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On the trace-manifold generated by the deformations of a body-manifold

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Abstract

In this paper, concerned to the study of continuous deformations of material media using some tools of modern differential geometry, a moving frame of Frenet type along the orbits of an one-parameter group acting on a so-called "trace-manifold", $\boldsymbol{M}$, associated to the deformations, is constructed. The manifold $\boldsymbol{M}$ is defined as an infinite union of non-disjoint compact manifolds, generated by the consecutive positions in the Euclidean affine 3-space of a body-manifold under deformations in a closed time interval. We put in evidence a skew-symmetric band tensor of second order, ω, which describes the deformation in a small neighborhood of any point along the orbits. The non-null components $\omega_{i,i+1}$, (i =1,2), of ω are assimilated as like curvatures at each point of an orbit in the planes generated by the pairs of vectors $(\tilde{\mathbf{e}}_{\mathrm{i}}, \tilde{\mathbf{e}}_{\mathrm{i+1}})$ of a moving frame in $\boldsymbol{M}$ associated to the orbit in a similar way as the Frenet's frame is. Also a formula for the energy of the orbits is given and its relationship with some stiffness matrices is established.

1 Introduction

The relationship between differential geometry and material structures is from a time well known. It was meanwhile underlined by consistently using of the notion of body-manifold, i.e. a material object considered

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as an oriented, connected and compact 3-dimensional submanifold of the Euclidean affine space $\mathcal{E}^3$. A material medium (body) $\mathcal{B}$ is placed in the Euclidean affine 3-space $\mathcal{E}^3$ by some regular embeddings, that is by injective differentiable mappings of rank 3 at every particle of $\mathcal{B}$. A mapping $\mathrm{p} : \mathcal{B} \rightarrow \mathcal{E}^3$ with the properties asked above is said to be a *placement*. Let $\mathrm{P}(\mathcal{B})$ be the set of all placements of $\mathcal{B}$ in $\mathcal{E}^3$. A class $C^k(\mathcal{B})(\subseteq \mathrm{P}(\mathcal{B}))$ of placements is said to be *admissible* if for any $\mathrm{p}, \mathrm{p}' \in \mathrm{C}^{\mathrm{k}}(\mathcal{B})$ the images $\mathrm{p}(\mathcal{B}) \doteq \mathbf{M}$ and $\mathrm{p}'(\mathcal{B}) \doteq \mathbf{M}'$ are both C^k- submanifolds of $\mathcal{E}^3$ and $\mathrm{p}' \circ \mathrm{p}^{-1}$ is a diffeomorphism of class $C^k, (k \geq 1)$, of $\mathrm{p}(\mathcal{B})$ onto $\mathrm{p}'(\mathcal{B})$. The images of the body particles by admissible placements are called admissible positions.

Further, for the geometry of continuous material media, we have in view only placements of $\mathrm{P}(\mathcal{B})$ in accordance with the following hypothesis ([13]):

H1. All placements of a material medium in an Euclidean affine space are admissible.

So, for $\mathrm{p} \in \mathrm{C}^{\mathrm{k}}(\mathcal{B})$, if $\lambda \in \mathcal{B}$ is a particle, $\mathrm{p}(\lambda) \doteq \mathrm{P}$ will denote its admissible position, and their union $\mathbf{M} = \{\mathrm{p}(\lambda) \mid \lambda \in \mathcal{B}\}$ can be endowed with a differentiable (even Riemannian) manifold structure.

Endowed with a real 3-dimensional differentiable manifold structure, an image $\mathbf{M}$ by the elements of $C^k(\mathcal{B})$ is called a *body-manifold* (see[1] for supplementary conditions).

The considered above embedding is regular, that is $\mathcal{B}$ and its image $\mathbf{M}(\subset \mathcal{E}^3)$ are homeomorphic, because the body-manifolds in our acceptance (see [1]) are compact manifolds. However, if $\mathcal{B}$ is a non-compact differentiable manifold there exists an open neighborhood $\mathbf{U}$ of every point of $\mathcal{B}$ such that the restriction $\mathrm{p} \mid \mathbf{U}$ is a regular embedding ([9]).

In order to describe a body-manifold deformation we consider the placement p to be "of reference" and choose another placement $\tilde{\mathrm{p}}$ of $\mathcal{B}$ in $\mathcal{E}^3$ as a new element of $C^k(\mathcal{B})$. Let us denote by $\mathbf{M}$ and $\tilde{\mathbf{M}}$ the images of $\mathcal{B}$ by p and $\tilde{\mathrm{p}}$, respectively.

By a *deformation* of $\mathbf{M}$ we mean a continuous differentiable mapping $\mathbf{M} \longmapsto \tilde{\mathbf{M}}$. Thus, the image of a body-manifold by a deformation is obtained by the composed mapping $\delta = \tilde{\mathrm{p}} \circ \mathrm{p}^{-1}$. Taking into account the fact that both p and $\tilde{\mathrm{p}}$ are bijective mappings, as regular embeddings, it results that $\delta : \mathrm{p}(\mathcal{B}) \rightarrow \tilde{\mathrm{p}}(\mathcal{B})$ is bijective; this mapping is usually called *displacement* of the body-manifold points in $\mathcal{E}^3$. The displacement δ is not necessarily an isometry of $\mathcal{E}^3$, but, in virtue of the enumerated above properties, it is a transformation of the domain of $\mathcal{E}^3$ occupied by $\mathbf{M}$ into another domain occupied by $\tilde{\mathbf{M}}$ by composing some of the following elementary transformations (see [10]): translations, rotations, and stretches; each of them defines an elementary state of local deformation of the continuum. However, by superposition can also be considered rigid-body motions, compressions, twisting, etc.

Starting with the next section we will have in view only deformations with respect to a time interval, that is the development in time of a body-manifold deformation.

2 One-parameter group of transformations associated to a deformation

Let us consider an one-parameter family of displacements $\Delta_{\mathbb{I}} = \{\delta_t := \mathrm{p}_t \circ \mathrm{p}^{-1} \mid t \in \mathbb{I}\}$, $(\mathrm{p}_0 = \mathrm{p})$ where $t \in \mathbb{I}(\subset \mathbb{R})$ is a time variable on an open interval $\mathbb{I}$ containing $0(\in \mathbb{R})$ and $\{\mathrm{p}_t\}_{t\in\mathbb{I}}$ is a family of admissible placements of $\mathcal{B}$, such that $\mathrm{p}_t(\lambda)$ designates the position of the image of $\lambda \in \mathcal{B}$ in $\mathcal{E}^3$ at the moment t. We say that the family of displacements $\Delta_{\mathbb{I}} = \{\delta_t \mid t \in\mathbb{I}\}$ defines a proper deformation ϕ of $\mathbf{M}$ if for at least one $t \in \mathbb{I}$ and for at least one pair of points $(\mathrm{P}, \mathrm{P}') \in \mathbf{M}\times\mathbf{M}$ we have $\mathrm{d}(\mathrm{P}, \mathrm{P}') \neq \mathrm{d}(\delta_\mathrm{t}(\mathrm{P}), \delta_\mathrm{t}(\mathrm{P}'))$, where d is the distance function in $\mathcal{E}^3$. Otherwise, ϕ is a motion of $\mathbf{M}$ in $\mathcal{E}^3$. So, with the help of the family $\Delta_{\mathbb{I}}$ one can describes any deformation $\phi : \mathbf{M} \to \tilde{\mathbf{M}}$ (see [2]). Here ϕ must be understood not as much as a description of the body state after deformation but as a mapping describing it continuously during the deformation process.

Thus, the image by ϕ of any point $\mathrm{P} \in \mathbf{M}$ is obtained as an infinite union of the consecutive positions of P by δ_t for all $t \in \mathbb{I}$. This will be explained below more precisely by introducing of a group of transformations acting on a differentiable manifold.

So, if $\mathbf{M}_t$ denotes the image by the placement p_t of the body $\mathcal{B}$ at the moment t, the union $\boldsymbol{M}= \cup_{t\in\mathbb{I}}\mathbf{M}_t$ can be endowed in a natural way with a differentiable manifold structure.

This is obtained as follows:

Consider for each $t \in \mathbb{I}$ a differentiable atlas $(\mathbf{U}_{t,\alpha}, h_{t,\alpha})_{\alpha\in\mathrm{A}}$ on the submanifold $\mathbf{M}_t$ of $\boldsymbol{M}$ (with embedding given by inclusion), such that

$h_{t,\alpha}$, (for all $\alpha \in$ A - a family of indices) map the open sets $\mathbf{U}_{t,\alpha} \subset \mathbf{M}_t$ on some open sets $\mathbf{D}_{t,\alpha} \subset \mathbb{R}^3$ and $\cup_{(t,\alpha)\in\mathbb{I}\times\mathrm{A}}\mathbf{U}_{t,\alpha} =\boldsymbol{M}$. Then, the union of these atlases for all $t \in \mathbb{I}$ is a differentiable atlas on $\boldsymbol{M}$, defining the differentiable manifold structure on it. We say that $\boldsymbol{M}$ is a *time deformed trace-manifold*.

In [2] and [3] were studied deformations when $\mathbf{M}$ and $\tilde{\mathbf{M}}$ are looked as submanifolds of $\boldsymbol{M}$. Namely, $\mathbf{M} = \mathbf{M}_0$ (at the moment $t = 0$) contains all the reference positions $\mathrm{p}_0(\lambda) \doteq \mathrm{P}$ of the particles λ of $\mathcal{B}$ in $\mathcal{E}^3$.

Now we associate to $\boldsymbol{M}$ an one-parameter group of transformations as follows.

Assume $\boldsymbol{M}$ corresponds in the previous definition to a symmetric (with respect to 0) closed subinterval of $\mathbb{I}$, $[-\varepsilon, \varepsilon] \subset\mathbb{I}$, and we restrict to it the family of displacements considered above, as

$$\Delta_\varepsilon = \{\delta_t = \mathrm{p}_t \circ \mathrm{p}^{-1}\}_{t\in[-\varepsilon,\varepsilon]}, \tag{1}$$

where $\mathrm{p}, \mathrm{p}_t \in C^k(\mathcal{B})$ are the reference placement and a current placement in $\mathcal{E}^3$ of a body $\mathcal{B}$, respectively, while $\mathrm{p}_t(\mathcal{B}) = \oslash$ for $t \in \mathbb{I}\backslash[-\varepsilon, \varepsilon]$.

Now we can prove the following assertion:

If for $t = 0$, $\mathbf{M}_0 = \mathbf{M} = \mathrm{p}(\mathcal{B})$, *and for every* $s \in [-\varepsilon, \varepsilon]$ *chosen such that* $| s + t |\leq \varepsilon$ *we define "the consecutive displacement* δ_s of δ_t *" acting on the points of the submanifold* $\mathrm{p}_t(\mathcal{B}) = \mathbf{M}_t(\subset \mathcal{E}^3)$ *by*

$$\delta_s = \mathrm{p}_{s+t} \circ \mathrm{p}_t^{-1} \tag{2}$$

then Δ_ε *forms an one-parameter group of transformations on the differentiable manifold* $\boldsymbol{M}= \cup_{t\in\mathbb{I}}\mathbf{M}_t$ *with respect to the product of consecutive displacements.*

Indeed, first we observe that the consecutive displacement δ_s of any displacement $\delta_t\in \Delta_\varepsilon$ also belongs to Δ_ε. In virtue of their definitions, δ_s and δ_t are both differentiable mappings on $\mathbf{M}_t$ and $\mathbf{M}$, respectively. Then $\delta_s \circ \delta_t$ is a differentiable mapping on $\mathbf{M}$ and we have

$$(\delta_s \circ \delta_t)(\mathrm{P}) = \delta_s(\delta_t(\mathrm{P})) = \delta_s(\mathrm{p}_t(\mathrm{p}^{-1}(\mathrm{P}))) = \delta_s(\mathrm{p}_t(\lambda)) = \delta_s(\mathrm{P}_t) = \\ = (\mathrm{p}_{s+t} \circ \mathrm{p}_t^{-1})(\mathrm{P}_t) = \mathrm{p}_{s+t}(\lambda) = \mathrm{P}_{s+t} = \delta_{s+t}(\mathrm{P}),$$

for every $\mathrm{P} = \mathrm{p}(\lambda)$, $\lambda \in \mathcal{B}$. It yields $\delta_s \circ \delta_t = \delta_{s+t}$.

For $t = \varepsilon$ the consecutive displacement of δ_s are all displacements $\delta_s \in \Delta_\varepsilon$ which correspond to $s \in [-\varepsilon, 0]$.

δ_0 is the identity transformation on each submanifold $\mathbf{M}_t$ because, if $t=0$ on $\mathbf{M}_0=\mathbf{M}$, $\delta_0=\mathrm{p}_0\circ\mathrm{p}^{-1}=\mathrm{Id}_{\mathbf{M}}$ and for $t\neq 0$, $\delta_0=\mathrm{p}_{0+t}\circ\mathrm{p}_t^{-1}$, on $\mathbf{M}_t$ we have $\delta_0(\mathrm{P}_t)=\mathrm{p}_t(\mathrm{p}_t^{-1}(\mathrm{P}_t)=\mathrm{p}_t(\lambda)=\mathrm{P}_t$, that implies $\delta_0=\mathrm{Id}_{\mathbf{M}_t}$. This ends the proof. But, we also have to remark:

Remark 1 *Δ_ε is a subgroup of $\Delta_{\mathbb{R}}$ when $\mathbb{I}$ is extended to $\mathbb{R}$. But only the group $\Delta_{\mathbb{R}}$ is isomorphic with the group $(\mathbb{R},+)$.*

Remark 2 *In the previous theorem to the real parameter $t\in[-\varepsilon,\varepsilon]$ it corresponds a time parameter $t^*\in[0,2\varepsilon]$, defined by*

$$t^*=\begin{cases} t, & t\in[0,\varepsilon] \\ \varepsilon-t, & t\in[-\varepsilon,0]\end{cases} . \tag{3}$$

We also can show that:

If $\mathbf{M}=\mathrm{p}(\mathcal{B})$ and $\tilde{\mathbf{M}}=\tilde{\mathrm{p}}(\mathcal{B})$ are two body-manifolds obtained by embedding of a body $\mathcal{B}$ in the Euclidean affine space with the help of two placements p *and* $\tilde{\mathrm{p}}(\in C^k(\mathcal{B}))$*, respectively, there exists an one-parameter group of transformation on the differentiable manifold $\boldsymbol{M}=\cup_{t\in\mathbb{I}}\mathbf{M}_t$ describing any deformation $\phi:\mathbf{M}\to\tilde{\mathbf{M}}$, where $\mathbf{M}$ and $\tilde{\mathbf{M}}(\subset\boldsymbol{M})$ are two submanifolds obtained for some $t\in\mathbb{I}$ and $\{\mathbf{M}_t\}$ are the body images by the placements of the family $\{\mathrm{p}_t\}_{t\in I}$.*

Indeed, let $\phi(t)$ be the corresponding deformation of $\mathbf{M}$ at the moment t in a time interval. Then, $\phi(t)(\mathbf{M})=\mathbf{M}_t(\subset\boldsymbol{M})$ is the deformed body-manifold at the moment t.

We have to show: $\phi(t)=\delta_t$ for $\delta_t\in\Delta_\varepsilon$.

The orbit of the point $\mathrm{P}(\in\mathbf{M})$ by the group is defined by the formula

$$\overline{\mathrm{OrbP}}:=\{\delta_t(\mathrm{P})|t\in[-\varepsilon,\varepsilon]\} \tag{4}$$

and it is obtained with the help of immersion

$$C_{\mathrm{P}}\colon[-\varepsilon,\varepsilon]\to\boldsymbol{M},\ \ \texttt{such that}\ \ C_{\mathrm{P}}(t):=\delta_t(\mathrm{P}),\ (\delta_t\in\Delta_\varepsilon).$$

It is an *orbit segment* of OrbP by the group $\Delta_{\mathbb{R}}$.

We can associate to Δ_ε a differentiable vector field X on $\boldsymbol{M}$, defined as usual by (see, for example, [9])

$$\mathrm{X}_{\mathrm{P}}f=\frac{df(\delta_t(\mathrm{P}))}{dt}|_{t=0}$$

for every differentiable function $f : \overline{\mathrm{OrbP}} \to \mathbb{R}$.

The field X defined by the group Δ_ε is complete, and it is the only one field on $\boldsymbol{M}$ defined by Δ_ε with such a property. The integral curves of X in $\boldsymbol{M}$ are the orbits of the points of $\mathbf{M}$ by the elements of Δ_ε, that is $X(C_\mathrm{P}(t)) = \mathrm{dC_P}(t)/\mathrm{dt}$, with $C_\mathrm{P}(0) = \mathrm{P}$. So, the point $\mathrm{P}(\in \mathbf{M})$ has not another orbit in $\boldsymbol{M}$ than $\{C_\mathrm{P}(t) \mid t \in [-\varepsilon, \varepsilon]\}$. Because ϕ and δ_t are both diffeomorphisms, this implies $\tilde{\mathrm{P}} = \phi(\mathrm{P})$ must belong to OrbP for all $t \in \mathbb{I}$.

Let $t = \varepsilon$ be the corresponding value of the final position $\tilde{\mathrm{P}}$ of P with respect to the deformation ϕ as a point of $\{C_\mathrm{P}(t) \mid t \in \mathbb{I}\}$. Thus, $\tilde{\mathrm{P}} = \delta_\varepsilon(\mathrm{P})$, that shows $\phi(\varepsilon) = \delta_\varepsilon$.

To finish the proof it is enough to repeat the previous argument for an arbitrary intermediate position $\mathrm{P}_t(\in \mathbf{M}_t)$ of P with respect to the deformation $\phi(t)$ for $t \in [-\varepsilon', \varepsilon']$, with $\varepsilon' < \varepsilon$. We will obtain again $\mathrm{P}_t \in \overline{\mathrm{OrbP}}$ and $\phi(\varepsilon') = \delta_{\varepsilon'}$ for $t = \varepsilon'$. This ends the proof.

Remark 1 *In order to describe only just finite deformations* $\mathbf{M} \to \tilde{\mathbf{M}}$ *we restricted the group action to a symmetric interval* $[-\varepsilon, \varepsilon] \subset \mathbb{R}$*, such that its right half interval* $[0, \varepsilon]$ *corresponds to a continuous differentiable deformation, while* $[-\varepsilon, 0]$ *corresponds to the relaxation of body from the final state to the initial one when the external loads that produced the deformation are eliminated.*

The point $\mathrm{P} \in \mathbf{M}$, defining the reference position in $\mathcal{E}^3$ of a particle $\lambda \in \mathcal{B}$, is called the *starting point (origin)* of its orbit in $\boldsymbol{M}$ by the group Δ_ε, denoted $\overline{\mathrm{OrbP}}$, in virtue of the relation $C_\mathrm{P}(0) = \mathrm{P}$. Then $\tilde{\mathrm{P}} = \phi(\mathrm{P})$ will be the *end point* of this orbit, according to the relation $\tilde{\mathrm{P}} = \delta_\varepsilon(\mathrm{P})$ (in the previous theorem), which implies $C_\mathrm{P}(\varepsilon) = \tilde{\mathrm{P}}$.

So, we also can prove that:

The manifold $\boldsymbol{M} = \cup_{t\in\mathbb{I}}\mathbf{M}_t$ *is a disjoint union of orbit segments of the one-parameter group of displacements* Δ_ε *obtained from* $\Delta_\mathbb{I}$ *by restriction to the closed subinterval* $[-\varepsilon, \varepsilon] \subseteq \mathbb{I}$, that is

$$\boldsymbol{M} = \cup_{\mathrm{P}\in\mathbf{M}}\overline{\mathrm{OrbP}} \tag{5}$$

Indeed, it is known the fact that if Q is a point of the orbit of P by the group $\Delta_\mathbb{R}$, i.e. if $\mathrm{Q} \in \mathrm{OrbP}$, then $\mathrm{OrbQ} = \mathrm{OrbP}$.

On the other hand, the meaning of the relation (2) in the first theorem is that for $t \in \mathbb{I} \diagdown [-\varepsilon, \varepsilon]$ the body $\mathcal{B}$ is not deformed anymore. So, $\mathbf{M}_t$ is not empty only for $t \in [-\varepsilon, \varepsilon]$.

Thus, according to the previous consideration, each point Q of the undeformed body-manifold $\mathbf{M}(\subset \mathcal{E}^3)$ may be considered as the starting point of an orbit in $\boldsymbol{M}$ by the group Δ_ε. To underline this assertion we observe that:

(i). If $\mathrm{Q} \in \overline{\mathrm{OrbP}}$, then there exists a positive number $t \in [0, \varepsilon]$ such that $C_{\mathrm{P}}(t) = \mathrm{Q}$ and we successively have

$$C_{\mathrm{Q}}(0) = \delta_0(\mathrm{Q}) = \delta_0(C_{\mathrm{P}}(t)) = \delta_0(\delta_t(\mathrm{P})) = \delta_t(\mathrm{P}) = C_{\mathrm{P}}(t) = \mathrm{Q},$$

which shows Q is the starting point of an orbit segment, $\overline{\mathrm{OrbP}} \subset \boldsymbol{M}$. Analogously, the end point of $\overline{\mathrm{OrbP}}$ is $\tilde{\mathrm{Q}}$.

$\overline{\mathrm{OrbP}}$ and $\overline{\mathrm{OrbQ}}$ have common points excepting the arcs $\overleftrightarrow{\mathrm{PQ}}$ and $\overleftrightarrow{\tilde{\mathrm{P}}\tilde{\mathrm{Q}}}$ of C_{P} and C_{Q}, respectively.

(ii). If $\mathrm{Q} \notin \overline{\mathrm{OrbP}}$, then $\mathrm{OrbQ} \cap \mathrm{OrbP} = \emptyset$, and there is an orbit of $\mathrm{Q}, C_{\mathrm{Q}}$, with Q as a starting point for $\overline{\mathrm{OrbQ}}$, that is $C_{\mathrm{Q}}(0) = \mathrm{Q}$. This follows by replacing P in the statements with any other point, Q, of $\boldsymbol{M}$, as reference position of a body particle.

Similarly will be treated the case when $P \in \overline{\mathrm{OrbQ}}$; then we refer again to (i). This ends the proof.

3 The vector field of displacements

With the help of the immersion $C_{\mathrm{P}} : [-\varepsilon, \varepsilon] \to \boldsymbol{M}$, defined by $C_{\mathrm{P}}(t) = \delta_t(\mathrm{P}) \doteq \mathrm{P}_t$, for all $\delta_t \in \Delta_\varepsilon$, we obtain the orbits of the points $\mathrm{P} \in \mathbf{M}_{t=0}$, as $\overline{\mathrm{OrbP}} = C_{\mathrm{P}}([-\varepsilon, \varepsilon])$. Let $\mathbf{c}$ be one of these curves in $\boldsymbol{M}$, which corresponds to a restriction of C_{P} to the positive semi-interval $[0, \varepsilon]$ and describes deformation (in the "direct sense"); the remaining negative semi-interval describes relaxation, in the elastic cases.

An analytical parametric representation of $\mathbf{c}$ (in its direct sense) can be obtained if we choose a local coordinate system in a neighborhood $\mathrm{U}(\subset \mathbf{M})$ of P, as $\{u^i \circ h\}, (i = 1, 2, 3)$, where (U, h) is a local chart on $\mathbf{M}$, and u^i are the local coordinate functions. The mapping h, that maps the open sets $\mathbf{U}_{t,\alpha}$, of $\boldsymbol{M}$ onto homeomorphic images $\mathbf{D}_{t,\alpha}$, in $\mathbb{R}^3$ for all $(t, \alpha) \in \mathbb{I} \times A$, is nothing else than passing to the coordinates of points of

$\boldsymbol{M}$ by means of the position vectors in $\mathbb{R}^3$ with respect to an orthonormal frame of this space. Thus, let us introduce the coordinates of the starting point P and of a current point P_t of the orbit $\mathbf{c}$ by $h(C_{\mathrm{P}}(0)) = h(\mathrm{P}) \doteq \mathbf{x}$ and $h(C_{\mathrm{P}}(t)) = h(\mathrm{P}_t) \doteq \bar{\mathbf{x}}$. Assume $\mathbf{x} = (x^1, x^2, x^3)$ and $\bar{\mathbf{x}} = (\bar{x}^1, \bar{x}^2, \bar{x}^3)$ with respect to the considered above frame.

Now we denote by $v^i = \bar{x}^i - x^i$, $(i = 1, 2, 3)$, the components of the related vector $v_{\mathrm{P}} = \overrightarrow{\mathrm{PP}_t}$ which describes the displacement of the point of $\mathrm{P}(\in \mathbf{M})$ in $\mathcal{E}^3$ at the moment t; it is called *displacement vector of deformation*.

Thus, the (parametric) equations of $\mathbf{c}$ are:

$$u^i(\bar{x}) = \bar{x}^i(t), \ (i = 1, 2, 3), \tag{6}$$

of which functions in the right-hand side of the equalities are

$$\bar{x}^i(t) = x^i + v^i(x^1, x^2, x^3, t), \ (i = 1, 2, 3), \tag{7}$$

with the initial conditions and that defines the second end point

$$t = 0 : v^i(\mathbf{x}, 0) = 0 \ \text{ and } \ t = \varepsilon : v^i(\mathbf{x}, \varepsilon) = \tilde{v}^i \tag{8}$$

Because $\mathbf{x} = (x^1, x^2, x^3)$ gives here the coordinates of the point $\mathrm{P}(\in \mathbf{M} \subset \mathcal{E}^3)$, the related vector $\mathbf{v}_{\mathrm{P}} = \overrightarrow{\mathrm{PP}_t}$, of components v^i, permits us to consider the associated mapping

$$\mathbf{v} : \Omega \times [0, \varepsilon] \to \mathbb{R}^3, \ \text{ defined by } \ \mathbf{v}(\mathbf{x}, t) := (v^1, v^2, v^3),$$

which is said to be the *displacement vector field*; here $\Omega = \cup_{\alpha \in A} D_{0,\alpha}, (t \in [0, \varepsilon])$, is a bounded domain of $\mathbb{R}^3$ with boundary $\partial\Omega$.

Finally, we observe that the deformation ϕ can also be looked upon, equivalently, as a diffeomorphism between two bounded domains of $\mathbb{R}^3$, Ω and $\Omega_t, (t \in [0, \varepsilon])$, that is $\phi \in C^1(\Omega, \mathbb{R}^3)$. So, this deformation is just described by the equations (7) of the orbits of Δ_ε in $\boldsymbol{M}$. But, for a more accurate approach of the "local deformation study" there are necessary, in addition, many other physical and geometrical elements. One of them, known as a main element of any deformation, is the *deformation gradient*. It is introduced as a continuous mapping

$$\mathbf{F} := \nabla\phi \in C^0(\Omega, \mathrm{M}_{3\times 3}(\mathbb{R})), \ \text{ defined by } \ \mathbf{F}(\mathbf{x}) := F_x,$$

where

$$F_x = \begin{bmatrix} 1+v^1_{,1} & v^1_{,2} & v^1_{,3} \\ v^2_{,1} & 1+v^2_{,2} & v^2_{,3} \\ v^3_{,1} & v^3_{,2} & 1+v^3_{,3} \end{bmatrix}, \ (v^i_{,j} = \frac{\partial v^i}{\partial x^j}), \tag{9}$$

is the matrix of the deformation gradient, with all entries computed at the point $\mathbf{x}$. The geometric meaning of the deformation gradient can be observed in the next section.

4 Moving frames associated to one - parameter groups

We associate to Δ_ε a moving frame consisting in a collection of three differentiable vector fields on $\boldsymbol{M}$, $\{\bar{\mathbf{X}}_1, \bar{\mathbf{X}}_2, \bar{\mathbf{X}}_3\}$, such that the vector fields are defined along to $\mathbf{c}$ as follows.

$\bar{\mathbf{X}}_1$ is the field of which integral curves in $\boldsymbol{M}$ are the orbits of the points of $\mathbf{M}$ by elements of Δ_ε:

$$\bar{\mathbf{X}}_1(C_{\mathrm{P}}(t)) = \frac{\mathrm{d}C_{\mathrm{P}}(t)}{\mathrm{dt}}, \quad C_{\mathrm{P}}(0) = \mathrm{P}. \tag{10}$$

Thus, with respect to a local coordinate system we expressed it under the form

$$\bar{X}_1 = F_{\mathbf{x}} \cdot X_1 \tag{11}$$

where X_1 and $\bar{X}_1$ denote the column-matrices of derivatives $\mathrm{dx}^i/\mathrm{dt}$ and $\mathrm{d}\bar{x}^i/\mathrm{dt}$, respectively, while the matrix $F_{\mathbf{x}}$ is the deformation gradient matrix at the position $\mathbf{x}$ of the point P; this is the matrix of the tangent mapping ϕ_*: $T_{\mathrm{P}}(\mathbf{M}) \rightarrow T\delta_t(\mathrm{P})(\mathbf{M}_t)$, i.e. $[\phi_*] = F_x$, associated to ϕ.

Now we consider the assignment (along $\overline{\mathrm{OrbP}}$) $\delta_t(\mathrm{P}) \rightarrowtail \mathbf{f}_1 = \beta\bar{\mathbf{X}}_1$, with $\beta = 1/||\dot{C}(t)||$- the so-called "speed factor".

The remaining two vector fields $\bar{\mathbf{X}}_2$ and $\bar{\mathbf{X}}_3$ along $\mathbf{c} = \overline{\mathrm{OrbP}}$ are defined by

$$\bar{X}_k = C^{(k)}_{\mathrm{P}}(t) - \sum_{j=1}^{k-1} \langle C^{(k)}_{\mathrm{P}}(t); f_j \rangle f_j, (k = 2, 3) \tag{12}$$

where $\mathbf{f}_j = \beta_j \bar{\mathbf{X}}_j, (\beta_j = 1/ < \bar{\mathbf{X}}_j; \bar{\mathbf{X}}_j >^{1/2})$, and all the vectors are considered at $\delta_t(\mathrm{P})$. We also define $\mathbf{f}_3 := \beta_3 \bar{\mathbf{X}}_3, (\beta_j = 1/||\bar{\mathbf{X}}_3||)$, and, so, we obtain a distinguished moving frame of Frenet type along $\overline{\mathrm{OrbP}}$.

This will serve to the study of the local behavior of the orbit $\mathbf{c}$ in a neighborhood of a current point $\delta_t(\mathrm{P})$ and, so, will provide an information about the body local deformation. Naturally, it will be not yet complete because of the absence of the body material properties, such as elastic constants, etc., which will appear below in 6. and in 7.

5 The components like curvatures of a deformation

We assume the components of the displacement vector $\mathbf{v}$ and their first order derivatives with respect to $t(\in [0, \varepsilon])$ are small and we neglect squares and products of these quantities in comparison with their first powers. With respect to such an approximation in the classical (infinitesimal) theory of elasticity, the covariant strain tensor becomes ([8], p.149)

$$\gamma_{ij} = \frac{1}{2}(v^i|_j + v^j|_i) \tag{13}$$

But, generally, it is introduced by the relation

$$\gamma_{ij} = \frac{1}{2}(\tilde{g}_{ij} - g_{ij}), \tag{14}$$

with the metric tensors defined by $g_{ij} =< \mathbf{e}_i; \mathbf{e}_j >$ at the point P and by $\tilde{g}_{ij} =< \tilde{\mathbf{e}}_i; \tilde{\mathbf{e}}_j >$ at $\delta_\varepsilon(\mathrm{P})$ of the original and strained bodies, respectively, where $\{\mathbf{e}_i\}$ and $\{\tilde{\mathbf{e}}_i\}$ form two bases at corresponding points in $\mathcal{E}^3$.

If we denote by $\mathbf{x} = (x^i)$ and $\bar{\mathbf{x}} = (\bar{x}^i)$ the position vectors of the origin and of a current point of an arbitrary orbit $\mathbf{c}$ of the group Δ_ε, respectively, the unit tangent vector at $\delta_t(\mathrm{P}) \in \mathbf{c}$, given by the relation

$$\mathbf{f}_1 = \mathbf{e}_1 + \beta \frac{\mathrm{d}\mathbf{v}}{\mathrm{dt}}, \quad (\beta = 1/||\dot{C}(t)|| = \frac{\mathrm{dt}}{\mathrm{ds}}), \tag{15}$$

is obtained by the differentiation with respect to arc length from the displacement equation at a current point:

$$\bar{\mathbf{x}} = \mathbf{x} + \mathbf{v}. \tag{16}$$

Computing the partial derivatives with respect to x^i, $(i = 1, 2, 3)$, from (16) we obtain the vectors

$$\tilde{\mathbf{e}}_i = \mathbf{e}_i + \mathbf{v}_{,i}, \tag{17}$$

which defines a moving frame in $\boldsymbol{M}$ along to the orbit $\mathbf{c}$. We observe that $(\mathbf{e}_i)$ is the standard basis at P, and, so $g_{ij} = \delta_{ij}$. Thus, the unit vector $\mathbf{e}_1$ in (15) is one and the same with the first vector of this basis. Now computing the tensor $\tilde{g}_{ij}$, for the linear approximation of the strain tensor, with (14), in the plane (i, j) at $\delta_t(\mathrm{P})$ we have

$$\gamma_{ij} = \frac{1}{2}(< \mathbf{e}_i;\mathbf{v}_{,j} > + < \mathbf{e}_j;\mathbf{v}_{,i} >). \tag{18}$$

Let us consider the skew-symmetric covariant tensor, associated to the previous one, defined by

$$\omega_{ij} = \frac{1}{2}(< \mathbf{e}_i;\mathbf{v}_{,j} > - < \mathbf{e}_j;\mathbf{v}_{,i} >) \tag{19}$$

Now we define a tensor ω of which non-null components are only $\omega_{i,i+1} = -\omega_{i+1,i}$, $(i = 1, 2)$, given by the formulas (19), and we will call it *tensor like curvature* of the orbit $\overline{\mathrm{OrbP}}$ projected on the planes generated by the pairs of vectors ($\tilde{\mathbf{e}}_i$,$\tilde{\mathbf{e}}_{i+1}$) of a moving frame in $\boldsymbol{M}$ associated to the orbit .

This can be explained by the similitude of the scalar components of the tensor ω with the coefficients (that define curvatures, [12]) in the Cartan's equations associated to the Frenet's moving frame $\{C(t); \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ along a curve C in $\mathcal{E}^3$:

$$\dot{C}(t) = \sum\nolimits_i \alpha^i(t)\mathbf{f}_i(t)$$

$$\dot{f}_j(t) = \sum\nolimits_i \omega^i_j(t)\mathbf{f}_i(t),$$

where the coefficients of the second equation are given by the scalar products $\omega^i_j = \dot{\mathbf{f}}_i \cdot \mathbf{f}_j (= -\omega^j_i)$ and satisfy the condition $\omega^i_j = 0$, when $j > i + 1$. Also we have $\alpha^i = 0$, for $i > 1$. In order to compute the components (19) of ω, we will start with the sums

$$\gamma_{ij} + \omega_{ij} =< \mathbf{e}_i;\mathbf{v}_{,j} > \tag{20}$$

and we will express the differential of the displacement vector in terms of covariant derivatives. So, if we consider the differential $d\mathbf{v} = \mathbf{v}_{,j}dx^j$, having in view the formula $\mathbf{v}_{,j} =< \mathbf{e}_i;\mathbf{v}_{,j} > \mathbf{e}_i$ consistent with respect to an orthonormal basis, we can write it under the form

$$d\mathbf{v} = \sum_{i,j=1}^{3} (\gamma_{ij} + \omega_{ij})dx^j \mathbf{e}_i. \tag{21}$$

On the other hand, we also have $d\mathbf{v} = v^i|_j dx^j \mathbf{e}_i$. Here the components of the differential are covariant derivatives of the components of $\mathbf{v}$, that is

$$v^i|_j = v^i_{,j} + \Gamma^i_{rj} v^r \tag{22}$$

where Γ^i_{rj} are the Christoffel symbols of second kind with respect to the coordinate system (x_1, x_2, x_3).

Thus, we have, from (21) and (22),

$$\gamma_{ij} + \omega_{ij} = v^i|_j \tag{23}$$

and combining (23), (19), (20) and (13) it results for the like curvature tensor components the following expressions :

$$\omega_{ij} = \frac{1}{2}(v^i|_j - v^j|_i), \quad (i, j = 1, 2, 3). \tag{24}$$

According to our definition of ω, we assume that the considered above small deformations are symmetric with respect to the plane $(i = 1, j = 3)$. Then $\omega_{13} = 0$, and we can associate to the deformation the following skew-symmetric matrix

$$\Omega = \begin{bmatrix} 0 & \omega_{12} & 0 \\ -\omega_{12} & 0 & \omega_{23} \\ 0 & -\omega_{23} & 0 \end{bmatrix} \tag{25}$$

where ω_{12} and ω_{23} measure the small "deviations" of the orbit projections on the planes generated by the pairs $(\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2)$ and $(\tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3)$ from the directions of $\tilde{\mathbf{e}}_1$ and $\tilde{\mathbf{e}}_2$, respectively. Finally, we remark that in the cases of orthogonal systems of curvilinear coordinates (x_1, x_2, x_3) at $\delta_t(\mathrm{P})$ in the manifold $\boldsymbol{M}$ the covariant derivatives can be replaced by the usual ones, and the matrix of components of the tensor ω can be obtained as half of the difference between the deformation gradient matrix and its transposed :

$$2\Omega = F_{\mathbf{x}} - F_{\mathbf{x}}^{T}. \tag{26}$$

6 Metric structure of a deformed body - manifold

Here we establish the following result:

Assume the body-manifold $\mathbf{M}$ *is endowed with a Riemannian metric* g *whose components with respect to a local coordinate system* $\{x^i\}, (i = 1, 2, 3)$, *on an open set* $\mathbf{U} \subset \mathbf{M}$ *by the entries of the matrix* D *of elastic constants of the body is defined.*

Then, the Gram matrix $\tilde{G}$ *defining the metric structure of the tangent spaces to the deformed body,* $T_{\tilde{\mathrm{P}}}(\tilde{\mathbf{M}})$, *at image points by a deformation* $\phi : \mathbf{M} \to \tilde{\mathbf{M}}$ *is of the form*

$$\tilde{G} = F_{\mathbf{x}}^{-T} \cdot D \cdot F_{\mathbf{x}}^{-1}, \tag{27}$$

where $F_{\mathbf{x}}$ *denotes the deformation gradient matrix at* $\mathbf{x} = (x^i)(\in \mathbb{R}^3)$, *which is the image of the point* $\mathrm{P} \in \mathbf{U}$ *with respect to a local cart* $(\mathbf{U}, h)$ *on* $\mathbf{M}$.

First we observe D is a symmetric nonsingular matrix of size $s \times s$, $(s = 2, 3, 6, \ldots)$. We can prove, it is also positive definite for many classes of body-manifolds. For instance, if we fix $s = 3$,

$$D = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda \\ \lambda & \lambda + 2\mu & \lambda \\ \lambda & \lambda & \lambda + 2\mu \end{bmatrix},$$

where λ and μ are the elastic constants of Lame:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)},$$

and, for $\nu < 1/2$, all the corner minors of D will be positive.

So, a body-manifold $\mathbf{M}$ corresponding to such a matrix D of elastic constants can be endowed with a Riemannian structure; otherwise, we may introduce a pseudo-Riemannian structure on it.

Now we consider a bijective differentiable mapping $\phi : \mathbf{M} \to \tilde{\mathbf{M}}$ of class $C^k, (k \geq 1)$, that is a deformation of $\mathbf{M}$.

At each point P of an open set $\mathbf{U} \subset \mathbf{M}$ we join to $\mathbf{M}$ its tangent space $T_P(\mathbf{M})$ whose Euclidean structure by the elements of D is defined. Let $(\mathbf{e}_i)$ be the corresponding basis of $T_P(\mathbf{M})$. The deformed basis $(\tilde{\mathbf{e}}_i)$ of $T_{\tilde{P}}(\tilde{\mathbf{M}})$ at the end point of the orbit $\mathbf{c}$ of P in the trace-manifold $\boldsymbol{M}$ is connected with the initial one by the relation

$$\mathbf{G}\mathbf{e}_i = \tilde{\mathbf{e}}_i, \quad (i = 1, 2, 3), \tag{28}$$

where $\mathbf{G} = \bar{x}^i_{,j}\mathbf{e}_i \otimes \varepsilon^j$ is the *deformation gradient tensor* of type (1,1), $\mathbf{G} : T_P(\mathbf{M}) \to T_{\tilde{P}}(\tilde{\mathbf{M}})$, of coordinates $\bar{x}^i_{,j} = \delta^i_j + v^i_{,j}$; here ε^j denotes the covectors of $\mathbf{e}_i$ (see, for instance, [1], pp.20-21).

From this it results the coordinate matrix of the tensor $\mathbf{G}$ is the gradient matrix F_x at $x = (x^1, x^2, x^3)$, given by (9).

Thus, if E and $\tilde{E}$ are the row-matrices of the canonical bases $(\mathbf{e}_i)$ and $(\tilde{\mathbf{e}}_i)$, respectively, the relation (28) can be translated into a matrix equation such as

$$\tilde{E} = E \cdot F_x. \tag{29}$$

According to the hypothesis we have $E^T \cdot E = D$.

So, $\tilde{G}$ defined by the similar matrix product $\tilde{E}^T \cdot \tilde{E}$, can be written under the form (27), if we take into account the relation (29). This ends the proof.

The basic idea of the previous theorem consists in the assertion that the Euclidean structure of each tangent space to the strained body is directly connected with the material properties of the body during the deformation, that is, for an isotropic elastic body it can be described by the matrix of elastic constants.

Thus, if $<,>_{\tilde{g}}$ denotes the inner-product on the Euclidean vector space $T_{\tilde{P}}(\tilde{\mathbf{M}})$ and $\tilde{q}$ is the associated quadratic form, then we have the equality $[\tilde{q}] = \tilde{G}$ with respect to the basis $(\tilde{\mathbf{e}}_i)$, where $\tilde{G}$ by the relation (27) is obtained.

7 The energy of deformation and stiffness matrices

Another facet of this development is that to consider the body-manifold **M** as an elemental component of a material structure with s nodes. Then we may associate to it the corresponding stiffness matrix of size $s \times s$, usually used in the Finite Element Method and expressing the rigidity properties of the body with respect to the deformation.

The energy of a continuous deformation of a body-manifold $\phi : \mathbf{M} \to \tilde{\mathbf{M}}$ *depends on the stiffness matrix* K *associated to the body by means of the relation*

$$E(\phi) = \frac{1}{2}[\delta]^T \cdot K \cdot [\delta], \tag{30}$$

where $[\delta]$ *denotes the column-matrix of the displacement at each point of* **M**.

To establish the relation (30) we start with the energy $E(\phi)$ associated to the diffeomorphism ϕ defined by the expression ([7], p.202)

$$E(\phi) = \frac{1}{2}\int_M \|\phi_*\|^2 d\omega, \tag{31}$$

where ϕ_* is the differential of ϕ and $d\omega$ is the volume element of M. The Euclidean norm $||.||$ here present acts on the tangent space $T_{\tilde{P}}(\tilde{\mathbf{M}})$ at each image $\tilde{P} = \phi(\mathrm{P}), (\mathrm{P} \in \mathbf{M})$.

Denote by $\mathbf{X}$ and $\tilde{\mathbf{X}}$ two vector fields on **M** and $\tilde{\mathbf{M}}$, respectively, which are ϕ-related in the sense that if X_P is the tangent vector at a point $\mathrm{P} \in \mathbf{M}$ to its orbit $\mathbf{c} \subset \boldsymbol{M}$, then during the deformation $\tilde{\mathbf{X}}_{\tilde{\mathrm{P}}} = \phi_* X_P$ is the strained vector tangent to the same curve at $\tilde{\mathrm{P}}$. Here we mention that the two considered above vectors fields are not unit.

Thus, if we denote by X and $\tilde{X}$ the column-matrices of the coordinates of the considered above vectors an equation of type (11),

$$\tilde{X} = F_{\mathbf{x}} \cdot X, \tag{32}$$

holds.

The expression (31) of the energy of ϕ can be successively transformed as

$$E(\phi) = \frac{1}{2}\int_M \langle \phi_* X_P ; \phi_* X_P \rangle_{\tilde{g}} d\omega = \frac{1}{2}\int_M \tilde{q}(\phi_* X_P) d\omega = \frac{1}{2}\int_M \tilde{X}^T \cdot \tilde{G} \cdot \tilde{X} d\omega$$

(33)

According to the relations (32) and (27) we also have

$$E(\phi) = \frac{1}{2}\int_M X^T \cdot (F_{\mathbf{x}} \cdot \tilde{G} \cdot F_{\mathbf{x}}) \cdot X d\omega, \tag{34}$$

or, still,

$$E(\phi) = \frac{1}{2}\int_M X^T \cdot D \cdot X d\omega. \tag{35}$$

Now we choose that vector field **X** on **M** of which value at each point $\mathrm{P}, \mathbf{X}(\mathrm{P}) = X_P$, will coincide with the vector $[\varepsilon]$ of principal strains. It is known the fact that this last one is a tangent vector to the integral curve of **X**.

Thus $E(\phi)$ will coincide with the strain energy of deformation U.

But the strain vector is connected with the displacement vector at each point of the strained body-manifold by means of the matrix equation

$$[\varepsilon] = B \cdot [\delta], \tag{36}$$

where B is a $s \times s$-matrix of derivatives of a vector-valued function of nodal coordinates. Because the strain energy of deformation associated to **M** is of the form

$$\mathrm{U} = \frac{1}{2}[\delta]^T(\int_M B^T \cdot D \cdot B d\omega)[\delta] \tag{37}$$

and we observe that the value under integral represents the stiffness matrix

$$K = \int_M B^T \cdot D \cdot B d\omega \tag{38}$$

of the body-manifold, from (35), (36), (37) and (38), for $X = [\varepsilon]$ it results the relation (30), which ends the proof.

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O mnogostrukosti traga odredjenog deformacijama mnogostrukosti tela

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U radu se razmatra deformacija materijalne sredine koristeci neke elemente moderne diferencijalne geometrije. Uvodi se pokretni sistem Freneovog tipa duž orbita jednoparametarske grupe koja deluje na takozvane "mnogostrukosti traga - trace-manifold" , M, a koja se odnodi na deformacije tela. Takva mnogostrukost se definiše kao beskonačna unija neprekidnih kompaktnih mnogostrukosti koju generišu uzastopni položaji (mnogostrukosti) tela usled njegove deformacije u afinom Euklidskom trodimenzionom prostoru u zatvorenom vremenskom intervalu. Uvodi se antisimetričan (povezujuci) tenzor drugog reda ω koji opisuje deformaciju u bliskoj okolini bilo koje tacke duž orbite. Komponente tenzora $\omega_{i,i+1}$, različite od nule, tenzora ω predstavljaju krivine u svakoj tački orbite a u ravnima odredjenim parom vektora $(\mathbf{e}_i, \mathbf{e}_{i+1})$ pokretnog sistema u M, a koji je asociran orbiti na sličan nacin kao klasičan Freneov sistem. Takodje je data formula za energiju orbita i odredjena njena veza sa nekom matricom krutosti.