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Restoration of the symmetries broken by reversible growth in hyperelastic bodies

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Abstract

In this paper, we interpret the development of material inhomogeneities in continuum, hyperelastic bodies in the presence of reversible growth in terms of broken symmetries [1]. By applying Noether's Theorem [1,2,3,4], we find a set of equations yielding the fields necessary to compensate for the broken symmetry. As growth occurs, these fields provide for an instantaneously *updated* reference configuration of the body, and are responsible for the dynamical restoring of the body symmetries. In addition, we propose to use these compensating fields in order to generalize the definition of the *transplant operator* given in [5,6]. This work has been motivated by the current theoretical investigations on the biomechanical aspects of growth in articular cartilage.

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1 Introduction

The starting-point of this paper is the article by Epstein and Maugin reported in [5]. In this work, we tackle the case of reversible growth in hyperelastic, uniform and inhomogeneous continuum bodies, because we aim at interpreting the development of material inhomogeneities in terms of broken symmetries. Symmetries are here meant as transformations, performed on fields and independent variables, corresponding to the invariance of a suitable defined Action functional [1].

We describe the growing body as a macroscopic open system in which new material is smoothly added [5,7]. Volumetric growth is modeled by means of a smooth field, the mass source, representing the rate of change of the volumetric mass density [5,7,8]. Even though the growing body is assumed to be uniform [5], this field acts as a source of inhomogeneities. The development of material inhomogeneities is interpreted as a continuous breaking of the body symmetries.

As far as growth is disregarded, the evolution of the body can be described by a Lagrangian density function. In this case, dynamic equations are found via Hamilton's Principle [9,10,11], and conservation laws are obtained by Noether's Theorem [1,2,3,4]. When growth occurs, the mass source brings about some polygenic momentum sources, that cannot be taken into account by a suitable Lagrangian density function [9]. In this case, Hamilton's Principle no longer applies, and Noether's Theorem ceases to be valid. This is because dynamics must take into account the polygenic momentum sources, and conservation laws are modified by the presence of additional source terms, which cause the non-vanishing of the four-divergence of the Noether's four-current. The transformations performed on independent coordinates and fields, which provide for the steadiness of the Action functional when there is no growth (these transformations are named *symmetries*), do not apply when additional sources act within the body. In this sense, we speak about *broken symmetries*. New classes of transformations are therefore required in order to restore the broken symmetries. In the following, we demonstrate that, by imposing some proper conditions on the transformations applied on fields and independent coordinates, we are able to obtain an *effective source term* which reduces to the virtual work density exerted by the polygenic momentum source. Under these conditions, we can state that the virtual work density exerted by the momentum source (which corrects the dynamics of the system as growth takes place) is balanced by the effective source term yielding the non-vanishing of the four-divergence of the Noether's four-current. This sets the condition for the variation of Action to vanish, and thus, the symmetries of the system to be restored. We believe that the fields necessary to compensate for the symmetry breaking allow for the generalization of the *transplant operator* defined in [5,6]. With respect to the work by Epstein and Maugin, the hypotheses of hyperelastic material and reversible growth simplify the description of the process of growth, because, assuming the body to be hyperelastic gives a constitutive framework in which mass-diffusive aspects of growth are neglected [5,12].

2 The continuity equation and the dynamic equation of a hyperelastic body with reversible growth

As new material is locally inserted into the body, the increment of inhomogeneities is "measured" by the variation of the volumetric mass density, which is related to the volumetric mass source through the continuity equation. This physical situation is usually described by defining the volumetric mass density as a function of time and referential coordinates (in the following we denote the Lagrangian form of fields by using "ˆ"). The continuity equation must take into account the mass source and the mass flux vector [5]. The latter represents the mass-diffusive effects within the body and describes some morphogenetic events related to growth [5,8]. In order to investigate these aspects, the mass flux is determined as a dependent variable within a second-order constitutive framework [5,12,13]. In the hypothesis of hyperelastic body, the mass flux vector must vanish identically and the continuity equation reads:

$$\partial_t \hat{\rho}_R = \hat{\Pi} \tag{2.1}$$

where $\hat{\rho}_R = \hat{\rho}\hat{J}$ and $\hat{\Pi} = \hat{\pi}\hat{J}$ are the Piola-transformed volumetric mass density and mass source, respectively, $\hat{J}$ being the determinant of the deformation gradient tensor, $\hat{F}$. If there were no growth, the mass of the system would be constant and Eq.(2.1) would reduce to $\partial_t \hat{\rho}_R = 0$ for any $(t, X_0) \in \mathcal{I} \times \mathcal{C}_0$, $\mathcal{I}$ being an interval of time and $\mathcal{C}_0$ a subset of $\mathcal{B}_0$, i.e. the reference configuration of the body at time $t = 0$. The field $\hat{\Pi}$ is the smooth field describing growth. Its values can be positive or negative, depending on whether the mass of the whole system is increased or decreased by the process of growth [5,8]. If the mass source is a given function of time and reference coordinates, the mass density of the body can be determined by integrating Eq.(2.1) with respect to time [7]. By introducing the notation $\hat{\rho}_R(0, X_0) = \hat{\rho}_0(X_0)$ and $\hat{\mathcal{G}}(t, X_0) = \int_0^t \hat{\Pi}(s, X_0) ds$, we obtain:

$$\hat{\rho}_R(t, X_0) = \hat{\rho}_0(X_0) + \hat{\mathcal{G}}(t, X_0), \quad \forall t \in \mathcal{I}. \tag{2.2}$$

The dynamics of a growing uniform body takes into account reversible and irreversible momentum sources and fluxes [5]. In the hypotheses of reversible growth and hyperelastic material, the dynamic equation is given by:

$$\partial_t(\hat{\rho}_R \hat{v}_i) = \hat{\rho}_R \hat{f}_i + \partial_k \hat{T}_{ik} + \hat{\Pi} \hat{v}_i \tag{2.3}$$

where $\hat{v}$ is the velocity of the material point, $\hat{f} : \mathcal{I} \times \mathcal{C}_0 \to \mathrm{R}^3$ represents the body forces per unit mass, $\hat{T} : \mathcal{I} \times \mathcal{C}_0 \to \mathrm{R}^{3,3}$ is the first Piola-Kirchhoff stress-tensor, and $\hat{\Pi}\hat{v} : \mathcal{I} \times \mathcal{C}_0 \to \mathrm{R}^3$ is the reversible momentum source. By invoking Eq.(2.1), the polygenic term, $\hat{\Pi}\hat{v}$, can be entirely taken into account by the time variation of the volumetric mass density. Therefore, the dynamic equation can be rearranged as:

$$\hat{\rho}_R(\partial_t \hat{v}_i) = \hat{\rho}_0 \hat{f}_i + \partial_k \hat{T}_{ik}, \quad \textit{or} \quad \hat{\rho}_0(\partial_t \hat{v}_i) + \hat{\mathcal{G}}(\partial_t \hat{v}_i) = \hat{\rho}_0 \hat{f}_i + \partial_k \hat{T}_{ik}. \tag{2.4}$$

If we were able to switch off the mass source at $t = 0$, the system would evolve according to the dynamic equation:

$$\hat{\rho}_0(\partial_t \hat{v}_i) = \hat{\rho}_0 \hat{f}_i + \partial_k \hat{T}_{ik}. \tag{2.5}$$

Such a dynamic equation can be derived by defining a suitable Lagrangian density function and requiring the Action functional of the body to be stationary [5,10]. This is known as the "weak formulation" of Hamilton's Variational Principle. However, even in the presence of growth, the body can be assigned a Lagrangian density function, $\mathcal{L}$, which takes into account the kinetic energy density, a generalized work-function, $-\mathcal{V} : \mathcal{I} \times \Omega \to \mathrm{R}$, and the Helmholtz free energy per unit mass, $\psi : \mathcal{I} \times \Omega \to \mathrm{R}$. In case of thermoelastic material, ψ depends on space coordinates, on the temperature field and the deformation gradient tensor [12,13]. The existence of $\mathcal{V}$, which depends on the configuration field only, is assumed in order to model the interaction between the system and the environment. Thus, with respect to the Eulerian forms of the fields, $\mathcal{L}$ is given by:

$$\begin{aligned}&\mathcal{L}(t, y, v(t, y), \Theta(t, y), F(t, y)) = \\ &\quad \frac{1}{2}\rho(t, y)[v(t, y)]^2 - \rho(t, y)\mathcal{V}(y) - \rho(t, y)\psi(t, y)\end{aligned} \tag{2.6}$$

where $(t, y) \in (\mathcal{I}, \Omega)$ ($\Omega \subset \mathcal{C}(t)$ being an open set), and $\Theta : \mathcal{I} \times \Omega \to \mathrm{R}$. In the reference configuration of the body, the Lagrangian form of the Helmholtz free energy density, $\hat{\psi}$, is defined by a functional $\mathcal{F}^{\psi}$ such that:

$$\mathcal{F}^{\psi}(X_0, \hat{\Theta}(t, X_0), \hat{F}(t, X_0)) = \hat{\psi}(t, X_0) \tag{2.7}$$

The first Piola-Kirchhoff stress tensor, $\hat{T}$, and the entropy density are determined as constitutive dependent variables [5,14]. The Action functional is obtained by integrating the Lagrangian density function, given in Eq.(2.6), over the time-space domain $\mathcal{I} \times \mathcal{C}(t)$. By applying the Piola-transformation, $\mathcal{L}_R = \hat{J}\mathcal{L}$, Action can be written with respect to the reference configuration of the body [5,15,16], i.e.:

$$\begin{aligned}&\mathcal{A}(\hat{x}, \hat{\Theta}) = \\ &\quad \int_{\mathcal{I}} \{\int_{\mathcal{C}_0} \mathcal{L}_R(t, X_0, \hat{x}(t, X_0), \hat{v}(t, X_0), \hat{\Theta}(t, X_0), \hat{F}(t, X_0)) d\tau\} dt, \\ &\mathcal{L}_R(t, X_0, \hat{x}(t, X_0), \hat{v}(t, X_0), \hat{\Theta}(t, X_0), \hat{F}(t, X_0)) = \\ &\quad \frac{1}{2}\hat{\rho}_R(t, X_0)[\hat{v}(t, X_0)]^2 - \hat{\rho}_R(t, X_0)[\mathcal{V}(\hat{x}(t, X_0))] - \\ &\quad \hat{\rho}_R \mathcal{F}^{\psi}(X_0, \hat{\Theta}(t, X_0), \hat{F}(t, X_0)).\end{aligned} \tag{2.8}$$

3 Thermoelastic body with reversible growth

The thermodynamic description of a growing body must take into account the sources of internal and kinetic energies due to the incoming material. In reference [5], the local form of the energy balance law and the Clausius-Duhem inequality are investigated in detail for the general case. By assuming reversible growth and thermoelastic material, the energy balance law and the Clausius-Duhem inequality are given by:

$$\hat{\rho}_R(\partial_t\hat{\mathcal{U}}) = \hat{r}_R + tr[\hat{T}(\mathrm{grad}\hat{v})^T] \quad \textit{and} \quad \hat{\rho}_R\hat{\Theta}(\partial_t\hat{\mathcal{S}}) \geq \hat{r}_R, \tag{3.1}$$

where $\hat{\mathcal{U}}$ is the internal energy density and $\hat{r}_R$ is the bulk heat supply. In the absence of $\hat{r}_R$, the Clausius-Duhem inequality is satisfied as an equality, which states that the entropy density does not vary in time. According to this further assumption, the time-variation of the internal energy density depends on mechanical factors only. This enables us to drop the dependence of the Lagrangian density function on the temperature field.

4 A Noether-like approach to the dynamical restoring of the broken symmetries in a growing hyperelastic body

The polygenic reversible momentum source, $\hat{\Pi}\hat{v}$, entails the increase in material inhomogeneities [5]. The mass density $\hat{\rho}_R$, which, in the absence of growth, was just a function of space coordinates, now depends on time as well [5,7]. The addition of new material is accompanied by a continuous rearrangement of the inhomogeneities in the reference configuration of the body. If there was no growth the material inhomogeneities would be fixed in the reference configuration of the body and would not vary in time. In the situation just depicted, the symmetries of the system are broken in the reference configuration of the body, i.e., in the configuration in which the system appears to be inhomogeneous. Nevertheless, we can determine a suitable reference configuration of the body in which the system appears homogeneous. This can be accomplished by applying Noether's Theorem. If we require Action to be stationary the equations of motion and conservation laws can be determined simultaneously [1,2,3]. The latter are expressed by the vanishing of the four-divergence of the Noether's four-current [1,2]. This generalized continuity equation does not necessarily reduce to the vanishing of the four-divergence of the energy-momentum tensor [1,17]. Rather, it imposes a set of conditions on the transformations performed on the independent variables. These transformations are called *external symmetries* [1], and, in this case, do not depend on time. It is worthwhile to remark that, throughout this section, we deal with the "weak formulation" of the variational principle. When growth is considered, symmetries are continuously broken and a new class of transformations on time-space coordinates must be found in order to be restored. However, in this case, Noether's Theorem cannot be applied directly [9,18]. Action is not stationary because $\hat{\Pi}\hat{v}$ entails the presence of non-trivial terms that modify the dynamics and introduce additional sources to the Noether's four-current. In order to restore the continuously broken symmetries, the sources must be reduced to an *effective source term*, which turns out to be equal to the virtual work density exerted by $\hat{\Pi}\hat{v}$. If there exists a class of fields which are able to eliminate the non-effective sources, these fields must be such that the divergence of the Noether's current is just equal to the effective source term. Then, the result of the topological integral provided by Noether's Theorem is the virtual work exerted by $\hat{\Pi}\hat{v}$. But, this work is also the negative of the term missing in order to retrieve the dynamic equation. Hence, there is a class of transformations for which dynamics and conservation laws compensate. As Action is stationary under these transformations, we say that they restore the broken symmetries of the system [1]. These fields belong to the class of external symmetries [1], which, in this case, are parameterized by time. By fixing a field belonging to this class, the original reference configuration of the body is smoothly transformed into a *new* admissible reference configuration in which the system appears to be homogeneous [2,3].

4.1 The admissible reference configuration set

Every physical quantity dealing with the body, whether or not growth is considered, is conceived as a map which, in the Lagrangian framework of Continuum Mechanics, is defined on $\mathcal{I} \times \mathcal{C}_0$ and valued in some normed space. Following some suggestions taken from the literature (cf. e.g.[2,3,4]), the most general transformation that can be performed on the independent coordinates is given by:

$$\hat{\gamma} : \mathcal{I} \times \mathcal{C}_0 \to \mathcal{I}^* \times \mathcal{C}_0^* \tag{4.1}$$

where $\mathcal{I}^*$ and $\mathcal{C}_0^*$ are the transformed sets. The vector field $\hat{\gamma}$ is a diffeomorphism, and its components are expressed by $\hat{\gamma}_\alpha$, $\alpha = 0, ..., 3$ (the 0-*th* component represents the time-component of the vector field). If we fix an instant in time, t, and consider only the space-components of $\hat{\gamma}$, we obtain $\hat{\gamma}_i(t, \cdot) : \mathcal{C}_0 \to \mathcal{C}_0^*$. This transforms each reference position, $X_0 \in \mathcal{C}_0 \subset \mathcal{B}_0$, into $X_0^* \in \mathcal{C}_0^*$. The set $\mathcal{C}_0^* = (\hat{\gamma}_i(t, \mathcal{C}_0))_{i=1}^3$ is parameterized by t and is a subset of the new reference configuration $\mathcal{B}_0^*$. The map $\hat{\gamma}_i(t, \cdot)$ belongs to a class of transformations, the external symmetries, spanning a family of sets which we call *admissible reference configuration set* of the body, $\mathbf{B}_0$. Each element of $\mathbf{B}_0$ represents a possible global reference configuration of the body.

4.2 Dynamic equation

The dynamic equation for a growing body is obtained by taking into account the Lagrangian density function given in Eq. (2.8), the virtual work exerted by the reversible source of momentum, $\hat{\Pi}\hat{v}$, and the surface forces, $\hat{\varphi}$, acting on the boundary, $\partial\mathcal{C}_0$, of $\mathcal{C}_0$ [10]. In order to treat the Lagrangian density function as dependent on time and space coordinates, we define an auxiliary map, $\hat{\eta}_\alpha : \mathcal{I} \times \mathcal{C}_0 \to \mathcal{I} \times \mathcal{C}_0 | (t, X_0) \mapsto \hat{\eta}_\alpha(t, X_0) = (t, X_{0k})$ and consider the composed map $\mathcal{L}_R \circ (\hat{\eta}_\alpha, \hat{x}_i, \partial_\beta \hat{x}_i)$. Nevertheless, in order to simplify the mathematical formalism, we refer to $\mathcal{L}_R \circ (\hat{\eta}_\alpha, \hat{x}_i, \partial_\beta \hat{x}_i)$ as to $\mathcal{L}_R$. The four-gradient, $\partial_\beta \hat{x}_i$, is $\partial_\beta \hat{x}_i = (\partial_t \hat{x}_i, \partial_k \hat{x}_i) = (\hat{v}_i, \hat{F}_{ik})$. By performing the transformation, $\hat{x}_i \mapsto \hat{x}_i^* = \hat{x}_i + \mu\hat{\xi}_i$ [1,10], on the configuration field, where μ is a real smallness parameter, and $\hat{\xi} : \mathcal{I} \times \mathcal{C}_0 \to \mathrm{R}^3$ is a smooth map such that

$\hat{\xi}(0, X_0) = \hat{\xi}(\bar{t}, X_0) = 0$ ($\{0, \bar{t}\} = \partial\mathcal{I}$) for every $X_0 \in \partial\mathcal{C}_0$, the dynamic equation is found by requiring the vanishing of the integral [10]:

$$\int_{\mathcal{I}} \{\int_{\mathcal{C}_0} (\delta\mathcal{L}_R) d\tau + \int_{\mathcal{C}_0} \hat{\Pi}\hat{v}_i(\mu\hat{\xi}_i) d\tau + \int_{\partial\mathcal{C}_0} \hat{\varphi}_i(\mu\hat{\xi}_i) da\} dt = 0. \tag{4.2}$$

This is the principle of virtual work in a disguise, and corresponds to the weak formulation of the boundary value problem governing dynamics. Equation (4.2) can be rearranged as:

$$\begin{gathered} \int_{\mathcal{I}} \{\int_{\mathcal{C}_0} [-\partial_t(\hat{\rho}_R\hat{v}_i) + \hat{\rho}_R\hat{f}_i + \partial_k\hat{T}_{ik} + \hat{\Pi}\hat{v}_i](\mu\hat{\xi}_i) d\tau + \\ \int_{\partial\mathcal{C}_0} [\hat{T}_{ik}\hat{n}_k - \hat{\varphi}_i](\mu\hat{\xi}_i) da\} dt = 0. \end{gathered} \tag{4.3}$$

By virtue of Eq.(4.3), Eq.(2.3) is retrieved under the constraint condition $\hat{T}_{ik}\hat{n}_k - \hat{\varphi}_i = 0$, on $\mathcal{I} \times \partial\mathcal{C}_0$.

4.3 Determination of the compensating fields for restoring the broken symmetries of the system

The transformation $\hat{\gamma}$ in Eq. (4.1) is defined by $\hat{\gamma} = \hat{\eta} + \lambda\hat{\Psi}$, where λ is a real smallness parameter and $\hat{\Psi} : \mathcal{I} \times \mathcal{C}_0 \to \mathcal{I}^* \times \mathcal{C}_0^*$ is a diffeomorphism. The reference positions, $X_0 \in \mathcal{C}_0$, are transformed into the new admissible reference positions, $X_0^* \in \mathcal{C}_0^*$, such that $X_{0k}^* = X_{0k} + \lambda\hat{\Psi}_k(t, X_0)$. The perturbations, $\mu\hat{\xi}$ and $\lambda\hat{\Psi}$, allow for the definition of the Noether's four-current [1,2], i.e.:

$$\hat{\mathcal{J}}_\alpha = -\mu\hat{\xi}_i \frac{\partial\mathcal{L}_R}{\partial(\partial_\alpha\hat{x}_i)} + \lambda\hat{\Psi}_\alpha\mathcal{L}_R. \tag{4.4}$$

According to Noether's Theorem, if a Classical Field Theory with a certain energy-momentum tensor admits a one-parameter family of symmetry transformations, then Noether's four-current $\hat{\mathcal{J}}_\alpha$, is conserved [1]. If no source is considered, the conservation of $\hat{\mathcal{J}}_\alpha$ is expressed by the vanishing of its four-divergence [1]. Yet, in case of growth, this is not true. Hence, the conservation laws are understood in a generalized meaning.

These laws acquire a general form and are no longer expressed by homogeneous equations. The system is open, and the continuity equation of a closed system cannot be re-established by suitably adjusting the source terms [17]. In order to find out the explicit form of the source term that appears in the energy-momentum continuity equation, we calculate the four-divergence of $\hat{\mathcal{J}}_\alpha$. In the most general case, we apply transformations on the configuration field and on independent time-space variables, i.e.:

$$\hat{x}_i \mapsto (\hat{x}_i^* \circ \hat{\gamma}_\alpha) = \hat{x}_i + (\delta \hat{x}_i) \circ \hat{\gamma}_\alpha \Rightarrow (\delta \hat{x}_i) \circ \hat{\gamma}_\alpha = (\hat{x}_i^* \circ \hat{\gamma}_\alpha) - \hat{x}_i \quad (4.5)$$

where $(\hat{x}_i^* \circ \hat{\gamma}_\alpha), [(\delta \hat{x}_i) \circ \hat{\gamma}_\alpha] : \mathcal{I} \times \mathcal{C}_0 \to \mathrm{R}^3$ and $\hat{x}_i^* : \mathcal{I}^* \times \mathcal{C}_0^* \to \mathrm{R}^3$. If we seek for an expression involving only linear terms with respect to μ and λ, the global variation in Eq.(4.5) becomes:

$$\delta \hat{x}_i = \mu \hat{\xi}_i + \lambda \hat{\Psi}_\alpha (\partial_\alpha \hat{x}_i). \quad (4.6)$$

Following the spirit of the local gauge transformation theory, we express $\delta \hat{x}_i$ in terms of a coordinate-dependent rotation of the original configuration field (cf. e.g.[19]). This is done by defining a skew-symmetric second-order tensor, $\hat{R}_{il}$, such that $\delta \hat{x}_i = \hat{R}_{il} \hat{x}_l$. The diffeomorphism $\lambda \hat{\Psi}$ is expressed by the superposition of a coordinate translation and a coordinate rotation. For this purpose, we define a skew-symmetric second-order four-tensor rotation field, $\hat{\mathcal{R}}_{\alpha\beta}$, and a four-vector translation field, $\hat{w}_\alpha$, such that $\lambda \hat{\Psi}_\alpha(t, X_0) = \hat{\mathcal{R}}_{\alpha\beta}(t, X_0) \hat{\eta}_\beta(t, X_0) + \hat{w}_\alpha(t, X_0)$. The components $\hat{\mathcal{R}}_{0k} = -\hat{\mathcal{R}}_{k0}$ of $\hat{\mathcal{R}}_{\alpha\beta}$ are assumed to be zero, because we only admit time to be translated. Consequently, the transformation performed on time reduces to $\lambda \hat{\Psi}_0(t, X_0) = \hat{w}_0(t, X_0)$, while, setting $\hat{\mathcal{R}}_{kl} = \hat{R}_{kl}$, space-coordinates are transformed into $\lambda \hat{\Psi}_k(t, X_0) = \hat{R}_{kl}(t, X_0) X_{0l} + \hat{w}_k(t, X_0)$, where $X_{0l} = \hat{\eta}_l(t, X_0)$. Noether's four-current can now be written in the form:

$$\hat{\mathcal{J}}_\alpha = \hat{\mathcal{E}}_{\alpha\beta} \hat{w}_\beta + \hat{\Xi}_{\alpha i l} \hat{R}_{il} + \hat{\chi}_{\alpha\beta\mu} \hat{\mathcal{R}}_{\beta\mu} \quad (4.7)$$

where $\hat{\mathcal{E}}_{\alpha\beta}$ is the energy-momentum tensor [1], $\hat{\Xi}_{\alpha i l}$ and $\hat{\chi}_{\alpha\beta\mu}$ are, respectively, the *intrinsic* and the *orbital* angular-momentum tensors [1]. These tensors are given by:

$$\hat{\mathcal{E}}_{\alpha\beta} = -(\partial_\beta \hat{x}_i)\frac{\partial \mathcal{L}_R}{\partial(\partial_\alpha \hat{x}_i)} + \delta_{\alpha\beta}\mathcal{L}_R,$$
$$\hat{\Xi}_{\alpha il} = \frac{1}{2}\left\{\hat{x}_l\frac{\partial \mathcal{L}_R}{\partial(\partial_\alpha \hat{x}_i)} - \hat{x}_i\frac{\partial \mathcal{L}_R}{\partial(\partial_\alpha \hat{x}_l)}\right\}, \tag{4.8}$$
$$\hat{\chi}_{\alpha\beta\mu} = \frac{1}{2}[\hat{\eta}_\mu \hat{\mathcal{E}}_{\alpha\beta} - \hat{\eta}_\beta \hat{\mathcal{E}}_{\alpha\mu}]$$

The component $\hat{\mathcal{E}}_{00}$ is the negative of the energy density of the system without sources, i.e. $\hat{\mathcal{E}}_{00} = -\hat{\mathcal{H}}_R = -\{\frac{1}{2}\hat{\rho}_R\hat{v}^2 + \hat{\rho}_R\mathcal{V}(\hat{x}) + \hat{\rho}_R\hat{\psi}\}$. The components $\hat{\mathcal{E}}_{k0}$ and $\hat{\mathcal{E}}_{0l}$ represent the energy density flux $\hat{\mathcal{E}}_{k0} = \hat{v}_i\hat{T}_{ik}$, and the canonical-momentum density $\hat{\mathcal{E}}_{0l} = -\hat{\rho}_R\hat{v}_i\hat{F}_{il} = \hat{P}_l$, respectively [5,20]. The components $\hat{\mathcal{E}}_{kl}$ give the negative of the Mandel stress tensor [5,21], $\hat{\mathcal{E}}_{kl} = \hat{T}^T_{ki}\hat{F}_{il} + \delta_{kl}\mathcal{L}_R = -\hat{b}'_{kl}$. This tensor generalizes the Eshelby stress tensor when growth is taken into account, and represent the canonical-momentum density flux. Similarly, the components $\hat{\Xi}_{0il}$ and $\hat{\Xi}_{kil}$ give the direct-dynamics angular momentum, $\hat{\Xi}_{0il} = \frac{1}{2}\{\hat{x}_l\hat{\rho}_R\hat{v}_i - \hat{x}_i\hat{\rho}_R\hat{v}_l\}$, and the angular momentum induced by the Piola-Kirchhoff stress tensor, $\hat{\Xi}_{kil} = \frac{1}{2}\{\hat{x}_l\hat{T}_{ik} - \hat{x}_i\hat{T}_{lk}\}$, while $\hat{\chi}_{0il} = \frac{1}{2}\{\hat{P}_i\hat{\eta}_l - \hat{P}_l\hat{\eta}_i\}$ and $\hat{\chi}_{kil} = \frac{1}{2}\{-\hat{\eta}_l\hat{b}'_{ik} + \hat{\eta}_i\hat{b}'_{lk}\}$ provide the inverse-dynamics angular momentum and the angular momentum induced by the Mandel stress tensor, respectively. Noether's current, written in Eq.(4.7), is split into the superposition of two independent contributions: the translational contribution is $\hat{\mathcal{J}}^{tr}_\alpha = \hat{\mathcal{E}}_{\alpha\beta}\hat{w}_\beta$, and the rotational contribution is $\hat{\mathcal{J}}^{rot}_\alpha = (\hat{\Xi}_{\alpha il}\hat{R}_{il} + \hat{\chi}_{\alpha\beta\mu}\hat{\mathcal{R}}_{\beta\mu})$. Therefore, rather than calculating the divergence $\partial_\alpha\hat{\mathcal{J}}_\alpha$, we investigate the contributions $\partial_\alpha\hat{\mathcal{J}}^{tr}_\alpha$ and $\partial_\alpha\hat{\mathcal{J}}^{rot}_\alpha$ separately.

The divergence $\partial_\alpha\hat{\mathcal{J}}^{tr}_\alpha$ yields:

$$\partial_\alpha\hat{\mathcal{J}}^{tr}_\alpha = \partial_\alpha(\hat{\mathcal{E}}_{\alpha\beta}\hat{w}_\beta) = \partial_\alpha(\hat{\mathcal{E}}_{\alpha 0}\hat{w}_0) + \partial_\alpha(\hat{\mathcal{E}}_{\alpha l}\hat{w}_l) \tag{4.9}$$

where the term $\partial_\alpha(\hat{\mathcal{E}}_{\alpha 0}\hat{w}_0)$ is given by:

$$\partial_\alpha(\hat{\mathcal{E}}_{\alpha 0}\hat{w}_0) = [-(\partial_t\hat{\mathcal{H}}_R) + \partial_k(\hat{v}_i\hat{T}_{ik})]\hat{w}_0 - \hat{\mathcal{H}}_R(\partial_t\hat{w}_0) + (\hat{v}_i\hat{T}_{ik})(\partial_k\hat{w}_0). \tag{4.10}$$

The time-derivative of $\hat{\mathcal{H}}_R$ is $\partial_t\hat{\mathcal{H}}_R = \hat{\Pi}\hat{\mathcal{H}}'_R + \hat{\rho}_R(\partial_t\hat{v}_i)\hat{v}_i - \hat{\rho}_R\hat{f}_i\hat{v}_i +$

$\hat{T}_{ik}(\partial_t \hat{F}_{ik})$, where $\hat{\mathcal{H}}'_R$ is given by $\hat{\mathcal{H}}'_R = \frac{1}{2}\hat{v}^2 + \mathcal{V}(\hat{x}) + \hat{\psi}$. By substituting $\partial_t \hat{\mathcal{H}}_R$ into Eq.(4.10), and applying Schwartz's Theorem, after some manipulations, we achieve:

$$\partial_\alpha(\hat{\mathcal{E}}_{\alpha 0}\hat{w}_0) = -\hat{\Pi}\hat{\mathcal{H}}'_R\hat{w}_0 - \hat{\mathcal{H}}_R(\partial_t \hat{w}_0) + (\hat{v}_i \hat{T}_{ik})(\partial_k \hat{w}_0). \tag{4.11}$$

By noticing that $\hat{\Pi}\hat{\mathcal{H}}'_R = \hat{\Pi}\hat{v}_i\hat{v}_i - \hat{\Pi}\mathcal{L}'_R$, where $\mathcal{L}'_R = \frac{1}{2}\hat{v}_i\hat{v}_i - \mathcal{V}(\hat{x}) - \hat{\psi}$, Eq.(4.11) can be rearranged as:

$$\partial_\alpha(\hat{\mathcal{E}}_{\alpha 0}\hat{w}_0) = -\hat{\Pi}\hat{v}_i\hat{v}_i\hat{w}_0 + \hat{\Pi}\mathcal{L}'_R\hat{w}_0 - \hat{\mathcal{H}}_R(\partial_t \hat{w}_0) + (\hat{v}_i \hat{T}_{ik})(\partial_k \hat{w}_0). \tag{4.12}$$

The term $\partial_\alpha(\hat{\mathcal{E}}_{\alpha l}\hat{w}_l)$ in Eq.(4.9) is given by:

$$\partial_\alpha(\hat{\mathcal{E}}_{\alpha l}\hat{w}_l) = (\partial_t \hat{P}_l - \partial_k \hat{b}'_{kl})\hat{w}_l + \hat{P}_l(\partial_t \hat{w}_l) - \hat{b}'_{kl}(\partial_k \hat{w}_l). \tag{4.13}$$

The time-derivative of the canonical momentum density, and the divergence of the generalized Mandel stress tensor, are such that:

$$\begin{aligned}(\partial_t \hat{P}_l - \partial_k \hat{b}'_{kl}) = &-(\partial_t \hat{\rho}_R)\hat{v}_i\hat{F}_{il} - \hat{\rho}_R(\partial_t \hat{v}_i)\hat{F}_{il} - \hat{\rho}_R\hat{v}_i(\partial_t \hat{F}_{il}) + \\ &+(\partial_k \hat{T}_{ik})\hat{F}_{il} + \hat{T}_{ik}(\partial_k \hat{F}_{il}) + \delta_{kl}\left[(\partial_k \mathcal{L}_R)_{\mathrm{exp}} + \right. \\ &\left. +\hat{\rho}_R\hat{v}_i(\partial_k \hat{v}_i) + \hat{\rho}_R\hat{f}_i(\partial_k \hat{x}_i) - \hat{T}_{nm}(\partial_k \hat{F}_{nm})\right]\end{aligned} \tag{4.14}$$

The derivative $(\partial_k \mathcal{L}_R)_{\mathrm{exp}}$ is named *explicit gradient* of the Lagrangian density function [5]. By invoking Eq.(2.1) and Schwartz's Theorem, Eq.(4.14) can be rearranged as:

$$(\partial_t \hat{P}_l - \partial_k \hat{b}'_{kl}) = -\hat{\Pi}\hat{v}_i\hat{F}_{il} + (\partial_l \mathcal{L}_R)_{\mathrm{exp}} \tag{4.15}$$

The equation (4.13) can be written as:

$$\partial_\alpha(\hat{\mathcal{E}}_{\alpha l}\hat{w}_l) = -\hat{\Pi}\hat{v}_i\hat{F}_{il}\hat{w}_l + (\partial_l\mathcal{L}_R)_{\exp}\hat{w}_l + \hat{P}_l(\partial_t\hat{w}_l) - \hat{b}'_{kl}(\partial_k\hat{w}_l). \quad (4.16)$$

Finally, the divergence of $\hat{\mathcal{J}}^{tr}_\alpha = \hat{\mathcal{E}}_{\alpha\beta}\hat{w}_\beta$ is equal to:

$$\begin{aligned}\partial_\alpha\hat{\mathcal{J}}^{tr}_\alpha = \partial_\alpha(\hat{\mathcal{E}}_{\alpha 0}\hat{w}_0) + \partial_\alpha(\hat{\mathcal{E}}_{\alpha l}\hat{w}_l) = -\hat{\Pi}\hat{v}_i\hat{v}_i\hat{w}_0 + \hat{\Pi}\mathcal{L}'_R\hat{w}_0 - \\ \hat{\mathcal{H}}_R(\partial_t\hat{w}_0) + (\hat{v}_i\hat{T}_{ik})(\partial_k\hat{w}_0) - \hat{\Pi}\hat{v}_i\hat{F}_{il}\hat{w}_l + \\ (\partial_l\mathcal{L}_R)_{\exp}\hat{w}_l + \hat{P}_l(\partial_t\hat{w}_l) - \hat{b}'_{kl}(\partial_k\hat{w}_l).\end{aligned} \quad (4.17)$$

The right-hand side of Eq.(4.17) can be manipulated by requiring the fields $\hat{w}_0$ and $\hat{w}_l$ to solve the following equations:

$$\hat{\Pi}\mathcal{L}'_R\hat{w}_0 - \hat{\mathcal{H}}_R(\partial_t\hat{w}_0) + (\hat{v}_i\hat{T}_{ik})(\partial_k\hat{w}_0) = 0 \quad (4.18)$$

$$(\partial_l\mathcal{L}_R)_{\exp}\hat{w}_l + \hat{P}_l(\partial_t\hat{w}_l) - \hat{b}'_{kl}(\partial_k\hat{w}_l) = 0. \quad (4.19)$$

If $\hat{w}'_0$ and $\hat{w}'_l$ are the solutions to Eqs.(4.18) and (4.19), respectively, Eq.(4.17) becomes:

$$\begin{aligned}\partial_\alpha\hat{\mathcal{J}}'^{tr}_\alpha = \partial_\alpha(\hat{\mathcal{E}}_{\alpha 0}\hat{w}'_0) + \partial_\alpha(\hat{\mathcal{E}}_{\alpha l}\hat{w}'_l) = \\ -\hat{\Pi}\hat{v}_i\hat{v}_i\hat{w}'_0 - \hat{\Pi}\hat{v}_i\hat{F}_{il}\hat{w}'_l = -\hat{\Pi}\hat{v}_i(\partial_\beta\hat{x}_i)\hat{w}'_\beta.\end{aligned} \quad (4.20)$$

The divergence of $\hat{\mathcal{J}}^{rot}_\alpha = (\hat{\Xi}_{\alpha il}\hat{R}_{il} + \hat{\chi}_{\alpha\beta\mu}\hat{\mathcal{R}}_{\beta\mu})$ yields:

$$\begin{aligned}\partial_\alpha\hat{\mathcal{J}}^{rot}_\alpha = [\partial_t\hat{\Xi}_{0il} + \partial_t\hat{\chi}_{0il} + \partial_k\hat{\Xi}_{kil} + \partial_k\hat{\chi}_{kil}]\hat{R}_{il} + \\ (\hat{\Xi}_{0il} + \hat{\chi}_{0il})(\partial_t\hat{R}_{il}) + (\hat{\Xi}_{kil} + \hat{\chi}_{kil})(\partial_k\hat{R}_{il}).\end{aligned} \quad (4.21)$$

By substituting the explicit forms of $\hat{\Xi}_{\alpha il}$ and $\hat{\chi}_{\alpha\beta\mu}$, Eq.(4.21) becomes:

$$\begin{aligned}\partial_\alpha \hat{\mathcal{J}}^{rot}_\alpha = \frac{1}{2}\{[\hat{\rho}_R\hat{f}_i\hat{x}_l - \hat{\rho}_R\hat{f}_l\hat{x}_i] + [(\partial_i\mathcal{L}_R)_{\exp}\hat{\eta}_l - (\partial_l\mathcal{L}_R)_{\exp}\hat{\eta}_i] + \\ [\hat{T}_{lk}\hat{F}_{ik} - \hat{T}_{ik}\hat{F}_{lk}] + [\hat{b}'_{il} - \hat{b}'_{li}]\}\hat{R}_{il} + \frac{1}{2}[(\hat{\rho}_R\hat{v}_i\hat{x}_l - \hat{\rho}_R\hat{v}_l\hat{x}_i) + \\ (\hat{P}_i\hat{\eta}_l - \hat{P}_l\hat{\eta}_i)](\partial_t\hat{R}_{il}) + \frac{1}{2}[(\hat{T}_{lk}\hat{x}_i - \hat{T}_{ik}\hat{x}_l) + (\hat{b}'_{kl}\hat{\eta}_i - \hat{b}'_{ki}\hat{\eta}_l)] \\ (\partial_k\hat{R}_{il}) + \frac{1}{2}[\hat{\Pi}\hat{v}_i\hat{x}_l - \hat{\Pi}\hat{v}_l\hat{x}_i]\hat{R}_{il} + \frac{1}{2}[-\hat{\Pi}\hat{v}_j\hat{F}_{ji}\hat{\eta}_l + \hat{\Pi}\hat{v}_j\hat{F}_{jl}\hat{\eta}_i]\hat{R}_{il}.\end{aligned} \tag{4.22}$$

The right-hand side of Eq.(4.22) can be modified by requiring the field $\hat{R}_{il}$ to satisfy the equation:

$$\begin{aligned}&\frac{1}{2}\{[\hat{\rho}_R\hat{f}_i\hat{x}_l - \hat{\rho}_R\hat{f}_l\hat{x}_i] + [(\partial_i\mathcal{L}_R)_{\exp}\hat{\eta}_l - (\partial_l\mathcal{L}_R)_{\exp}\hat{\eta}_i] + \\ &+[\hat{T}_{lk}\hat{F}_{ik} - \hat{T}_{ik}\hat{F}_{lk}] + [\hat{b}'_{il} - \hat{b}'_{li}]\}\hat{R}_{il} + \\ &+\frac{1}{2}[(\hat{\rho}_R\hat{v}_i\hat{x}_l - \hat{\rho}_R\hat{v}_l\hat{x}_i) + (\hat{P}_i\hat{\eta}_l - \hat{P}_l\hat{\eta}_i)](\partial_t\hat{R}_{il}) + \\ &+\frac{1}{2}[(\hat{T}_{lk}\hat{x}_i - \hat{T}_{ik}\hat{x}_l) + (\hat{b}'_{kl}\hat{\eta}_i - \hat{b}'_{ki}\hat{\eta}_l)](\partial_k\hat{R}_{il}) = 0.\end{aligned} \tag{4.23}$$

By indicating with $\hat{R}'_{il}$ the solution to Eq.(4.23) and applying Eq.(4.6), Eq.(4.22) reduces to:

$$\begin{aligned}\partial_\alpha \hat{\mathcal{J}}'^{rot}_\alpha = \tfrac{1}{2}[\hat{\Pi}\hat{v}_i\hat{x}_l - \hat{\Pi}\hat{v}_l\hat{x}_i]\hat{R}'_{il} + \tfrac{1}{2}[-\hat{\Pi}\hat{v}_j\hat{F}_{ji}\hat{\eta}_l + \hat{\Pi}\hat{v}_j\hat{F}_{jl}\hat{\eta}_i]\hat{R}'_{il} = \\ \hat{\Pi}\hat{v}_i\hat{R}'_{il}\hat{x}_l - \hat{\Pi}\hat{v}_j\hat{F}_{ji}\hat{R}'_{il}\hat{\eta}_l = \hat{\Pi}\hat{v}_i(\mu\hat{\xi}'_i) + \hat{\Pi}\hat{v}_j(\partial_\beta\hat{x}_j)\hat{w}'_\beta.\end{aligned} \tag{4.24}$$

Finally, by summing Eqs.(4.20) and (4.24) the divergence of $\hat{\mathcal{J}}'_\alpha$ becomes:

$$\partial_\alpha \hat{\mathcal{J}}'_\alpha = \hat{\Pi}\hat{v}_i(\mu\hat{\xi}'_i) + \hat{\Pi}\hat{v}_j(\partial_\beta \hat{x}_j)\hat{w}'_\beta - \hat{\Pi}\hat{v}_i(\partial_\beta \hat{x}_i)\hat{w}'_\beta = \hat{\Pi}\hat{v}_i(\mu\hat{\xi}'_i). \quad (4.25)$$

The quantity $\hat{\Pi}\hat{v}_i(\mu\hat{\xi}'_i)$ is the *effective source term* to be considered when stating the continuity equation for the whole Noether's current. By suitable coordinate-transformations, this physical quantity has been reduced to the virtual work density exerted by the reversible source of momentum.

We know from Noether's Theorem that the variation of Action yields [1]:

$$(\delta\mathcal{A})(\hat{x}) = \int_{\mathcal{I}} \{\int_{\mathcal{C}_0} [\frac{\partial \mathcal{L}_R}{\partial \hat{x}_i} - \partial_\alpha \frac{\partial \mathcal{L}_R}{\partial(\partial_\alpha \hat{x}_i)}] d\tau\} dt + \int_{\mathcal{I}} \{\int_{\mathcal{C}_0} \partial_\alpha \hat{\mathcal{J}}_\alpha d\tau\} dt =$$

$$= \int_{\mathcal{I}} \{\int_{\mathcal{C}_0} [-\partial_t(\hat{\rho}_R \hat{v}_i) + \hat{\rho}_R \hat{f}_i + \partial_k \hat{T}_{ik}] \mu\hat{\xi}_i d\tau\} dt + \int_{\mathcal{I}} \{\int_{\mathcal{C}_0} \partial_\alpha \hat{\mathcal{J}}_\alpha d\tau\} dt. \quad (4.26)$$

Since the first integrand does not satisfy the dynamic equation, the second integrand is equal to the sources, and these two terms do not compensate, in general, the variation, $(\delta\mathcal{A})(\hat{x})$, is not zero. In this sense we say that the symmetries of the system are broken [1]. Yet, by invoking the dynamic equation and Eq.(4.25), the variation of Action reduces to:

$$(\delta\mathcal{A})(\hat{x}) = \int_{\mathcal{I}} \{\int_{\mathcal{C}_0} -\hat{\Pi}\hat{v}_i \mu\hat{\xi}_i d\tau\} dt + \int_{\mathcal{I}} \{\int_{\mathcal{C}_0} \hat{\Pi}\hat{v}_i \mu\hat{\xi}'_i d\tau\} dt. \quad (4.27)$$

As the field $\mu\hat{\xi}_i$ is arbitrary, we can assume it to be equal to $\mu\hat{\xi}'_i$. Therefore, we obtain:

$$(\delta\mathcal{A})(\hat{x}) = \int_{\mathcal{I}} \{\int_{\mathcal{C}_0} -\hat{\Pi}\hat{v}_i \mu\hat{\xi}'_i d\tau\} dt + \int_{\mathcal{I}} \{\int_{\mathcal{C}_0} \hat{\Pi}\hat{v}_i \mu\hat{\xi}'_i d\tau\} dt = 0. \quad (4.28)$$

The vanishing of $(\delta\mathcal{A})(\hat{x})$ occurs for those fields, $\hat{w}'_0$, $\hat{w}'_l$ and $\hat{R}'_{il}$, such that Eqs. (4.18), (4.19), and (4.23) are satisfied. These equations must be respected in order to restore the broken symmetries of the system. The transformations $\hat{w}'_0$, $\hat{w}'_l$ and $\hat{R}'_{il}$ are the compensating fields.

4.4 Transversal time

The compensating field $\hat{w}'_0$ performs a translation on time (time, t, is transformed into $t^* = t + \hat{w}'_0(t, X_0)$). We named the interval of time $\hat{w}'_0(t, X_0) = t^* - t$ *transversal time*. Transversal time distinguishes between the time that measures the process of growth, and the time that measures the "true" dynamics of the system. As growth occurs, the admissible reference configuration of the body $\mathcal{C}^*_0$, determined by the field $\hat{\gamma}_k(t, \cdot)$, is parameterized by t, but time is unaffected by this transformation. On the other hand, the mass source acting within the body provides an energy source. Therefore, the energy of the system is not conserved, and time-symmetry is destroyed. This symmetry breaking is reflected by the inhomogeneity of time, but the map $\hat{w}'_0(t, \cdot)$ provides an instant of time, t^*, in which the time-symmetry is restored. As time is conjugated to energy so the quantity $\hat{w}'_0(t, X_0)$ can be regarded as a "measure" of the amount of energy carried by the incoming mass. Since in the Lagrangian view of Continuum Mechanics, the reference coordinates, X_0, are labels attached to the material evolving particles, the explicit presence of space-coordinates in $\hat{w}'_0(t, X_0)$ means that the symmetry breaking must be compensated for on each curve. Thus, while time t is associated with the dynamic evolution of the generic curve labeled by X_0, time $\hat{w}'_0(t, X_0)$ is a "measure" of growth.

4.5 A generalization of the transplant operator

The fields $\hat{w}'_l$ and $\hat{R}'_{il}$ allow to fix a transformation, $\hat{\gamma}'_k(t, \cdot) : \mathcal{C}_0 \to \mathcal{C}^*_0$, which displaces the reference configuration of the body into an admissible new one, in which the body symmetries are restored. This transformation may be regarded as a deformation, but it is not accomplished by the dynamical evolution of the system. Rather, it provides for an instantaneously "updated" reference configuration set, which could be assumed as referential for the dynamical evolution of the system. Note that the evolution of $\mathcal{C}_0$ into $\mathcal{C}^*_0$, interpreted as dynamical restoring of broken symmetries, is formally consistent with the concept of an evolving stress-free configuration [5,22,23,24], although methodologically distinct. By considering the field $\hat{x}_i(t,\cdot) : \mathcal{C}_0^* \to \mathrm{R}^3 | X_0^* \mapsto y_i = \hat{x}_i(t, X_0^*)$, and the composed map $\hat{x}_i(t,\cdot) \circ \hat{\gamma}'_k(t,\cdot)$, we obtain $y_i = \hat{x}_i(t, X_{0k}^*) = \hat{x}_i(t, \hat{\gamma}'_k(t, X_0))$, where $X_{0k}^* = \hat{\gamma}'_k(t, X_0)$. The calculation of the gradient of $[\hat{x}_i(t,\cdot) \circ \hat{\gamma}'_k(t,\cdot)]$ yields:

$$\frac{\partial \hat{x}_i}{\partial X_{0j}}(t, \hat{\gamma}_k(t, X_0)) = \left[\frac{\partial \hat{x}_i}{\partial X_{0k}^*}(t, X_l^*)\right]\left[\frac{\partial \hat{\gamma}_k}{\partial X_{0j}}(t, X_0)\right]. \tag{4.29}$$

Denoting by $\hat{F}_{ij}(t, \hat{\gamma}_k(t, X_0))$, $\hat{G}_{ik}(t, X_0^*)$ and $\mathbf{K}_{kj}^{-1}(t, X_0)$, the gradients

$$\begin{aligned}
\hat{F}_{ij}(t, \hat{\gamma}_k(t, X_0)) &= \frac{\partial \hat{x}_i}{\partial X_{0j}}(t, \hat{\gamma}_k(t, X_0)),\\
\hat{G}_{ik}(t, X_0^*) &= \frac{\partial \hat{x}_i}{\partial X_{0k}^*}(t, X_0^*),\\
\mathbf{K}_{kj}^{-1}(t, X_0) &= \frac{\partial \hat{\gamma}_k}{\partial X_{0j}}(t, X_0)
\end{aligned} \tag{4.30}$$

we rearrange Eq.(4.29) in the form:

$$\begin{aligned}
&\hat{F}_{ij}(t, \hat{\gamma}_k(t, X_0)) = \hat{G}_{ik}(t, X_0^*)\mathbf{K}_{kj}^{-1}(t, X_0) \Rightarrow\\
&\hat{G}_{ik}(t, X_0^*) = \hat{F}_{ij}(t, \hat{\gamma}_k(t, X_0))\mathbf{K}_{jk}(t, X_0).
\end{aligned} \tag{4.31}$$

The tensor map $\mathbf{K}_{kj}^{-1}(t, X_0)$ is formally equivalent to the *transplant operator* found in the literature [5,6]. In the cited cases, the transplant operator was defined as a linear operator "connecting" the reference crystal (i.e. a fixed undistorted stress-free configuration of a material point, and not a reference configuration of a material body) with the reference configuration of the body. We believe that, within this first-order theory, a generalization of the concept of transplant operator can be given, because the reference crystal is substituted by a time-varying reference configuration in which the symmetries broken by growth are continuously restored. Recalling that $\hat{\gamma}_i(t, X_0) = X_{0i} + \hat{R}_{ij}(t, X_0)X_{0j} + \hat{w}_i(t, X_0)$, the proposed generalization of the transplant operator is given by:

$$\begin{aligned}\mathbf{K}^{-1}_{ik}(t,X_0) = (\partial_k\hat{\gamma}_i)(t,X_0) = \\ \delta_{ik} + [(\partial_k\hat{R}_{ij})(t,X_0)]X_{0j} + \\ \hat{R}_{ik}(t,X_0) + (\partial_k\hat{w}_i)(t,X_0).\end{aligned} \tag{4.32}$$

The last three terms on the right-hand side of Eq.(4.32) can be interpreted as time-dependent first-order corrections to the unit tensor, supplied by the fields $\hat{R}_{ij}$ and $\hat{w}_i$ which solve Eqs.(4.19) and (4.23). The idea of writing the transplant operator as the gradient of a smooth change of reference configuration was present in the work by Epstein and Maugin [5]; note that in our work, the fields that allow for calculating $\mathbf{K}^{-1}_{ik}$ are found by restoring the symmetries broken by growth.

5 Conclusions

The equations (4.18), (4.19) and (4.23) are the conditions that the transformations on time and reference coordinates must respect in order to restore the symmetries broken by the development of material inhomogeneities due to growth. Although dynamics plays a central role in these equations, they are not dynamic equations. They constitute a set of linear and homogeneous equations whose solutions provide a field transforming the original configuration of the body into a new admissible one. Such a configuration is an instantaneous reference configuration in which the symmetries of the system are re-established. This result "translates" the idea of describing growth as the evolution of the stress-free configuration in the language of symmetries. The fields $\hat{w}'_0$, $\hat{w}'_l$ and $\hat{R}'_{il}$ are not uniquely determined by Eqs.(4.18), (4.19) and (4.23). Actually, while $\hat{w}'_0$ is a scalar field found via a scalar equation, $\hat{w}'_l$ and $\hat{R}'_{il}$ have three independent components that must be determined by solving two independent scalar and homogeneous equations. Hence, each equation admits two undetermined variables which can be chosen arbitrarily, regardless of the initial and boundary conditions that must be associated with each equation. Thus, Eqs.(4.18), (4.19) and (4.23) provide a class of equivalence of transformations. Each coordinate-transformation belonging to this class compensates for the broken symmetries, is such that the variation of Action vanishes, and contains the class of transformations that solve the variational problem. As a consequence to this indetermination, we find a family of equivalent admissible reference configurations. Since in each admissible reference configuration the symmetries of the system are continuously restored, we believe that $\hat{w}'_l$ and $\hat{R}'_{il}$ allow for a generalization of the concept of transplant operator [5,6].

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Uspostavljenje simetrije narušene reverzibilnim rastom u hiperelastičnim telima

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U ovom radu pomoću narušene simetrije interpretiramo razvoj materijalnih nehomogenosti u neprekidnim hiperelastičnim telima [1]. Primenom teoreme Emi Neter [1,2,3,4] nalazimo skup jednačina iz kojih slede polja potrebna za kompenzaciju narušene simetrije. Pri rastu ova polja nam daju trenutnu promenljivu referentnu konfiguraciju tela, a ona su odgovorna i za dinamičko ene uspostavljanje simetrije kompenzacionih polja u cilju uopštenja definicije operatora transplanta datog u [5,6]. Ovaj rad je motivisan nedavnim teorijskim istraživanjima o biomehaničkim aspektima rasta u zglobnoj hrskavici.