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The Green's and the Eshelby's identities in generalised continua and in dielectrics *

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Abstract

In 1973, A. E. Green pointed out several interesting formulae, which hold true in finite elasticity [1]. One of them (formula (2.10), p.75) is repeatedly quoted in the literature as the *Green identity*. This remarkable identity has been successfully employed in several contexts. We only mention here its central role in theorems on uniqueness in elastostatics [2,3]. A deeper insight to the Green identity shows in evidence an intimate link of this formula with the Eshelby tensor and with the *material balance law* of equilibrium [4,5]. In homogeneous hyperelastic materials, this law turns out to an identity, the *Eshelby identity*, and one can easily prove that the Green identity stems straightforwardly from the Eshelby identity. These identities possibly extend to generalized continua, such as continua with microstructure and elastic dielectrics. Hereafter, the validity of the Eshelby identity is discussed for these materials. Basing on the novel extended Eshelby-like identity, the corresponding extended Green-like identity can be also established, under specific assumptions. In the case of dielectrics, two equivalent forms for the Eshelby tensor emerge from the treatment, both satisfying the Eshelby identity. One of them is more appropriate for deriving the desired Green-like identity. The second one, which is a reduced form of the first one, represents the physical Eshelby tensor in dielectrics [4-5].

Keywords: *Elasticity. Dielectrics. Microstructure.*

*Dedicated to the memory of Rastko Stojanovic.

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1 Introduction

The name of Eshelby is customarily associated with the definition of the force acting upon an elastic singularity [6,7]. This force is intimately related to a configurational force, which in turn addresses a tensor field. This tensor is known as the Eshelby stress or the Eshelby energy-momentum tensor. In the absence of elastic singularities or in homogeneous materials the aforementioned force identically vanishes, whereas the notion of Eshelby stress survives and this stress turns out to be divergence-free in the mechanical equilibrium. The latter property expresses the *Eshelby identity* in elasticity. Remarkably, this identity holds true either in linear or in finite elasticity. However, the Eshelby tensor also appears in a quite different context and in a slightly different form. The context is that of non-regular solutions in variational problems [8]. The form is that of a jump condition across a discontinuity surface for the first derivatives of the possible solution. This condition is known as the 'Erdmann-Weierstrass corner condition' for the variational equation and turns out to correspond to the Eshelby's identity across the discontinuity surface. All these remarks have stimulated a great interest in the Eshelby identity and many people wished to enquire on the very nature of the Eshelby tensor [1,4,5,9-17]. Based on the variational approach, most of the authors pointed out that the Eshelby tensor could be recovered by appealing to a Noether-like theorem for a continuum [9,10,13-15,17]. This result encouraged conceiving the Eshelby identity as a conservation law for homogeneous elastic materials.

This view, which was adopted by Knowles and Sternberg in their 1972's paper [9], happily finds a correspondence in the dynamical framework. In fact, in dynamics, the divergence of the Eshelby stress does not vanish, even in homogeneous materials, but rather balances the *material momentum* [4,5,12-15]. The notion of material momentum is meaningful in Solid State Physics, as it corresponds to the *crystal momentum*. It is also meaningful in optics, as it represents the *pseudo-momentum* of light [18]. In this view, should the configurational force (which is related to the divergence of the Eshelby stress) vanish, the *material momentum* would be conserved. The literature on this topic is very extensive and covers many fields of interest (phase transitions, electromagnetism, criteria on progressing cracks et cetera). The interested reader is addressed to the papers [4,5,12,13] for an exhaustive list of references. Here, we will be only concerned with some mathematical results, either in infinitesimal or finite elasticity, which have been proposed by Knowles and Sternberg, by Green and by Knops. Among the other results, Knowles and Sternberg established an interesting formula in infinitesimal elasticity (formula (3.19), p. 198, [9]). A. E. Green subsequently recovered the validity of such a formula in finite elasticity (formula (2.10), p. 75, [1]). In the quoted paper Green also proposed other useful formulas. However, the formula that extends the Knowles and Sternberg' result turned out to be very promising and is known in the literature as the *Green identity* [2]. An extension of the Green identity was first proposed for an elastic material in terms of the complementary energy [3]. Hereafter, a possible extension of the Green identity is investigated for generalised continua such as materials with microstructure [19] or nonlinear deformable dielectrics [10,11,20-24]. Dielectrics represent a special case of non-classical continua. In fact, though the polarisation is a field defined only in the domain occupied by the material, which is possibly of finite extent, the electric field invades the whole physical space. Hence, the mathematical problem needs to be stated in the whole space, which is currently understood as an Euclidean space.

An extension of the Green formula in nonlinear dielectrics is presented in section 5. This Green-like formula turns out to give also some interesting insight on the Eshelby tensor. In fact, several equivalent forms for the Eshelby tensors emerge in dielectrics, only one of which seems to lead straightforwardly to the extended Green formula.

Hopefully, all these Green-like formulae represent a useful tool for a deeper insight in mathematical problems. Specifically, for investigating on uniqueness theorems whose energy response is a non-convex one [2,3].

2 The Eshelby tensor and the Green identity in elastostatics

Hereafter, a stored energy density is assumed to exist for the hyperelastic body of interest. As usual, a reference configuration V is associated with this body [25]. W denotes the energy density per unit volume of this configuration. We also assume that the reference configuration is a natural global configuration in the Euclidean space E_3 [11,25]. The deformation from this configuration to the current configuration ν is described by the mapping $\boldsymbol{k} \in C^3(V, \mathcal{V})$. $\mathbf{X} \in V$, $\mathbf{x} \in \mathcal{V}$. $V, \mathcal{V} \subset E_3$. The deformation gradient is denoted by $\mathbf{F}$ and $(det\mathbf{F}) \equiv J$ is assumed to be strictly positive, as usual.

As in this context $W = \hat{W}(\mathbf{F})$, the Eshelby tensor reads [4-7]

$$\mathbf{b} = \hat{W}\mathbf{I} - \mathbf{F}^T(\partial\hat{W}/\partial\mathbf{F}). \tag{1}$$

With reference to the expression (1), the following identities hold true in equilibrium:

$$Div\ \mathbf{b} = 0, \tag{2}$$

$$Div\ (\mathbf{b}^T\mathbf{X}) = tr\ \mathbf{b}, \tag{3}$$

$$tr\ \mathbf{b} = 3\hat{W} - Div\ [(\partial\hat{W}/\partial\mathbf{F})^T\mathbf{x}]. \tag{4}$$

'Div' stands for the divergence operator in the reference configuration, whereas $(tr\ \mathbf{b})$ denotes the spur of the second order tensor $\mathbf{b}$.

The validity of the identities (2), (3) and (4) can be easily checked, by taking into account the Cauchy equilibrium equation, which reads

$$Div\ \mathbf{T}_R = 0. \tag{5}$$

$\mathbf{T}_R \equiv (\partial\hat{W}/\partial\mathbf{F})$ represents the 1^{st} Piola-Kirchhoff stress tensor [25].

By combining the formulae (3) and (4) and by re-arranging the terms, one can eventually write the following remarkable equality:

$$3\int_V \hat{W} dV = \int_{\partial V} \{\hat{W}(\mathbf{N}\cdot\mathbf{X}) + [(\partial\hat{W}/\partial\mathbf{F})\mathbf{N}]\cdot(\mathbf{x} - \mathbf{F}\mathbf{X})\} dS, \tag{6}$$

where $\mathbf{N}$ is the outward unit normal to ∂V, the regular boundary of V. The relationship (6) corresponds to the Green identity, though it is here written in slightly different notation with respect the original one [1].

3 In the presence of microstructure

An additional field, which accounts for the possible presence of microstructure [19], is possibly defined on the reference configuration of the continuum of interest. This additional field is here assumed to be a vector field

$\mathbf{d}(\mathbf{X})$. Such a vector field may account for the behavior of the liquid crystals and also for that of a broad class of non-classical materials, once the notion of reference configuration is defined in these cases. The extension to higher order tensor fields is omitted hereafter, in order to simplify the treatment and to avoid unnecessary cumbersome formulas.

It is also worth noting that here $\mathbf{d}$ depends on $\mathbf{X}$ through the deformation, though it cannot be excluded also an explicit dependence on $\mathbf{X}$ in the most general case. The stored energy W for such a continuum possibly depends on $\mathbf{F}$, on $\mathbf{d}$ and on $\nabla_R\mathbf{d}$, where ∇_R denotes the gradient operator in the reference frame. As we wish that the reference configuration be a natural configuration, we assume that $\mathbf{d}$ is uniform in this configuration. In the presence of a stored energy density, the equilibrium equations that govern the behavior of the continuum with microstructure can be derived from a variational formulation as Euler Lagrange equations. These equations stem from the following variational equation, in which the fields $\mathbf{d}$ and $\boldsymbol{k}$ are independently varied:

$$\delta_X \int_V \hat{W}(\mathbf{F}, \mathbf{d}, \nabla_R\mathbf{d})dV = 0. \tag{7}$$

The related Euler-Lagrange equations read

$$Div\ (\partial\hat{W}/\partial\mathbf{F}) = 0 \tag{8}$$

$$Div[\partial\hat{W}/\partial(\nabla_R\mathbf{d})] - (\partial\hat{W}/\partial\mathbf{d}) = 0. \tag{9}$$

The following Eshelby-like tensor can be associated with this continuum:

$$\mathbf{b} = \hat{W}\mathbf{I} - \mathbf{F}^T(\partial\hat{W}/\partial\mathbf{F}) - (\nabla_R\mathbf{d})^T[\partial\hat{W}/\partial(\nabla_R\mathbf{d})]. \tag{10}$$

In order to not overloading the notation, we agree on using the same symbol for the tensor $\mathbf{b}$ in all the different cases that are here examined. With reference to the formula (10), one is first concerned with the equality

$$Div\ \mathbf{b} = 0. \tag{11}$$

This equality holds true as an identity, provided that the field $\mathbf{d}$ possesses the same regularity of the deformation field. Eventually, the proof of the equality (11) stems straightforwardly by appealing to the equations (8) and (9). We can note that additional identities that are similar to

(3) and (4) respectively, also hold true in the present case. In order to recover these identities one can explicitly write the spur of **b** as

$$tr\ \mathbf{b} = 3\hat{W} + \mathbf{d}\cdot(\partial\hat{W}/\partial\mathbf{d}) - \\ Div\{(\partial\hat{W}/\partial\mathbf{F})^T\mathbf{x} - [\partial\hat{W}/\partial(\nabla_R\mathbf{d})]^T\mathbf{d}\}, \tag{12}$$

whence, by combining all previous results, one achieves the following final expression:

$$\textstyle\int_V[3\hat{W} + \mathbf{d}\cdot(\partial\hat{W}/\partial\mathbf{d})]dV = \int_{\partial V}\{\mathbf{b}(\mathbf{N}\cdot\mathbf{X}) + \\ [(\partial\hat{W}/\partial\mathbf{F})\mathbf{N}]\cdot\mathbf{x} + [\partial\hat{W}/\partial(\nabla_R\mathbf{d})\mathbf{N}]\cdot\mathbf{d}\}dS. \tag{13}$$

This formula can be further enhanced by the explicit evaluation of the quantity

$$\mathbf{b}(\mathbf{N}\cdot\mathbf{X}) = \hat{W}(\mathbf{N}\cdot\mathbf{X}) - [(\partial\hat{W}/\partial\mathbf{F})\mathbf{N}]\cdot(\mathbf{F}\mathbf{X}) - \\ [\partial\hat{W}/\partial(\nabla_R\mathbf{d})\mathbf{N}]\cdot[(\nabla_R\mathbf{d})\mathbf{X}]. \tag{14}$$

Hence, the formula (13) eventually reads

$$\textstyle\int_V[3\hat{W} + \mathbf{d}\cdot(\partial\hat{W}/\partial\mathbf{d})]dV = \int_{\partial V}\{\hat{W}(\mathbf{N}\cdot\mathbf{X}) + \\ [(\partial\hat{W}/\partial\mathbf{F})\mathbf{N}]\cdot(\mathbf{x} - \mathbf{F}\mathbf{X}) + \\ [\partial\hat{W}/\partial(\nabla_R\mathbf{d})]\mathbf{N}\cdot[\mathbf{d} - (\nabla_{\mathbf{R}}\mathbf{d})\mathbf{X}]\}dS. \tag{15}$$

This formula stands as an extension of the Green identity, if compared with the formula (6). Notice that, whenever the energy density does not depend explicitly on **d** (a case which frequently occurs in materials with specific microstructure), the volume integral at l.h.s. of formula (15) exactly coincides with the corresponding term of formula (6).

4 In nonlinear dielectrics

The polarization is the vector field that essentially defines a dielectric material. Accordingly, the energy density of an elastic dielectric W is currently assumed to depend on the polarization **P**, possibly on its gradient and on the deformation gradient **F** [20-23]. The polarization enters into the Maxwell equations through the electric displacement. Thus, as one wishes to recover the Maxwell equations of electrostatics along with the mechanical equilibrium equations, the appropriate functional for the variational formulation is not simply the volume integral of W, but the so-called electric enthalpy [10,11,26,27]. Notice that this quantity is defined all over the space, inside and outside a dielectric body of finite extent [5,20-23]. Hereafter, the dependence of W on the gradient of polarization is disregarded as this case could be recovered subsequently, without severe difficulties. Therefore, the electric enthalpy per unit volume of the reference configuration of the material reads

$$H = J(\ w - \frac{1}{2}\varepsilon_o E^2 - \mathbf{E}\cdot\mathbf{P}), \qquad \text{in}\ \ V, \tag{16}$$

where ε_o represents the dielectric constant of a vacuum, $Jw = W$ and $\mathbf{E}$ is the electric field. $\mathbf{E}$ is a curl-free field in electrostatics.

In finite deformations, Maxwell equations need to be properly expressed in the reference configuration. This can be done by assuming that Maxwell equations preserve their form in all configurations, whereas the electromagnetic fields are transformed accordingly [4,5,13,17,26,27]. The polarization in the reference frame and per unit volume of the reference configuration is $\mathcal{P} = J\mathbf{F}^{-1}\mathbf{P}$, which represents the proper argument of W. Details can be found in references [5,13,15,17,26,27]. In terms of the transformed fields, the electric enthalpy density (16) equivalently reads

$$\hat{H} = \hat{W}(\mathbf{F}, \mathbf{F}\mathcal{P}) - \frac{1}{2}\ \varepsilon_0 J\ \mathcal{E}\cdot\mathbf{C}^{-1}\mathcal{E} - \mathcal{E}\cdot\mathcal{P}, \tag{17}$$

where $\mathcal{E} \equiv \mathbf{F}^T\mathbf{E} = -(\nabla_R\phi)$ and $\mathbf{C}^{-1} \equiv \mathbf{F}^{-1}\mathbf{F}^{-T}$.

In the variational equation that is related to H, the three fields $\boldsymbol{k}$, ϕ and $\mathcal{P}$ are independently varied. The corresponding Euler Lagrange equations read

$$Div(\partial\hat{H}/\partial\mathbf{F}) = 0, \tag{18}$$

$$Div(\varepsilon_0 J\ \mathbf{C}^{-1}\mathcal{E} + \mathcal{P}) = 0, \tag{19}$$

$$\partial\hat{H}/\partial\mathcal{P} \equiv (\partial\hat{W}/\partial\mathcal{P}) - \mathcal{E} = 0. \tag{20}$$

With reference to the equation (19), note that the quantity

$$(\varepsilon_0 J\ \mathbf{C}^{-1}\mathcal{E} + \mathcal{P}) \equiv \partial \hat{H}/\partial \mathcal{E}\ \equiv\ \mathcal{D} \tag{21}$$

represents the electric displacement properly convected in the referential frame.

The above equations are defined in the open domain V, which represents the region occupied by the dielectric. Should the dielectric body be of finite extent, an extension of H *to* $(E_3 - V)$ would be needed, as ϕ is defined also outside V. By introducing a continuous extension of the electric enthalpy to $(E_3 - V)$, the variational problem can be formulated on the whole physical space. An additional equation for ϕ would stem from it, which simply states that the electrostatic potential has to be a harmonic function in a vacuum. Additional equations for the boundary conditions also stem in a natural way from the variational problem.

With reference to equation (18), it is worth evaluating explicitly the quantity

$$(\partial \hat{H}/\partial \mathbf{F}) = (\partial \hat{W}/\partial \mathbf{F})\ |_{expl} + \mathbf{E}\otimes\mathcal{P} + J\varepsilon_o\ \mathbf{T}_M \mathbf{F}^{-T}, \tag{22}$$

where the following definitions have been used:

$$\begin{aligned}(\partial \hat{H}/\partial \mathbf{F}) = (\partial \hat{W}/\partial \mathbf{F})\ |_{expl} + (\partial \hat{W}/\partial(\mathbf{F}\mathcal{P}) \cdot \partial(\mathbf{F}\mathcal{P})/\partial \mathbf{F}) \equiv \\ (\partial \hat{W}/\partial \mathbf{F})\ |_{expl} + \mathbf{E}\otimes\mathcal{P},\end{aligned} \tag{23}$$

$$\mathbf{T}_M \equiv \varepsilon_o \left(\mathbf{E}\otimes\mathbf{E} - \frac{1}{2}E^2\ \mathbf{I} \right) \qquad \text{(Maxwell stress tensor)}. \tag{24}$$

5 Eshelby's tensors and a Green identity in nonlinear dielectrics

In the following we will be mainly concerned with an unbounded domain, which is identified with E_3. With reference to all quantities that have been introduced in the previous section, the Eshelby-like tensor for an elastic dielectric reads [5]

$$\mathbf{b} = (\hat{W} - \mathcal{E}\cdot\mathcal{P})\ \mathbf{I} - \mathbf{F}^T(\partial \hat{W}/\partial \mathbf{F}) + \mathcal{E}\otimes\mathcal{P}, \tag{25}$$

or, equivalently,

$$\mathbf{b} = (\hat{W} - \mathcal{E} \cdot \mathcal{P})\ \mathbf{I}\ -\ \mathbf{F}^T(\partial \hat{W}/\partial \mathbf{F})\ |_{expl}, \tag{26}$$

having in mind the definition (22). As in the cases expounded in the previous sections, the expressions (25) and (26) satisfy the identities (2) and (3). Thus, with reference to these expressions, one can re-write the mentioned identities, which in the electrostatic equilibrium read

$$Div\ \mathbf{b} = 0, \tag{27}$$

$$Div\ (\mathbf{b}^T \mathbf{X}) = tr\ \mathbf{b}. \tag{28}$$

However, along with the formulas (25) and (26), a third equivalent form for the Eshelby-like tensor in a dielectric is [10,11]

$$\boldsymbol{b}\ = \hat{H}\mathbf{I} - \mathbf{F}^T(\partial \hat{W}/\partial \mathbf{F} +\ J\ \mathbf{T}_M \mathbf{F}^{-T}) + \mathcal{E} \otimes \mathcal{D}. \tag{29}$$

Of course, this tensor satisfies the equalities (27) and (28) as well. The identity (27) can be checked by taking into account the equations (18)-(21) and by evaluating explicitly (Div $\mathbf{b}$). With reference to the identity (28), it is worth writing explicitly the spur of $\mathbf{b}$

$$tr\ \boldsymbol{b}\ = 3\hat{H} - Div[(\partial \hat{W}/\partial \mathbf{F} + J\mathbf{T}_M \mathbf{F}^{-T})^T \mathbf{x}] - Div[\phi \mathcal{D}]. \tag{30}$$

Based on the formulas (27)-(30), one achieves the Green-like identity by following the expounded procedure

$$3 \int_V \hat{H} dV = \int_{\partial V} \{\hat{H}(\mathbf{N} \cdot \mathbf{X}) + [(\partial \hat{H}/\partial \mathbf{F})\mathbf{N}] \cdot (\mathbf{x} - \mathbf{F}\mathbf{X}) + (\mathcal{D} \cdot \mathbf{N})[\phi - (\nabla_R \phi) \cdot \mathbf{X}]\}\ dS. \tag{31}$$

One can note that $\mathbf{b}$ is naturally defined only in the region occupied by the dielectric, according to the definition (25) or (26). In contrast, the formula (29) may address an undue extension of the notion of Eshelby tensor to a vacuum. Nevertheless, the formula (29), properly interpreted, seems to be more useful in some mechanical problems. In fact, the quantities $(\partial \hat{H}/\partial \mathbf{F})\mathbf{N}$ and $(\mathcal{D} \cdot \mathbf{N})$, which appear at r.h.s. of the integral formula (31), directly address the full set of boundary conditions.

6 Final comment

The possible extension of the Green's identity to liquid crystals stems straightforwardly from sections **3**, under additional minor assumptions and provided that the notion of reference configuration is suitably introduced in this case [28]. A Green-like identity for elastic dielectrics is proposed in section **5**. Basing on the Eshelby's identity, one can extend the Green formula to other non-classical continua (polar media, magnetic materials, materials with intrinsic spin and others), by following a similar procedure. The Green-like integral-formulae proposed here relate (like the original one) the boundary conditions with the functional that is associated with the energetic material response. This link is important, as these conditions are typically prescribed in boundary value problems. As mentioned, the original Green formula (and its extension in terms of complementary energy) has been successfully employed in establishing uniqueness theorems for non-convex energy response [2,3]. Therefore, these formulae seem to be promising for investigating possible solutions of problems in generalised continua.

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Green-ove i Eshelby-jeve identičnosti u generalisanim kontinuumima i dielektricima

UDK 531.01; 539.3

A. E. Green je 1973. godine ukazao na nekoliko interesantnih formula, koje važe u elastičnosti konačnih deformacija. Jedna od njih (formula (2.10), p.75 [1]) se često u literaturi citira kao *Green-ova identičnost*. Ova veoma značajna identičnost je uspešno primenjivana u različitim kontekstima. Ovde pominjemo samo njenu centralnu ulogu u teoremama jedinosti u elastostatici [2,3]. Dublje sagledavanje Green-ove identičnosti pokazuje suštinsku vezu ove formule sa Eshelby-jevim tenzorom kao i sa *zakonom materijalnog balansa* ravnoteže [4,5]. U homogenim hiperelastičnim materijalima , ovaj zakon prelazi u *Eshelby-jevu identičnost*, te se može dokazati da Green-ova identičnost direktno sledi iz Eshelby-jeve identičnosti. Moguće je da se ove identičnosti protežu na generalisane kontinuume, kao što su kontinuumi sa mikrostructurom i elastični dielektrici.

U ovom radu se diskutuje mogućnost važenja Eshelby-jeve identičnosti za takve materijale. Na osnovu nove proširene Eshelby-jeve identičnosti može se, pri specifičnim pretpostavkama, takodje uspostaviti odgovarajuća Green-ova identičnost. U slučaju dielektrica, dva ekvivalentna oblika Eshelby-jevog tenzora slede iz predloženog pristupa pri čemu oba zadovoljavaju Eshelby-jevu identičnost. Jedan od njih je podesniji za izvodjenje željene identičnosti Green-ovskog tipa. Drugi tenzor, koji je redukovani oblik prvoga, reprezentuje fizički Eshelby-jev tenzor u dielektricima [4-5].