k\pi\xi T_k(t))_\xi^{(i)}| \leq M|A_k|(k\pi)^i \leq \frac{MN}{k^{\beta+\gamma-i}},$$

for $i = 0, ..., \beta - 1$. Thus $u_\xi^{(i)}(\xi, t)$ exists and

$$u_\xi^{(i)}(\xi, t) = \sum_{k=k_0}^{\infty} A_k (\sin k\pi\xi)^{(i)} T_k(t),\quad i = 1, ..., \beta - 1,$$

and is a continuous function on $[0,1] \times [0,\infty)$. It is easily seen that $u(\xi, t)$ satisfy boundary condition (2.2).

This completes the proof.

Remark. The function $u(\xi, t)$ satisfies also the initial condition

$$u(\xi, 0) = T_0 \sum_{k=k_0}^{\infty} A_k \sin k\pi\xi.$$

This is more extensive class of functions then the class $u_k(\xi, 0) = A_k T(0) \sin k\pi\xi$.

3 Concluding remarks

In applications, the most important case corresponds to the first mode vibration of the rod. Thus, we choose $k = 1$ in (2.3) and we compare the results obtained here, with the results about stability of the same rod, obtained by static (Euler) method.

1. Suppose that the dimensionless force $\lambda(t)$ (see (1.9) is larger than the Euler buckling force [1, p. 111], that is, $\lambda(t) > \pi^2$. Theorem 1 in Section 2.2 specifies the properties of the solution, indicting the *instability* of the rod, since T_{2k} is an increasing function. The static method (the method of adjacent equilibrium configuration) also predicts instability for $\lambda(t) = const. > \pi^2$, independently of α since rotary inertia does not play any role in the static method.

2. If the axial force $\lambda(t)$ is smaller than the Euler buckling force $\lambda(t) < \pi^2$ (assumption 4) in Theorem 1, Section 3) than the Theorem 1 in Section 2.3 predicts stability if the value of the axial force is increased from zero (assumption 2) in Theorem 1, Section 2.3). This is in agreement with the predictions of static method. We note that above conclusion is independent of the value of the rotary inertia parameter α. Thus this result is in agreement with the static method where there is no influence of rotary inertia on stability boundary.

Our results show equivalence in prediction of stability boundary by dynamic and static methods. Also our results show some of the properties of the dynamic solutions and their relation to regular varying functions.

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O poprečnim oscilacijama elastičnog štapa sa promenljivom pritiskujućom silom

UDK 531.01

U radu se proučavaju poprečne oscilacije slobodno oslonjene, aksijalno pritisnute elastične grede, bez zanemarivanje efekta rotacione inercije. Usvojeno je da je aksijalna sila poznata funkcija vremena. Ispitana je stabilnost grede u linearnoj aproksimaciji i odredjeni su uslovi pod kojima je vremenska evolucija sistema opisana sporo promenljivim i regularno sporo promenljivim funkcijama. Osim toga, analizirana su i neka svojstva uopštenih rešenja problema. Dobijeni rezultati ovde iznete dinamicke analize, u saglasnosti su sa rezultatima koji se dobijaju statičkim (Ojlerovim) metodom.