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Rayleigh-Bénard convection with magnetic field

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Abstract

We discuss the solution of the small perturbation equations for a horizontal fluid layer heated from below with an applied magnetic field either in vertical or in horizontal direction. The magnetic field stabilizes, due to the Lorentz force, more or less Rayleigh-Bénard convective cellular motion. The solution of the eigenvalue problem shows that the critical Rayleigh number increases with increasing Hartmann number while the corresponding wave length decreases. Interesting analogies to solar granulation and black spots phenomena are obvious. The influence of a horizontal field is stronger than that of a vertical field. It is easy to understand this by discussing the influence of the Lorentz force on the Rayleigh-Bénard convection. This result corrects earlier calculations in the literature.

Keywords: *Rayleigh-Bénard convection, Lorentz force, solar granulation, black spots.*

Nomenclature

a	wave number
A	Ampére, constant
$\vec{B} = \{B_x, B_y, B_z\}$	magnetic induction $[V s\ m^{-2}] \equiv$ Tesla

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$\vec{F}_L = \{F_{L,x}, F_{L,y}, F_{L,z}\}$	Lorentz force $[N]$
F_V	viscous force $[N]$
g	gravity acceleration $[ms^{-2}]$
h	fluid layer thickness $[m]$
$\vec{H} = \{H_x, H_y, H_z\}$	magnetic field $[Am^{-2}]$
$\vec{i} = \{i_x, i_y, i_z\}$	current density $[Nm^2s^{-1}V^{-1}]$
k	thermal conductivity $[m^2s^{-1}]$
p	pressure $[Pa]$
$Ra = \frac{g\alpha\beta h^4}{k\nu}$	Rayleigh number
$R_h = Bh(\frac{\sigma}{\rho\nu})^{1/2}$	Hartmann number
t	time $[s]$
$\vec{v} = \{u, v, w\}$	fluid velocity $[ms^{-1}]$
V	Volt
x, y, z	coordinates $[m]$

Greek symbols

α	thermal expansion $[K^{-1}]$
$\beta(< 0)$	vertical temperature gradient $[Km^{-1}]$
$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$	Laplace operator $[m^{-2}]$
$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$	two-dimensional operator $[m^{-2}]$
Θ	temperature disturbance $[K]$
λ	wave length $[m]$
μ	magnetic permeability $[VsA^{-1}m^{-1}]$

ν	kinematic viscosity $[m^2 s^{-1}]$
ρ	density $[kgm^{-3}]$
σ	electrical conductivity$[Nm^3V^{-2}s^{-1}]$

1 Fundamental equations

We start with the *Lorentz force* $\vec{F}_L$ due to a magnetic field $\vec{H}$ with magnetic induction $\vec{B} = \mu\vec{H}$, μ = magnetic permeability, in a fluid field $\vec{v} = \{u, v, w\}$. With the electrical conductivity σ and the current density $\vec{i} = \sigma(\vec{v} \times \vec{B})$ we have

$$\vec{F}_L = \vec{i} \times \vec{B} = \sigma(\vec{v} \times \vec{B}) \times \vec{B}, \tag{1}$$

or explicit in cartesian coordinates

$$\vec{F}_L = \sigma \begin{cases} wB_xB_z - uB_z^2 - uB_y^2 + vB_xB_y, \\ uB_yB_x - vB_x^2 - vB_z^2 + wB_yB_z, \\ vB_zB_y - wB_y^2 - wB_x^2 + uB_zB_x. \end{cases} \tag{1a,b,c}$$

The linearized equations of the *Rayleigh Theory* with general magnetic field $\vec{B} = \{B_x, B_y, B_z\}$ are the momentum, energy and continuity equations

$$\frac{\partial u}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial x} - \nu\Delta u = \frac{1}{\rho}F_{L,x} = \frac{\sigma}{\rho}(wB_xB_z - \underline{uB_z^2} - \underline{\underline{uB_y^2}} + vB_xB_y), \tag{2a}$$

$$\frac{\partial v}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial y} - \nu\Delta v = \frac{1}{\rho}F_{L,y} = \frac{\sigma}{\rho}(uB_yB_x - vB_x^2 - \underline{vB_z^2} + wB_yB_z), \tag{2b}$$

$$\frac{\partial w}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial z} - g\alpha\Theta - \nu\Delta w = \frac{1}{\rho}F_{L,z} = \frac{\sigma}{\rho}(vB_zB_y - \underline{\underline{wB_y^2}} - wB_x^2 + vB_zB_x), \tag{2c}$$

$$\frac{\partial \Theta}{\partial t} = k\Delta\Theta - \beta w, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{2d,e}$$

Herein we underlined once the case of a vertical magnetic field and twice that with a horizontal field.

The *viscous dissipation* $\sim \nu(w^2/h^2)$ and *Joule's heat* $\sim (\sigma/\rho)B^2w^2$ are both $\sim w^2$ and therefore they don't appear separately in the linearized energy equation $(2d)$. Nevertheless their ratio is important as we will see later on.

2 Direction of the Lorentz force in a special case

For the moment we consider only a vertical velocity component $\vec{v} = \{0, 0, w\}$ and apply a horizontal magnetic field $\vec{B} = \{0, B_y, 0\}$. The electric current is $\vec{i} = \sigma\ (\vec{v} \times \vec{B}) = \{-\sigma w B_y, 0, 0\}$. Therefore we get finally for the Lorentz force $\vec{F}_L = \vec{i} \times \vec{B} = \{0, 0, -\sigma w B_y^2\}$.

The direction of $\vec{F}_L$ is according to the left hand three fingers rule (Fig.1). This means: *if we have an upward motion* $(w > 0)$ *and a magnetic field lies in y-direction then we get an electric current* $\vec{i}$ *in* $(-x)$*-direction.* This leads to a downward $(-z)$ Lorentz force that acts in the same direction as the viscous force and opposite to the buoyancy force. Therefore the magnetic field stabilizes the Rayleigh-Bénard instability.

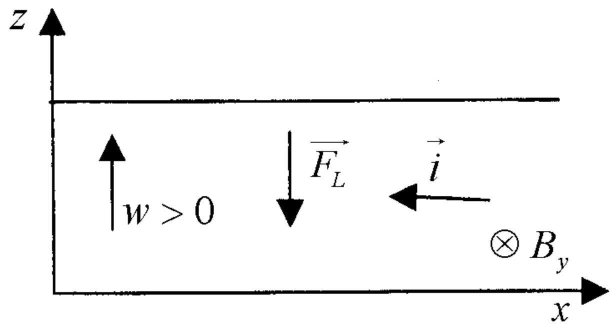

Figure 1: Vertical motion in a fluid layer with horizontal magnetic field

3 Heuristic discussion of the instability

The forces per unit mass in z-direction - by absolute value and order of magnitude - are given as follows:

$$\text{buoyancy force:}\quad F_B = g\alpha\Theta \sim \frac{g\alpha\beta w h^2}{k}, \tag{3a}$$

$$\text{viscous force:}\quad F_V = \nu\Delta w \sim \nu\frac{w}{h^2} \tag{3b}$$

$$\text{Lorentz force:}\quad F_L = \frac{\sigma B^2}{\rho}\, w. \tag{3c}$$

Here we have for stationary flow

$$k\Delta\Theta = \beta w,\ \Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \sim \frac{1}{h^2},\ k\Delta\Theta \sim k\frac{\Theta}{h^2} \sim \beta w.$$

The ratios of these forces (3a,b,c) lead to the following non-dimensional parameters:

$$\frac{F_B}{F_V} = \frac{g\alpha\Theta}{\nu\Delta w} \sim \frac{g\alpha\beta h^4}{k\nu} = Ra = \text{Rayleigh number}, \tag{4}$$

$$\frac{F_B}{F_L} = \frac{g\alpha\Theta}{\frac{\sigma B^2}{\rho}w} \sim \frac{\frac{g\alpha\beta h^4}{kv}}{\frac{\sigma}{\rho\nu}B^2h^2} = \frac{Ra}{R_h^2},$$

where

$$R_h = Bh\sqrt{\frac{\sigma}{\rho\nu}} = \text{Hartmann number.} \tag{5}$$

The Rayleigh number (4) and the Hartmann number (5) are the key - parameters for the Rayleigh -Bénard convection with magnetic field. The Hartmann number itself has the following meaning:

$$R_h^2 = \frac{\text{Joule's heat}}{\text{viscous dissipation}} \sim \frac{\frac{\sigma B^2 w^2}{\rho}}{\nu\frac{w^2}{h^2}} = B^2h^2\frac{\sigma}{\rho\nu} \sim \frac{F_L}{F_V}. \tag{6}$$

We see that although the both dissipation terms (Joule's heat, viscous dissipation) do not appear in the linearized energy equation ($2d$) their ratio will be one of the fundamental parameters of our problem.

4 Rayleigh theory with magnetic field $\vec{B}$

4.1 Vertical magnetic field with steady flow

Taking into account that $\vec{B} = \{0, 0, B_z\}$ elimination of u, v, p and Θ from $(2, a, b, c, d, e)$ leads - similar to the well known classical theory - to:

$$\Delta\Delta\Delta w + \frac{g\alpha\beta}{k\nu}\Delta_2 w - \frac{\sigma B_z^2 w^2}{\rho\nu}\Delta\frac{\partial^2 w}{\partial z^2} = 0 \tag{7}$$

where Δ = three-dimensional Laplace operator and Δ_2 = two-dimensional (x, y) operator.

We note that (7) is a six order linear partial differential equation. For a periodic solution of the form:

$$w = f(x, y) \cdot F(z),\ \Delta_2 f + \frac{a^2}{h^2} f = 0,\ F(z) = A \sin \frac{\pi z}{h} \quad (8)$$

we get the eigenvalue relation

$$\boxed{\text{Ra}{=}-\frac{g\alpha\beta h^4}{k\nu} = \frac{(\pi^2 + a^2)^3}{a^2} + R_h^2 \frac{\pi^2(\pi^2 + a^2)}{a^2}} \quad (9)$$

relating Rayleigh number with Hartmann number $R_h = Bh(\sigma/\rho\nu)^{1/2}$ and a = wave number. This means that any point on the curve (9) leads to a periodic convective motion of the form (8) (cf Figs. 2,3).

The eigencurve (9) Ra=$f_1(a, R_h)$ has a typical minimum Ra_{crit} for a_{crit}. For $R_h = 0$ we get the well known values: $\text{Ra}_{crit}{=}\frac{27}{4}\pi^4$, $a_{crit} = \frac{\pi}{\sqrt{2}}$. For $R_h > 0$, Ra_{crit} and a_{crit} increase. For $R_h^2 \gg 1$ we have the remarkable asymptotic relation:

$$\text{Ra}_{crit} \to \pi^2 R_h^2, \qquad a_{crit}^2 \to \left(\frac{\pi^4}{2}\right)^{\frac{1}{3}} R_h^{\frac{2}{3}}. \quad (10)$$

This means that the magnetic field stabilizes the Bénard convection while the wave length $\lambda = 2\pi\frac{h}{a}$ decreases with increasing a. This reminds us of some kind of solar granulation. The convection cells degenerate to vertical orientated needles parallel to the vertical magnetic field lines.

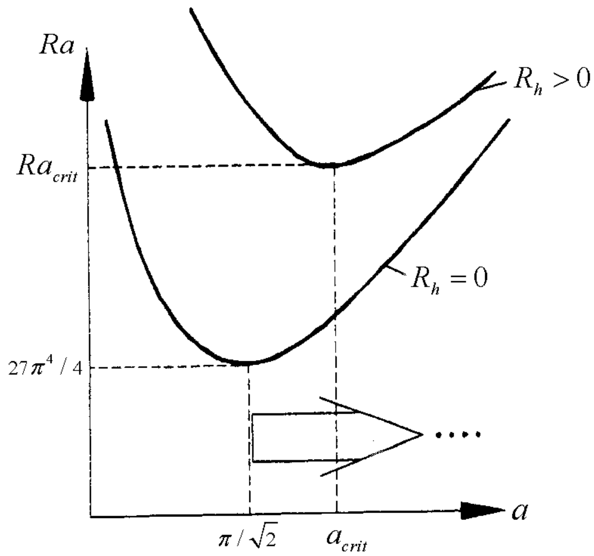

Figure 2: Eigenvalue Ra=$f_1(a, R_h)$ for a magnetic field in z-direction (qualitative)

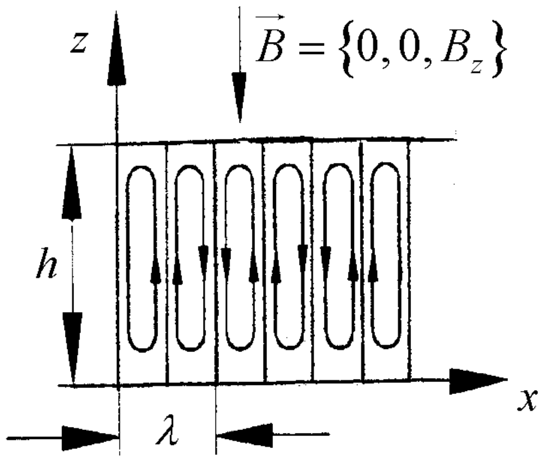

Figure 3: $2\frac{\pi}{a}=\lambda\to 0$ for $R_h\gg 1$ cell geometry for $R_h\gg 1$

4.2 Horizontal magnetic field steady flow

In this case we have $\vec{B}=\{0,B_y,0\}$,. For vortices along the horizontal magnetic field lines $\left(\frac{\partial}{\partial y}=0\right)$ the explained elimination leads now to the following differential equation

$$\Delta\Delta\Delta w + \frac{g\alpha\beta}{k\nu}\frac{\partial^2 w}{\partial x^2} - \frac{\sigma B_y^2}{\rho\nu}\Delta\Delta w = 0 \tag{11}$$

For a periodic solution similar to (8) we get a corresponding eigenvalue relation

$$\boxed{\mathrm{Ra} = -\frac{g\alpha\beta h^4}{k\nu} = \frac{(\pi^2+a^2)^3}{a^2} + R_h^2\frac{(\pi^2+a^2)^2}{a^2}} \tag{12}$$

relating Rayleigh number with R_h and a. This time the domains of the variables are restricted to

$$\frac{\pi^2}{2} \leq a_{crit}^2 \leq \pi^2, \qquad \frac{27}{4}\pi^4 \leq \mathrm{Ra}_{crit} \leq 4\pi^2 R_h^2. \tag{13}$$

Again we have the characteristic minimum of the eigencurve (12), Fig.4. But in contrast to the case with a vertical magnetic field analyzed in section 4.1 now the wave length decreases only moderately (Fig.5) while Ra_{crit} and R_h tend to infinity. The convection rolls are orientated along the horizontal magnetic field lines. This simulates the fluid flow at the outer rim of the so called solar black spots.

5 Conclusions and comments

1. An applied magnetic field stabilizes more or less the Rayleigh-Bénard convection in a horizontal fluid layer heated from below.

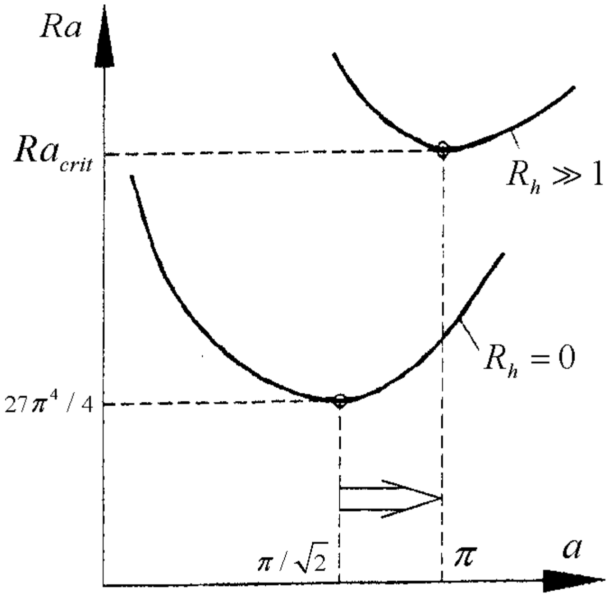

Figure 4: Eigencurve Ra=$f_2(a, R_h)$ for a magnetic field in y-direction (qualitative)

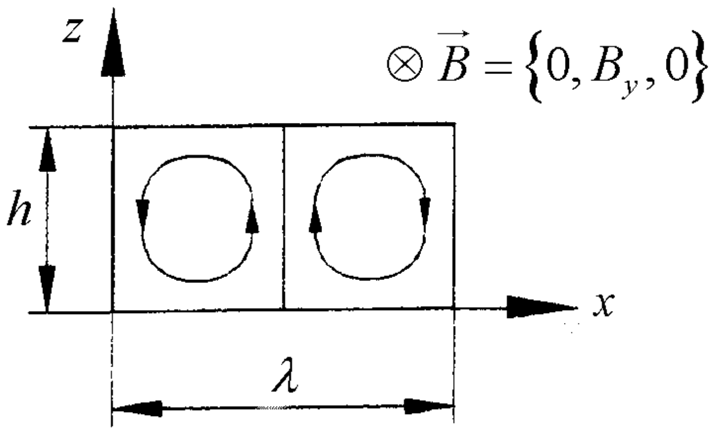

Figure 5: $2\pi \frac{h}{a} = \lambda \to 2h$ for $R_h \gg 1$ cell geometry for $R_h \gg 1$

2. A vertical magnetic field leads to extremely decreasing wave length with increasing Hartmann number (6). This looks like solar granulation [6].

3. A horizontal magnetic field gives moderately decreasing wave length (7). We get vortices along the magnetic field lines. This looks like fluid flow at the outer rim of the solar black spots.

4. The results for a horizontal field are new and correct earlier calculations in the literature [4]. The influence of the Lorentz force in this case is much stronger than that for a vertical field. This means that a horizontal field stabilizes the convection much more than a vertical field.

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Rayleigh-Bénard-ova konvekcija sa magnetskim poljem

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Diskutuje se rešenje jednačina sa malom perturbacijom za horizontalni fluidni sloj zagrevan odozdo sa magnetnim poljem u vertikalnom ili horizontalnom pravcu. Magnetsko polje je stabilizovalo, zbog Lorentz-ove sile, manje ili više Rayleigh-Bénard-ovo konvektivno celularno kretanje. Rešenje problema sopstvenih vrednosti pokazuje da kritični Rayleigh-jev broj raste sa porastom Hartmann-ovog broja dok odgovarajuća talasna dužina opada. Interesantne analogije sa fenomenima solarne granulacije i crnih mrlja su očigledne. Uticaj horizontalnog polja je jači od uticaja vertikalnog polja. Ovo je lako je razumeti pomoću diskusije uticaja Lorentz-ove sile na Rayleigh-Bénard-ovu konvekciju. Ovaj rezultat ispravlja ranija izračunavanja raspoloživa u literaturi.