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Properties of the first eigenfunctions of the clamped column equation

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Abstract

We show that the clamped column equation may not possess a positive first eigenfunction. This result discovers the anomalies of some papers in determining the shape of the strongest clamped-clamped column.

Key words. eigenvalue, clamped column equation, optimal shape

1 Introduction

Let $\sigma \in L^{\infty}(0,1)$ be a nonnegative function and consider the fourth order linear differential equation

$$(\sigma(x)u'')'' + \lambda u'' = 0, \qquad 0 < x < 1, \tag{1}$$

subject to the boundary conditions

$$u(0) = u'(0) = 0, \qquad u(1) = u'(1) = 0. \tag{2}$$

This boundary value problem describes the equilibrium of an axially symmetric column clamped at the extremities $x = 0$ and $x = 1$ and

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having a cross sectional area equal to $\sigma(x)$. We denote by U the set of all functions in $L^\infty(0,1)$ bounded below by positive constants, that is

$$U = \{\sigma \in L^\infty(0,1), \exists\, h > 0 \;\text{ such that } \sigma(x) \geq h \;\; a.e. \text{ in } (0,1)\}.$$

The weak formulation of the clamped column equation for $\sigma \in U$ is

$$\int_0^1 \sigma u'' v''\, dx = \lambda \int_0^1 u'v'\, dx, \qquad \forall v \in H_0^2(0,1). \tag{3}$$

For each $\sigma \in U$, its spectrum consists of a sequence of positive eigenvalues $\{\lambda_k\}$ which, counted with their multiplicities, can be arranged as follows [4]

$$0 < \lambda_1(\sigma) \leq \lambda_2(\sigma) \leq \lambda_3(\sigma) \leq \cdots \to +\infty.$$

These eigenvalues correspond to a sequence of eigenfunctions $\{u_k(\sigma)\} \subset H_0^2(0,1)$, which can be chosen orthonormal with respect to the bilinear form associated with the right side of (3). It is not difficult to show that the multiplicity of each eigenvalue of (1)-(2) is at most two, see for instance [7] or [4].

By employing the change of variable $w = \sigma u''$ [12], Equation (1) becomes

$$w'' + \lambda\sigma^{-1} w = 0, \tag{4}$$

If we integrate twice the following equation $w'' = -\lambda u''$, and use the clamped conditions imposed on u at $x = 0$, we obtain the following relations

$$u'(x) = [w'(0) - w'(x)]/\lambda, \qquad u(x) = [xw'(0) - w(x) + w(0)]/\lambda. \tag{5}$$

By using the clamped conditions at $x = 1$, we derive the boundary conditions for w

$$w(0) = w'(1), \qquad w(1) - w(0) = w'(1). \tag{6}$$

Equation (4) and boundary conditions (6) form a self-adjoint eigenvalue problem for the determination of λ. This eigenvalue problem admits an infinite sequence of nonnegative eigenvalues increasing without limit. Once an eigenvalue $\lambda \neq 0$ and an associated eigenfunction $w(x)$ are found, we can use (5) to uniquely determine an eigenfunction $u(x)$ associated with λ, and vice versa. However, the two problems in question are not completely equivalent for the reason that the second one has a double zero eigenvalue with associated eigenfunctions $w(x) = 1$ and $w(x) = 1 - x$. The first eigenvalue $\lambda_1(\sigma)$ requires therefore to be equal to the smallest positive (i.e. *third*) eigenvalue of (4)-(6), see [11].

In [4], Cox and Overton claim in Theorem 2.2 that if $\sigma \in U$ is even (about 1/2), i.e., $\sigma(x) = \sigma(1-x)$, then there exists a positive even eigenfunction corresponding to $\lambda_1(\sigma)$. The idea of their proof is inspired from a finite-dimentional technique for computing the least eigenvalue and eigenvector of symmetric matrix. They tried then to approximate the least eigenfunction $u_1(x)$ by a solution of a related non-homogeneous boundary value problem. For a given function $v_0 \in H_0^2(0,1)$ they considered its expansion in the complete set of eigenfunctions $\{u_k\}$,

$$v_0(x) = \bar{v}(x) + \sum_{k=m+1}^{\infty} a_k u_k(x),$$

where m is the least integer for which $\lambda_1(\sigma) < \lambda_{m+1}(\sigma)$ and $\bar{v}$ is an eigenfunction corresponding to $\lambda_1(\sigma)$. Next they produced a function v_0 whose corresponding $\bar{v}$ is even. However, the remainder of the proof assume implicitely that $\bar{v} \neq 0$, which is not true if the eigenfunctions corresponding $\lambda_1(\sigma)$ are all odd. Cox and Overton's claim has been implicitely used in [2] and later in [5], and led to erroneous results. The aim of this paper is to show that symmetric clamped-clamped columns may not possess positive first eigenfunctions. To our knowledge this fact was missing in all papers dealing with the optimal shape of the clamped-clamped column.

2 Oscillation of the first eigenfunction

Let F be the functional defined by

$$F[\sigma, u] = \int_0^1 \sigma u''^2 \, dx \Big/ \int_0^1 u'^2 \, dx. \tag{7}$$

According to a variational principle, $\lambda_1(\sigma)$ can be represented as

$$\lambda_1(\sigma) = \inf_u F[\sigma, u],$$

where the inf is taken over all functions u in $H_0^2(0,1)$. It is easy to verify that $\lambda_1(\sigma)$ can also be found as

$$\lambda_1(\sigma) = \inf_u G[\sigma, u], \tag{8}$$

where G is defined by

$$G[\sigma, u] = \int_0^1 \sigma u'^2 \, dx \Big/ \int_0^1 u^2 \, dx,$$

and the inf is taken over the set $\mathcal{H}$ of all functions $u \in H_0^1(0,1)$ and satisfying the condition $\int_0^1 u \, dx = 0$. Notice than each element of $\mathcal{H}$ has at least one zero in $(0,1)$. Throughout the paper, we denote by $\mu_n(\sigma)$ the nth eigenvalue of the following problem

$$(\sigma v')' + \mu v = 0, \qquad 0 < x < 1. \tag{9}$$

$$v(0) = v(1) = 0. \tag{10}$$

Since $\mu_1(\sigma) = \inf_u G[\sigma, u]$, and the inf is taken over of $u \in H_0^1(0,1)$, it follows that $\mu_1(\sigma) < \lambda_1(\sigma)$. The next theorem provides an upper bound for $\lambda_1(\sigma)$.

Theorem 1. *Let σ be a member of U (σ is not necessarily even). Then we have*

$$\lambda_1(\sigma) \leq \mu_2(\sigma).$$

Proof. Let v_2 be an eigenfunction corresponding to $\mu_2(\sigma)$. Then it is known that v_2 has exactly one zero (call it a) in the interval $(0,1)$. Moreover, $\mu_2(\sigma)$ is the first eigenvalue of the two problems

$$(\sigma v')' + \mu v = 0, \qquad v(0) = v(a) = 0,$$

$$(\sigma v')' + \mu v = 0, \qquad v(a) = v(1) = 0,$$

Let $\bar{v}_2$ be the function equal to v_2 in $[0,a]$ and equal to ξv_2 in $[a,1]$, where the real number ξ is chosen such that $\int_0^a v_2 \, dx + \xi \int_a^1 v_2 \, dx = 0$.

That is $\int_0^1 \bar{v}_2\,dx = 0$. Since $\bar{v}_2$ is an eigenfunction corresponding to $\mu_2(\sigma)$ in both intervals $(0,a)$ and $(a,1)$, we have

$$\int_0^a \sigma \bar{v}_2'^2\,dx = \mu_2(\sigma)\int_0^a \bar{v}_2^2\,dx,$$

and

$$\int_a^1 \sigma \bar{v}_2'^2\,dx = \mu_2(\sigma)\int_a^1 \bar{v}_2^2\,dx.$$

By summing, we obtain $\int_0^1 \sigma\bar{v}_2'^2\,dx = \mu_2(\sigma)\int_0^1 \bar{v}_2^2\,dx$. Therefore, in view of (8), we have $\lambda_1(\sigma) \le \mu_2(\sigma)$. $\square$

Notice that when $\xi \neq 1$, the function $\bar{v}_2$ is not an eigenfunction of (9). In such a case, we have $\lambda_1(\sigma) < \mu_2(\sigma)$ as the function $\bar{v}_2$ is not an eigenfunction of (1)-(2). Notice however that the case $\xi = 1$ does not necessary imply that $\lambda_1(\sigma) = \mu_2(\sigma)$. It only informs us that $\mu_2(\sigma)$ is a member of the spectrum of (1)-(2) and the function $\int_0^x v_2\,dt$ is an associated eigenfunction. We will see later that when $\lambda_1(\sigma)$ is simple, $\mu_2(\sigma)$ may be equal to the second eigenvalue $\lambda_2(\sigma)$ of (1)-(2).

We also notice that the strict inequality $\lambda_1(\sigma) < \mu_2(\sigma)$ implies that the functional $G[\sigma,\cdot]$ attains its minimum over $\mathcal{H}$ at some function v satisfying the Euler-Lagrange equation $(\sigma v')' + \lambda_1(\sigma)v = C$, where C is a *nonzero* constant.

Theorem 2. *Let σ be an arbitary member of U. If $\lambda_1(\sigma)$ is double, then $\lambda_1(\sigma) = \mu_2(\sigma)$.*

Proof. If $\lambda_1(\sigma)$ is double, then there exists two linearly independent eigenfunctions u_1 and u_2 correponding to $\lambda_1(\sigma)$. Let $v_1 = u_1'$ and $v_2 = u_2'$. Then, v_1 and v_2 are linearly independent in $\mathcal{H}$, and satisfy

$$(\sigma v_1')' + \lambda_1(\sigma)v_1 = C_1,$$

$$(\sigma v_2')' + \lambda_1(\sigma)v_2 = C_2,$$

where C_1 and C_2 are constants. If $C_1C_2 = 0$, then as v_1 and v_2 change sign in (0,1), we have by Sturm's theorem $\lambda_1(\sigma) \ge \mu_2(\sigma)$. From Thereom 1 we deduce that $\lambda_1(\sigma) = \mu_2(\sigma)$. If $C_1C_2 \neq 0$, then the function $\hat{v} = v_1 - (C_1/C_2)v_2$ belongs to $\mathcal{H}$ and satisfies the equation $(\sigma\hat{v}')' + \lambda_1(\sigma)\hat{v} = 0$. From Sturm's theorem and Theorem 1, we deduce again that $\lambda_1(\sigma) = \mu_2(\sigma)$. $\square$

We notice that the equality $\lambda_1(\sigma) = \mu_2(\sigma)$ does not imply that $\lambda_1(\sigma)$ is double. This is the case for the uniform column ($\sigma \equiv 1$). In what follows, we examine in detail the case where σ is even. We have the following lemma.

Lemma 1. *Let $\sigma \in U$ be an even function. If the eigenfunctions associated with $\lambda_1(\sigma)$ are not all odd, then there exists, up to a scalar multiple,a unique even eigenfunction associated with $\lambda_1(\sigma)$. This eigenfunction is positive and symmetrically decreasing.*

Proof. Assume that there is an eigenfunction u associated with $\lambda_1(\sigma)$, and which is not odd, i.e., $u(x) \neq -u(1-x)$. Then, the function ψ defined by $\psi(x) = u(x) + u(1-x)$ is a member of $H_0^2(0,1)$ and is not identically zero. As σ is even, ψ is also an eigenfunction to $\lambda_1(\sigma)$. Let us define a new function $\bar{u}$ as follows: $\bar{u} = \int_0^x v(t)\,dt$, where

$$v(t) = \begin{cases} |\psi'(x)| & \text{for } 0 \leq x \leq 1/2, \\ -|\psi'(x)| & \text{for } 1/2 < x \leq 1. \end{cases}$$

Then $v \in H_0^1(0,1)$, v is odd and the integral $\int_0^1 v(t)\,dt$ is zero. It follows that $\bar{u}$ is even, increasing on the interval $(0,1/2)$ (i.e. positive in $(0,1)$) and belongs to $H_0^2(0,1)$. On the other hand, since $F[\sigma,u] = F[\sigma,\bar{u}]$, we deduce that $\bar{u}$ is an eigenfunction associated with $\lambda_1(\sigma)$. Assume now that $\tilde{u}$ is another even first eigenfunction. As $\bar{u}'$ and $\tilde{u}'$ are odd, they are both solutions of the problem

$$(\sigma v')' + \lambda_1(\sigma)v = 0, \qquad v(0) = v(1) = 0.$$

Since the eigenvalues of this problem are simple, $\bar{u}'$ and $\tilde{u}'$ are linearly dependent, and so $\bar{u}$ and $\tilde{u}$ are. $\square$

The following lemma provides a necessary and sufficient condition for the equality $\lambda_1(\sigma) = \mu_2(\sigma)$.

Lemma 2. *Let $\sigma \in U$ be an even function. Then $\lambda_1(\sigma) = \mu_2(\sigma)$ if and only if $\lambda_1(\sigma)$ admits an even eigenfunction.*

Proof. Since σ is even, $\mu_2(\sigma)$ admits an odd eigenfunction v_2, and therefore the function u_1 defined by $u_1 = \int_0^x v_2\,dt \in H_0^2(0,1)$, is even and satisfies

$$(\sigma u_1'')'' + \mu_2(\sigma)u_1'' = 0.$$

If $\lambda_1(\sigma) = \mu_2(\sigma)$, then u_1 is a first eigenfunction of (1)-(2) corresponding to $\lambda_1(\sigma)$. Conversely, if u_1 is an even first eigenfunction corresponding to $\lambda_1(\sigma)$, then $(\sigma u_1'')' + \lambda_1(\sigma)u_1' = C$, for some constant C. As u_1' is odd, C is zero and therefore $\lambda_1(\sigma) \leq \mu_2(\sigma)$. By Theorem 1, we conclude that $\lambda_1(\sigma) = \mu_2(\sigma)$. $\square$

It follows from this lemma that, if $\lambda_1(\sigma) < \mu_2(\sigma)$ then $\lambda_1(\sigma)$ is simple and admits an odd eigenfunction.

In [4], Cox and Overton considered the problem of maximizing the first eigenvalue of $\lambda_1(\sigma)$ under the conditions

$$0 < a < \sigma < b < \infty, \qquad \int_0^1 \sigma^\alpha(x)\,dx = 1,$$

where a, b and α are given numbers. In Theorem 2.2, they claimed that, if $\sigma \in U$ is even, then there exists a positive even eigenfunction corresponding to $\lambda_1(\sigma)$. They also proved, basing on this theorem, that for each $\sigma \in U$ there exists an even function $\sigma^* \in U$ satisfying the above conditions and such that $\lambda_1(\sigma) \leq \lambda_1(\sigma^*)$. We notice that this result can also be proved independent of Theorem 2.2, using different approaches. According to Lemma 2, if Cox and Overton's claim is true, then the problem of maximizing $\lambda_1(\sigma)$ will be reduced to the one of maximizing $\mu_2(\sigma)$, or equivalently, to the problem of maximizing the first eigenvalue of

$$(\sigma v')' + \mu v = 0, \qquad 0 < x < 1/2, \tag{11}$$

$$v(0) = v(1/2) = 0, \tag{12}$$

under the conditions

$$0 < a < \sigma < b < \infty, \qquad \int_0^{1/2} \sigma^\alpha(x)\,dx = 1/2.$$

This is a classical problem since it involves only simple eigenvalues which vary smoothly with σ. It can also be proved that Cox and Overton's claim leads to the optimality of Tadjbakhsh and Keller's shape [12], which was shown to be wrong, numerically in [9], and analytically in [8], [11], [4] and [6].

Theorem 3. *Let σ be an even member of U. If $\lambda_1(\sigma)$ is double, then*

$$\lambda_1(\sigma) = \mu_2(\sigma),$$

and $\lambda_1(\sigma)$ admits, up to a scalar multiple, a unique odd and a unique even eigenfunction.

Proof. From the previous results, when $\lambda_1(\sigma)$ is double, $\lambda_1(\sigma) = \mu_2(\sigma)$, and according to Lemma 1, λ_1 admits an even eigenfunction u which is not even. From u, we may construct the odd first eigenfunction $\hat{u} = u(x) - u(1-x)$. This function satisfies the equation

$$(\sigma\hat{u}'') + \lambda_1(\sigma)\hat{u} = C(x - 1/2),$$

for some constant C. At $x = 1/2$, we obtain $(\sigma\hat{u}'')(1/2) = 0$ as $\hat{u}(1/2) = 0$. Hence, $\hat{u}$ restricted to the interval $(0, 1/2)$ is a solution for the follwoing problem

$$(\sigma u'')'' + \lambda_1(\sigma)u'' = 0, \qquad 0 < x < 1/2, \tag{13}$$

$$u(0) = u'(0) = 0, \qquad u(1/2) = (\sigma u'')(1/2) = 0. \tag{14}$$

Equation (13) with boundary conditions (14) describe the equilibrium of a column clamped at 0 and hinged at 1/2. To complete the proof, it suffices to show that the eigenvalues of this boundary value problem are simple. For this, we introduce the change of variable $w = \sigma u''$ and we obtain the Sturm-Liouville problem

$$w'' + \lambda\sigma^{-1}w = 0, \qquad 0 < x < 1/2, \tag{15}$$

$$2w(0) + w'(0) = 0, \qquad w(1/2) = 0, \tag{16}$$

having only simple eigenvalues, and zero as a first eigenvalue, with a corresponding eigenfunction $w_1(x) = 1/2 - x$. This completes the proof. □

Theorem 3 states that when $\sigma \in U$ is even and $\lambda_1(\sigma)$ is double, $\lambda_1(\sigma)$ is equal to $\mu_2(\sigma)$ and also equal to the second eigenvalue of (15)-(16). Recall that $\mu_2(\sigma)$ is equal to the first eigenvalue of (11)-(12) and also to the second eigenvalue $\nu_{(2)}(\sigma)$ of

$$w'' + \nu\sigma^{-1}w = 0, \qquad 0 < x < 1/2, \tag{17}$$

$$w'(0) = 0, \qquad w'(1/2) = 0. \tag{18}$$

Now we address the crucial question. When σ is even and $\lambda_1(\sigma)$ is simple, is an odd first eigenfunction expected? The answer is however positive. From our analysis, we see that the possibility of having a first eigenfunction which changes sign in $(0, 1)$, is due to the fact that we cannot in general compare the second eigenvalue of problem (15)-(16), which we denote by $\lambda_{(2)}(\sigma)$, and the second one of problem (17)-(18). Consider for instance the case where

$$\sigma(x) = \left(x - \frac{1}{4}\right)^2 + 0.1$$

on $[0, 1/2]$. Then, numerical computations using COLSYS [1] give us $\lambda_{(2)}(\sigma) = 9.465$ and $\nu_{(2)}(\sigma) = 5.175$, and corresponding eigenfunctions are displayed in Figure 1.

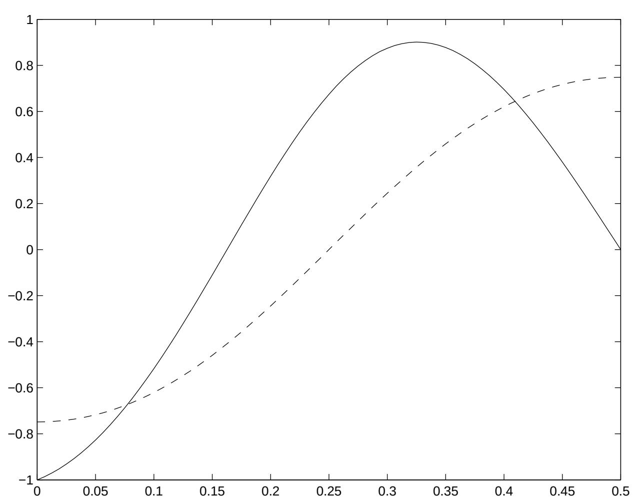

Figure 1: Eigenmodes corresponding to $\lambda_{(2)}$ (solid line) and $\nu_{(2)}$ (dashed line).

It follows that we have three different cases:

(i) if $\lambda_{(2)}(\sigma) > \nu_{(2)}(\sigma)$ then $\lambda_1(\sigma)$ is simple and $u_1(\sigma)$ is odd; $u_1(\sigma)$ restricted to the interval $(0, 1/2)$ is a first eigenfunction of (13)-(14).

(ii) if $\lambda_{(2)}(\sigma) < \nu_{(2)}(\sigma)$ then $\lambda_1(\sigma)$ is simple and $u_1(\sigma)$ is even; we have $\lambda_1(\sigma) = \mu_2(\sigma)$.

(iii) if $\lambda_{(2)}(\sigma) = \nu_{(2)}(\sigma)$ then $\lambda_1(\sigma)$ is double.
We have in other words

$$\lambda_1(\sigma) = \min\{\lambda_{(2)}(\sigma), \nu_{(2)}(\sigma)\}. \tag{19}$$

In case (i), it is possible to show that $\mu_2(\sigma)$, which is a member of the spectrum of problem (1)-(2), coincides with the second eigenvalue $\lambda_2(\sigma)$ of (1)-(2). Indeed, let u_2 be an eigenfunction associated with $\lambda_2(\sigma)$. If u_2 is even, then u_2' is a solution of $(\sigma v')' + \lambda_2(\sigma)v = 0,\ v(0) = v(1) = 0$, and since u_2' vanishes in $(0,1)$, we conclude that $\mu_2(\sigma) \le \lambda_2(\sigma)$.

If u_2 is odd, then $\lambda_2(\sigma)$ will be equal to the third eigenvalue $\lambda_{(3)}(\sigma)$ of problem (15)-(16). Recall that $\lambda_{(3)}(\sigma)$ is greater than the second eigenvalue $\mu_{(2)}(\sigma)$ of the boundary value problem

$$w'' + \mu\sigma^{-1}w = 0, \qquad 0 < x < 1/2,$$

$$w(0) = w(1/2) = 0$$

see for instance [11], since the latter is equivalent, except for the presence a simple zero eigenvalue, to the following problem

$$(\sigma u'')'' + \lambda_1(\sigma)u'' = 0, \qquad 0 < x < 1/2,$$

$$u(0) = (\sigma u'')(0) = 0, \qquad u(1/2) = (\sigma u'')(1/2) = 0,$$

which describes the equilibrium of a column hinged at both extremities $x = 0$ and $x = 1/2$. On the other hand, since the inequalty $\mu_{(2)}(\sigma) \ge \nu_{(2)}(\sigma)$ holds for any $\sigma \in U$, we have

$$\mu_2(\sigma) = \nu_{(2)}(\sigma) \le \mu_{(2)}(\sigma) \le \lambda_{(3)}(\sigma) = \lambda_2(\sigma),$$

and finally, in view of Theorem 1, we have $\mu_2(\sigma) = \lambda_2(\sigma)$. It now follows that in case (i) the first eigenfunction of (1)-(2) changes sign in $(0,1)$ while the second eigenfunction is of one sign, which seems an unusual phenomenon. This explains in part why Tadjbakhsh and Keller's best clamped-clamped column has a much lower buckling load than $16\pi^2/3$, the value they claimed. This was first revealed in [9], as an evidence to the erroneous of Tadjbakhsh and Keller's optimal shape. According to our analysis, the value $16\pi^2/3$ corresponds to the second eigenvalue of the clamped column equation.

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Osobine prvih sopstvenih funkcija jednačine stuba slobodno oslonjenog na krajevima

UDK 534.16

Pokazano je da jednačina stuba slobodno oslonjenog na krajevima ne sme da ima pozitivnu prvu sopstvenu funkciju. Ovaj rezultat otkriva nepravilnosti nekih radova u odredjivanju oblika najjačeg stuba oslonjenog na krajevima.