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Small elastic strains in finite elasto-plastic materials with continuously distributed dislocations

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Abstract

In this paper we propose a macroscopic model for elasto-plastic materials with continuously distributed dislocations, when we restrict to small elastic strains, but the elastic rotations and plastic distorsions remain large. The material is not homogeneous and it behaves like an elastic material element with respect to non-holonomic configuration. The elastic curvature tensor vanishes. Consequently to complete the definition of the mathematical model it is sufficient to prescribe the evolution equation for the plastic distorsion (i.e.for time-dependent non-holonomic configuration), as well as the for the internal variables.

1 Introduction

In this paper we propose a macroscopic model in the continuum theory of an elasto-plastic material with continuously distributed dislocations, under the hypothesis of small elastic strains, but large elastic rotations and plastic deformations. There are several approaches

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to the bahaviour of finite elasto-plastic crystalline materials based on different physical motivations, developed for instance by Kondo and Yuki [12], Bilby [2], Kröner [13], Teodosiu [23], Naghdi ans Srinivasa [19], Shizawa and Zbib [22], Le and Stumpf [15], [16]. The plastically deformed body is thought as a manifold, embedded in the euclidian space. Consequently, the differential geometrical problems (see Vrânceanu [24], Schouten [21], Kondo and Yuki [12]) are closely related to the mathematical fundations for the theory of continuously distributed dislocations, developed by Noll [20] and Wang [26].

In Beju, So os, Teodorescu [1] (for elastic materials only), Le and Stumpf [17], Epstein and Maugin [9] the problem of the decomposition of the appropriate connection into the cotorsion and the Riemann metric connection plays a principal role. The models of material with dislocations constructed by Kondo and Beju, Soós, Teodorescu [1], Kröner and Lagoudas, [14] are based on Riemann-Christoffel curvature. A Cosserat theory for plastic single crystals at finite deformations is elaborated in [19], [15], where the additional momentum-like laws are associated with the irreversible deformations. In [11] and [8] the couple stresses account for the dislocation density.

In our approach to small elastic strains first we consider the constitutive framework of finite elasto-plastic materials with continuously distributed dislocations proposed by Cleja-Ţigoiu [8]. We derive the peculiar feature of the model based on the assumptions of small elastic strains (see for instance our analyze developed in [7] for anisotropic finite elasto-plasticity). The linearization of the equations derived in the papers [23], [1](herein elastic materials with continuously distributed dislocations), [17], [11] are carried out in order to compare with the well-known Kröner theory. In these papers the elastic distorsion as well as the plastic distorsion are in the neighborhood of the identity tensor. From our point of view the case of small elastic strains but with large elastic rotations, considered by Mandel [18], is compatible with the objectivity assumption in the finite approach to elasto-plastic materials with continuously distributed dislocations.

The mathematical model proposed by Cleja-Ţigoiu [8] and [4] extend the rezults developed in [6], [3] for the elasto-plastic materials with relaxed configurations to the materials with continuously distributed dislocations. The fundamental idea is that the crystalline body is not

homogeneous in Noll's sense [20] and it has a non-holonomic time dependent configuration $\mathcal{K}$, i.e. which can not be generated by a global configuration of $\mathcal{B}$. The plastic distorsion and the plastic connections, supposed to be invariant with respect to a change of frame in the actual configuration, are compatible. This means the Riemann-Christoffel curvature tensor vanishes.

Dynamical balance equations involve **non-symmetric Cauchy stress tensor** and **couple stresses**. Based on energetic arguments developed in Cleja-Ţigoiu and Maugin [5], we choose the set of variables involved in the model. The crystalline body behaves as an **elastic material element** with respect to $\mathcal{K}(t)$ following an idea from [25]. Therefore the stress and the stress momentum defined as a third order tensor attached to the couple stresses are functions of the elastic distortion and its gradients.

The **irreversible behaviour** of the material is described by the evolution equations for the plastic distortion as well as for the gradient of plastic distortion. The evolution equations have to be compatible with the requirement of zero plastic curvature.

In the case of small elastic strains the appropriate elastic curvature introduced here vanishes. Consequently now it is sufficient to prescribe the evolution equations for time-dependent non-holonomic configuration through the rate for plastic distorsion only. Here we impose the invariance requirement to the constitutive and evolution equations derived with respect to the non-holonomic configuration. When we push the model forward to the actual configuration the corotational rates with respect to the elastic spin replace the material derivative of the appropriate fields.

Notations. We use the following notations:
$\mathcal{E}-$ the three dimensional Euclidian space, with the vector space of translations $\mathcal{V}$; $\mathcal{V}_k$, $\mathcal{V}_\chi$, $\mathcal{V}_\mathcal{K}$ are the tangent vector spaces of the body $\mathcal{B}$ related to the reference k, actual χ and non-holonomic $\mathcal{K}$ configurations;
ρ, ρ_0 and $\tilde{\rho}$ are mass densities in the actual, initial and non-holonomic $\mathcal{K}$ configurations;
Lin, Lin^+, Lin_s^+- the set of the linear mappings from $\mathcal{V}$ to $\mathcal{V}$, its subset of invertible transformations and the set of symmetric and positive definite second order tensors, respectively; $Ort-$ the set of all orthogonal

mappings;
$\mathbf{A}^s$ and $\mathbf{A}^a$ are the symmetrical and skew-symmetrical parts of the tensor $\mathbf{A}$ while $\mathbf{A}^T$ denotes the transpose of $\mathbf{A}$;
$\mathrm{d}\phi(x)$ denotes the differential of the function ϕ.
$Lin(\mathcal{V}, Lin) = \{\mathbf{N} : \mathcal{V} \longrightarrow Lin \quad \text{linear}\}-$ defines the set of the third order tensors or (1,2)-tensors, with $\mathbf{N} = N_{ijk}\mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^k$. As usual we have $\mathbf{N} \cdot \mathbf{M} = N_{ijk}M^{ijk}$.
For any second order tensor field $\mathbf{A}$ we denote by $\nabla\mathbf{A}$ the third order tensor field characterizing its differential, i.e. $\nabla\mathbf{A} = \dfrac{\partial A_{ij}}{\partial x^k}\mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^k$; $\epsilon_{ijk}-$ the permutation symbol. We have the formula $\bigtriangledown_\chi \mathbf{L} \equiv \dfrac{\partial}{\partial x^k}(\dfrac{\partial v_i}{\partial x^j})\mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^k$.
$\nabla^2_{\mathcal{K}}\boldsymbol{\epsilon}^e$ denotes the second order differential of the field $\boldsymbol{\epsilon}^e$;
$\mid \boldsymbol{\epsilon}^e \mid_{(4)}, \mid \boldsymbol{\epsilon}^e \mid_{(3)}$, denotes the modulus of the fourth and third order, respectively, tensors.

2 Constitutive framework

We give a brief review of some of the basic assumptions to our approach to elasto-plastic materials with continuously distributed dislocations in [8].

Let k and $\chi(,\cdot)$ be the reference and actual configurations of the body $\mathcal{B}$, respectively. Let $\mathcal{N}_X$ be an open simple connected part of $\mathcal{B}$, for X a given material point of the body.
A1. For any motion χ there exists an invertible second order tensor field $\mathbf{F}^p(\mathbf{Z}, t)$ at time t, defined for all $\mathbf{Z} \in \mathcal{N}_X$, which can not be derived from a (smooth) mapping.

A2. There exists an affine (plastic) connection $\overset{(\mathrm{p})}{\mathbf{\Gamma}}$, i.e. (1,2)-tensor field on $\mathcal{V}_k$, with the property: the Pfaffian system

$$\mathbf{F}^p(\overset{(\mathrm{p})}{\mathbf{\Gamma}}\, d\mathbf{Z}) = d\,\mathbf{F}^p \tag{1}$$

has solution or is integrable.
C1. As a first consequence we derived the necessary and sufficient conditions (Frobenius theorem) to have the complete integrability con-

dition of (1)

$$\begin{aligned}(\overset{(p)}{\mathcal{R}}(\mathbf{Z})\mathbf{u})\mathbf{v} &\equiv (\overset{(p)}{\mathbf{\Gamma}}(\mathbf{Z})\mathbf{u})(\overset{(p)}{\mathbf{\Gamma}}(\mathbf{Z})\mathbf{v}) - (\overset{(p)}{\mathbf{\Gamma}}(\mathbf{Z})\mathbf{v})(\overset{(p)}{\mathbf{\Gamma}}(\mathbf{Z})\mathbf{u}) + \\ &+ (d\overset{(p)}{\mathbf{\Gamma}}(\mathbf{Z})\mathbf{u})\mathbf{v} - (d\overset{(p)}{\mathbf{\Gamma}}(\mathbf{Z})\mathbf{v})\mathbf{u} = 0\end{aligned} \tag{2}$$

written for all $\mathbf{u}, \mathbf{v} \in \mathcal{V}_k$.

$\overset{(p)}{\mathcal{R}}$ is called the Riemann-Christoffel (fourth order) curvature tensor.

$\mathbf{F}^p$ is called the plastic distortion and we define the elastic distortion by

$$\mathbf{F}^e = \mathbf{F}(\mathbf{F}^p)^{-1} \quad \Longleftrightarrow \quad \mathbf{F} = \mathbf{F}^e\mathbf{F}^p,\ \mathbf{F}(\mathbf{Z}, t) = \nabla(\chi(\cdot, t) \circ k^{-1})(\mathbf{Z}), \tag{3}$$

where $\mathbf{F}$ is the deformation gradient relative to k, defined for all $\mathbf{Z} \in k(\mathcal{N}_X)$.

Hence the multiplicative decomposition of the deformation gradient relative to the reference configuration holds.

A non-holonomic configuration of $\mathcal{N}_X$

$$\mathcal{K}(t) = \{\mathbf{K}(X, t) = \mathbf{F}^p(\mathbf{Z}, t) \circ \nabla k(Z), \quad Z \in \mathcal{N}_X, \quad k(Z) = \mathbf{Z}\} \tag{4}$$

at any time t can be defined; here $\mathbf{K}(X, t)$ is a **local configurations** in Noll's sense [20] which can not be generated by a global configuration of $\mathcal{B}$.

Several affine connections of the deformation gradient, elastic and plastic distorsions, respectively, can be introduced related to different possible configurations of the body.

Proposition 1. 1. Two elastic connections

$$\overset{(e)}{\mathbf{\Gamma}} \equiv \overset{(e)}{\mathbf{\Gamma}}_{\chi} = \mathbf{F}^e\nabla_{\chi}(\mathbf{F}^e)^{-1}, \quad \overset{(e)}{\mathbf{\Gamma}}_{\mathcal{K}} = (\mathbf{F}^e)^{-1}\nabla_{\mathcal{K}}\mathbf{F}^e \tag{5}$$

can be defined with respect to the actual and non-holonomic configuration, respectively.

2. $\overset{(e)}{\mathbf{\Gamma}}_{\mathcal{K}}$ is a third order tensor field invariant with respect to a change of frame in the actual configuration, while $\mathbf{F}^e$ and $\overset{(e)}{\mathbf{\Gamma}}_{\chi}$ are objective fields, i.e.

$$\begin{aligned}&(\mathbf{F}^e)^* = \mathbf{Q}(t)\mathbf{F}_{\mathcal{K}(t)}, \quad \overset{(e)}{\mathbf{\Gamma}}_{\chi^*}{}^* = \mathbf{Q}(t)\overset{(e)}{\mathbf{\Gamma}}_{\chi}[\mathbf{Q}^T(t), \mathbf{Q}^T(t)], \text{ or} \\ &((\overset{(e)}{\mathbf{\Gamma}}_{\chi^*}{}^*)\mathbf{u})\mathbf{v} = \mathbf{Q}(t)(\overset{(e)}{\mathbf{\Gamma}}_{\chi}(\mathbf{Q}^T(t)\mathbf{u}))\mathbf{Q}^T(t)\mathbf{v},\end{aligned} \tag{6}$$

$\forall\ \mathbf{u},\mathbf{v}\in\mathcal{V}_\chi$ and for any time dependent orthogonal mapping $\mathbf{Q}(t)$ characterizing the change of frame in the actual configuration.
Proof. The Pfaffian system (1) is integrabil and then there exists an invertible and differentiable mapping $\mathbf{F}^p$ such that the equality

$$\overset{(\mathrm{p})}{\mathbf{\Gamma}} = (\mathbf{F}^p)^{-1}\nabla_k\,\mathbf{F}^p \tag{7}$$

holds on the neighborhood $\mathcal{N}_X$. Thus from (3) the existence of the differentiable elastic distorsion follows. Moreover the elastic connections (5) satisfy the condition (2) to have zero curvature.

The objectivity properties (6) are direct consequences of **The objectivity assumption** given in [8] under the form:
A5. The pair $(\mathbf{F}^p, \nabla_{\mathcal{K}}(\mathbf{F}^p)^{-1})$ which characterizes the relaxed state, attached to the motion χ by assumptions **A1** and **A2**, corresponds to any $\chi^* : \mathcal{N}_X\times\mathbf{R}\longrightarrow\mathcal{E}$, related to χ, by a change of observer.
Remark 1. In the theory we can assume the existence of $\mathbf{F}^e$ or of $(\mathbf{F}^e)^{-1}$ such that the Pfaffian systems similar to (1) have solutions. In this case we define the plastic distortion in term of the elastic distortion by

$$\mathbf{F}^p = (\mathbf{F}^e)^{-1}\mathbf{F}, \tag{8}$$

and so on.

We put into evidence the rule of the calculus for the representation of the differential of the tensor field $\bar{\mathbf{F}}$ when we pass from one configuration to the non-holonomic one

$$\begin{aligned}(\nabla_{\mathcal{K}}\bar{\mathbf{F}})(\tilde{\mathbf{u}}) &= (\nabla_k\bar{\mathbf{F}})((\mathbf{F}^p)^{-1}\tilde{\mathbf{u}}),\\ (\nabla_{\mathcal{K}}\bar{\mathbf{F}})(\tilde{\mathbf{u}}) &= (\nabla_\chi\bar{\mathbf{F}})(\mathbf{F}^e\tilde{\mathbf{u}})\quad\forall\ \tilde{\mathbf{u}}\in\mathcal{V}_{\mathcal{K}}\end{aligned} \tag{9}$$

Let $\boldsymbol{\mu}$ be a third order or (1,2)- tensor, respectively, tensor field. We introduce the useful notation $\boldsymbol{\mu}[\mathbf{F}_1,\mathbf{F}_2]$ for the third order tensor related to $\boldsymbol{\mu}$ by the following relationship

$$(\boldsymbol{\mu}[\mathbf{F}_1,\mathbf{F}_2]\mathbf{u})\mathbf{v} = (\boldsymbol{\mu}(\mathbf{F}_1\mathbf{u}))\mathbf{F}_2\mathbf{v} \tag{10}$$

that holds for all vectors $\mathbf{u},\mathbf{v}$. Here $\mathbf{F}_1$ and $\mathbf{F}_2$ are two arbitrary given second order tensors.

Proposition 2. *1. The connections* $\mathbf{\Gamma}$, $\overset{(e)}{\mathbf{\Gamma}}_{\mathcal{K}}$ and $\overset{(p)}{\mathbf{\Gamma}}_{\mathcal{K}}$ *are related by*

$$\begin{gathered}\overset{(e)}{\mathbf{\Gamma}}_{\mathcal{K}}\ \tilde{\mathbf{u}} = \mathbf{F}^p(\mathbf{\Gamma}(\mathbf{F}^p)^{-1}\tilde{\mathbf{u}})(\mathbf{F}^p)^{-1} + \overset{(p)}{\mathbf{\Gamma}}_{\mathcal{K}}\ \tilde{\mathbf{u}}, \forall\ \tilde{\mathbf{u}} \in \mathcal{V}_{\mathcal{K}},\\ \mathbf{\Gamma} = \mathbf{F}^{-1}(\nabla_k \mathbf{F}),\ \text{or}\\ \overset{(e)}{\mathbf{\Gamma}}_{\mathcal{K}} = \mathbf{F}^p\mathbf{\Gamma}[\mathbf{F}^p)^{-1}, (\mathbf{F}^p)^{-1}] + \overset{(p)}{\mathbf{\Gamma}}_{\mathcal{K}},\ \text{where} \quad \overset{(p)}{\mathbf{\Gamma}}_{\mathcal{K}} = \mathbf{F}^p\nabla_{\mathcal{K}}(\mathbf{F}^p)^{-1},\\ ((\mathbf{\Gamma}[(\mathbf{F}^p)^{-1}, (\mathbf{F}^p)^{-1}])\tilde{\mathbf{u}})\tilde{\mathbf{v}} = (\mathbf{\Gamma}(\mathbf{F}^p)^{-1}\tilde{\mathbf{u}})((\mathbf{F}^p)^{-1}\tilde{\mathbf{v}}) \quad \forall\ \tilde{\mathbf{u}} \in \mathcal{V}_{\mathcal{K}}.\end{gathered} \tag{11}$$

2. The both connections $\overset{(e)}{\mathbf{\Gamma}}_{\mathcal{K}}$ and $\overset{(p)}{\mathbf{\Gamma}}_{\mathcal{K}}$ *have the same torsion*

$$\begin{gathered}(\mathbf{S}_{\mathcal{K}}\tilde{\mathbf{u}})\tilde{\mathbf{v}} \equiv \mathbf{F}^p[((\nabla_{\mathcal{K}}(\mathbf{F}^p)^{-1})\tilde{\mathbf{u}})\tilde{\mathbf{v}} - ((\nabla_{\mathcal{K}}(\mathbf{F}^p)^{-1})\tilde{\mathbf{v}})\tilde{\mathbf{u}}] =\\ = (\mathbf{F}^e)^{-1}[((\nabla_{\mathcal{K}}\mathbf{F}^e)\tilde{\mathbf{u}})\tilde{\mathbf{v}} - ((\nabla_{\mathcal{K}}\mathbf{F}^e)\tilde{\mathbf{v}})\tilde{\mathbf{u}}],\ \forall \tilde{\mathbf{v}}, \tilde{\mathbf{u}} \in \mathcal{V}_{\mathcal{K}}.\end{gathered} \tag{12}$$

Now we recall the **decomposition theorem** for the connection that can be found in [24], [21], [1] and which appears in [17], [12], but written here for the plastic connection:

Theorem of the decomposition of the plastic connection *1. The plastic connection* $\overset{(p)}{\mathbf{\Gamma}}$ *is expressed in terms of the metric tensor* $\mathbf{C}^p$ *and* the **cotorsion** $\mathbf{W}^p$ *by*

$$\overset{(p)}{\mathbf{\Gamma}} = \boldsymbol{\gamma}^p + \mathbf{W}^p. \tag{13}$$

The cotorsion, third order tensor field $\mathbf{W}^p$ *and the torsion* $\mathbf{S}$ *determine each other by*

$$\begin{gathered}(\mathbf{W}^p\mathbf{u})\mathbf{v} = \frac{1}{2}((\mathbf{S}\mathbf{u})\mathbf{v} - (\mathbf{S}\mathbf{u})^T\mathbf{v} - (\mathbf{S}\mathbf{v})^T\mathbf{u}),\\ (\mathbf{S}\mathbf{u})\mathbf{v} = (\mathbf{W}^p\mathbf{u})\mathbf{v} - (\mathbf{W}^p\mathbf{v})\mathbf{u}.\end{gathered} \tag{14}$$

2. The following skew-symmetry holds

$$(\mathbf{S}\mathbf{u})\mathbf{v} = -(\mathbf{S}\mathbf{v})\mathbf{u}, \quad (\mathbf{W}^p\mathbf{u})^T = -\mathbf{W}^p\mathbf{u}, \quad \forall\ \mathbf{u}, \mathbf{v} \in \mathcal{V}_k. \tag{15}$$

3. The torsion $\mathbf{S}$ is defined as skew-symmetric part of the plastic connection

$$(\mathbf{S}\mathbf{u})\mathbf{v} = (\overset{(p)}{\mathbf{\Gamma}}\ \mathbf{u})\mathbf{v} - (\overset{(p)}{\mathbf{\Gamma}})\mathbf{u}, \quad \forall\, \mathbf{u}, \mathbf{v} \in \mathcal{V}_k. \tag{16}$$

In (13) $\boldsymbol{\gamma}^p$ *denotes the Riemann connection, with components given by the second order Christoffel symbols corresponding to the plastic metric tensor* $\mathbf{C}^p$. We use here an intrinsic formula, see also [17],

$$\begin{aligned}(\boldsymbol{\gamma}^p\mathbf{u})\mathbf{v}\cdot\mathbf{w} \ &\equiv \frac{1}{2}(\mathbf{C}^p)^{-1}\mathbf{w}\cdot[((\nabla_k\mathbf{C}^p)\mathbf{v})\mathbf{u} + (\nabla_k\mathbf{C}^p\mathbf{u})\mathbf{v}] - \\ &-\frac{1}{2}\nabla_k\mathbf{C}^p((\mathbf{C}^p)^{-1}\mathbf{w})\mathbf{u}\cdot\mathbf{v}, \quad \mathbf{C}^p = (\mathbf{F}^p)^T\mathbf{F}^p,\end{aligned} \tag{17}$$

$(\mathbf{C}^p)^{-1}$ denotes the inverse tensor of $\mathbf{C}^p$.

3 Small elastic strains and elastic connections

In our approach to small elastic strains first we consider the constitutive framework of finite deformation and subsequently we derive the peculiar feature involved in the model by assuming that the elastic strains are small. The linearization of the equations derived in the papers [23], [1](herein elastic materials with continuously distributed dislocations), [17], [11] are carried out in order to compare with the well-known Kröner theory. In these papers the elastic distorsion as well as the plastic distorsion have the form

$$\mathbf{F}^e = \mathbf{I} + \boldsymbol{\beta}^e, \quad |\, \boldsymbol{\beta}^e \,| \ll 1, \quad \mathbf{F}^p = \mathbf{I} + \boldsymbol{\beta}^p, \quad |\, \boldsymbol{\beta}^p \,| \ll 1, \tag{18}$$

where $\boldsymbol{\beta}^e$ and $\boldsymbol{\beta}^p$ can be neglected in the presence of the identity tensor. In contrast with these paper we adopt here the point of view developed by Mandel [18], based on the polar decomposition of the *elastic* distorsion represented under the form

$$\mathbf{F}^e = \mathbf{R}^e\mathbf{U}^e, \quad \mathbf{R}^e \in Ort, \quad \mathbf{U}^e = \mathbf{I} + \boldsymbol{\epsilon}^e, \quad \text{with} \quad |\, \boldsymbol{\epsilon}^e \,| \ll 1. \tag{19}$$

Hence the elastic strains are small, since

$$\boldsymbol{\Delta}^e = \frac{1}{2}(\mathbf{C}^e - \mathbf{I}) \simeq \boldsymbol{\epsilon}^e, \quad \mathbf{C}^e \equiv (\mathbf{F}^e)^T\mathbf{F}^e = (\mathbf{U}^e)^2 \simeq \mathbf{I} + \boldsymbol{\epsilon}^e, \quad |\, \boldsymbol{\epsilon}^e \,| \ll 1. \tag{20}$$

The invariant elastic connection $\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}}$ introduced in (5) can be expressed via the decomposition theorem (13) by

$$\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}} = \boldsymbol{\gamma}^e_{\mathcal{K}} + \mathbf{W}_{\mathcal{K}}, \tag{21}$$

with $\boldsymbol{\gamma}^e_{\mathcal{K}}$ given by

$$\begin{aligned}(\boldsymbol{\gamma}^e_{\mathcal{K}}\mathbf{u})\mathbf{v}\cdot\mathbf{w} \quad &\equiv \frac{1}{2}(\mathbf{C}^e)^{-1}\mathbf{w}\cdot[((\nabla_k\mathbf{C}^e)\mathbf{v})\mathbf{u} + (\nabla_k\mathbf{C}^e\mathbf{u})\mathbf{v}] - \\ &-\frac{1}{2}((\mathbf{C}^e)^{-1}\mathbf{w})\mathbf{u}\cdot\mathbf{v}, \quad \mathbf{C}^e = (\mathbf{F}^e)^T\mathbf{F}^e.\end{aligned} \tag{22}$$

In order to compare the order of magnitude of the different terms we introduce a dimensionless parameter $\delta > 0$ such that

$$\boldsymbol{\epsilon}^e = \delta\boldsymbol{\epsilon} \quad \text{with} \quad |\,\delta\,| \ll 1, \quad \nabla_{\mathcal{K}}\boldsymbol{\epsilon}^e = \frac{\delta}{l}\nabla_{\mathcal{K}}\boldsymbol{\epsilon}, \tag{23}$$

where l is a characteristic length. Then

$$\begin{aligned}((\nabla^2_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{u})\mathbf{v} &= \frac{\delta}{l^2}((\nabla^2_{\mathcal{K}}\boldsymbol{\epsilon})\mathbf{u})\mathbf{v}, \\ (\nabla_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{u})(\nabla_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{v} &= \frac{\delta^2}{l^2}((\nabla_{\mathcal{K}}\boldsymbol{\epsilon})\mathbf{u})(\nabla_{\mathcal{K}}\boldsymbol{\epsilon})\mathbf{v}\end{aligned} \tag{24}$$

and thus the following inequalities hold

$$\begin{aligned}&|\,((\nabla^2_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{u})\mathbf{v}\,| \le \frac{\delta}{l^2}\,|\,\nabla^2_{\mathcal{K}}\boldsymbol{\epsilon}\,|_{(4)}|\,\mathbf{u}\,||\,\mathbf{v}\,| \quad \text{while} \\ &|\,(\nabla_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{u})(\nabla_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{v}\,| \le \frac{\delta^2}{l^2}(|\,\nabla_{\mathcal{K}}\boldsymbol{\epsilon}\,|_{(3)})^2\,|\,\mathbf{u}\,||\,\mathbf{v}\,|\,.\end{aligned} \tag{25}$$

Based on (24) we justified that $((\nabla_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{u})(\nabla_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{v}$ can be neglected in the presence of $((\nabla^2_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{u})\mathbf{v}$.

Proposition 3. As a consequence of the appropriate exact formulae defining the properties of the elastic connection we obtain that

$$\overset{(e)}{\boldsymbol{\Gamma}}_{\mathcal{K}} \simeq \nabla_{\mathcal{K}}\boldsymbol{\epsilon}^e + (\mathbf{R}^e)^T\nabla_{\mathcal{K}}\mathbf{R}^e. \tag{26}$$

The elastic metric connection $\boldsymbol{\gamma}^e_{\mathcal{K}}$ can be reduced to

$$
\begin{aligned}
&(\boldsymbol{\gamma}^e_{\mathcal{K}}\mathbf{u})\mathbf{v}\cdot\mathbf{w}\simeq\\
&(((\nabla_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{v})\mathbf{u}+((\nabla_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{u})\mathbf{v})\cdot\mathbf{w}-((\nabla_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{w})\mathbf{u}\cdot\mathbf{v}.
\end{aligned}
\tag{27}
$$

The torsion $\mathbf{S}_{\mathcal{K}}$ defined for all $\mathbf{u},\mathbf{v}\in\mathcal{V}_{\mathcal{K}}$ by the second form written in (12)

$$
(\mathbf{S}_{\mathcal{K}}\widetilde{\mathbf{u}})\widetilde{\mathbf{v}}=(\mathbf{F}^e)^{-1}[((\nabla_{\mathcal{K}}\mathbf{F}^e)\widetilde{\mathbf{u}})\widetilde{\mathbf{v}}-((\nabla_{\mathcal{K}}\mathbf{F}^e)\widetilde{\mathbf{v}})\widetilde{\mathbf{u}}]
\tag{28}
$$

contains two part

$$
\begin{aligned}
(\mathbf{S}_{\mathcal{K}}\widetilde{\mathbf{u}})\widetilde{\mathbf{v}}\ &=((\nabla_{\mathcal{K}}\boldsymbol{\epsilon}^e)\widetilde{\mathbf{u}})\widetilde{\mathbf{v}}-((\nabla_{\mathcal{K}}\boldsymbol{\epsilon}^e)\widetilde{\mathbf{v}})\widetilde{\mathbf{u}}+\\
&+(\mathbf{R}^e)^T[((\nabla_{\mathcal{K}}\mathbf{R}^e)\widetilde{\mathbf{u}})\widetilde{\mathbf{v}}-((\nabla_{\mathcal{K}}\mathbf{R}^e)\widetilde{\mathbf{v}})\widetilde{\mathbf{u}}].
\end{aligned}
\tag{29}
$$

The first part of the torsion $\mathbf{S}_{\mathcal{K}}$ is characterized by the skew-symmetric part of the gradient of $\boldsymbol{\epsilon}^e-$ the small elastic strain tensor, while the second part is the torsion attached to the affine connection generated by the elastic rotation only.

Now we pass to the cotorsion $\mathbf{W}_{\mathcal{K}}$ from (21) taking into account the appropriate formula $(14)_1$ together with (28):

$$
(\mathbf{W}_{\mathcal{K}}\mathbf{u})\mathbf{v}\cdot\mathbf{z}\simeq((\nabla_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{z})\mathbf{v}\cdot\mathbf{u}-((\nabla_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{v})\mathbf{u}\cdot\mathbf{z}+(\mathbf{W}^R\mathbf{u})\mathbf{v}\cdot\mathbf{z},
\tag{30}
$$

where the cotorsion attached to the elastic rotation is expressed by

$$
\begin{aligned}
(\mathbf{W}^R\mathbf{u})\mathbf{v}\cdot\mathbf{z}\ &=\frac{1}{2}(\mathbf{R}^e)^T[((\nabla_{\mathcal{K}}\mathbf{R}^e)\widetilde{\mathbf{u}})\widetilde{\mathbf{v}}-((\nabla_{\mathcal{K}}\mathbf{R}^e)\widetilde{\mathbf{v}})\widetilde{\mathbf{u}}]\cdot\mathbf{z}-\\
&-\frac{1}{2}(\mathbf{R}^e)^T[((\nabla_{\mathcal{K}}\mathbf{R}^e)\widetilde{\mathbf{u}})\widetilde{\mathbf{z}}-((\nabla_{\mathcal{K}}\mathbf{R}^e)\widetilde{\mathbf{z}})\widetilde{\mathbf{u}}]\cdot\mathbf{v}-\\
&-\frac{1}{2}(\mathbf{R}^e)^T[((\nabla_{\mathcal{K}}\mathbf{R}^e)\widetilde{\mathbf{v}})\widetilde{\mathbf{z}}-((\nabla_{\mathcal{K}}\mathbf{R}^e)\widetilde{\mathbf{z}})\widetilde{\mathbf{v}}]\cdot\mathbf{u}.
\end{aligned}
\tag{31}
$$

The differential of the elastic metric connection has the following representation

$$
\begin{aligned}
(((d\,\boldsymbol{\gamma}^e_{\mathcal{K}})\mathbf{z})\mathbf{u})\mathbf{v}\cdot\mathbf{w}\ &\simeq\mathbf{w}\cdot(((\nabla^2_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{z})\mathbf{v})\mathbf{u}+\mathbf{w}\cdot(((\nabla^2_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{z})\mathbf{u})\mathbf{v}-\\
&-\mathbf{v}\cdot(((\nabla^2_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{z})\mathbf{w})\mathbf{u},
\end{aligned}
\tag{32}
$$

written for all $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z} \in \mathcal{V}_{\mathcal{K}}$.

We remark that the Riemann-Christoffel curvature tensor attached to the metric connection (27) has the expression

$$\begin{aligned}((\overset{(\gamma)}{\mathcal{R}}\,\mathbf{v})\mathbf{u})\mathbf{w}\cdot\mathbf{z} &= \mathbf{z}\cdot[((\nabla^2_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{v})\mathbf{w})\mathbf{u} - ((\nabla^2_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{u})\mathbf{w})\mathbf{v}] \\ &\quad + \mathbf{w}\cdot[((\nabla^2_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{u})\mathbf{z})\mathbf{v} - ((\nabla^2_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{v})\mathbf{z})\mathbf{u}].\end{aligned} \tag{33}$$

In the case of small elastic strains the above formulae allow us to emphasize the differences between the cases of small and large elastic rotations, respectively. When the elastic rotations are also in the vecinity of the identity tensor we get

$$\overset{(e)}{\mathbf{\Gamma}}_{\mathcal{K}} \simeq \nabla_{\mathcal{K}}\boldsymbol{\beta}^e, \quad (\mathbf{S}_{\mathcal{K}}\widetilde{\mathbf{u}})\widetilde{\mathbf{v}} = ((\nabla_{\mathcal{K}}\beta^e)\widetilde{\mathbf{u}})\widetilde{\mathbf{v}} - ((\nabla_{\mathcal{K}}\beta^e)\widetilde{\mathbf{v}})\widetilde{\mathbf{u}} \tag{34}$$

with the elastic metric connection described by (27) written for $\boldsymbol{\epsilon}^e$, while the cotorsion is expressed in terms of small spin measure $\boldsymbol{\omega}^e$. Here

$$\begin{aligned}&(\mathbf{W}_{\mathcal{K}}\mathbf{u})\mathbf{v}\cdot\mathbf{w} \simeq \\ &((\nabla_{\mathcal{K}}\boldsymbol{\omega}^e\mathbf{u})\mathbf{v}\cdot\mathbf{w} + ((\nabla_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{w})\mathbf{u}\cdot\mathbf{v} - ((\nabla_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{v})\mathbf{u})\cdot\mathbf{w}, \\ &\boldsymbol{\epsilon}^e = \frac{1}{2}(\boldsymbol{\beta}^e + (\boldsymbol{\beta}^e)^T), \quad \boldsymbol{\omega}^e = \frac{1}{2}(\boldsymbol{\beta}^e - (\boldsymbol{\beta}^e)^T).\end{aligned} \tag{35}$$

Remark 2. The Riemann-Christoffel curvature tensor, defined in the left hand side of (2) and attached to the elastic connection given in (34), and the related problems with the linearized framework can be found in [23], [1], [17],[11], [12].

Proposition 4. 1. The Riemann-Christoffel curvature tensor, defined in the left hand side of (2) and attached to the non-symmetric elastic connection given in (26) can be expressed

$$
\begin{aligned}
&(\overset{(e)}{\mathcal{R}}\ \mathbf{v})\mathbf{u} = (\overset{(R)}{\mathcal{R}}\ \mathbf{v})\mathbf{u} \equiv \\
&(\nabla_{\mathcal{K}}(\mathbf{R}^e)^T\mathbf{v})(\nabla_{\mathcal{K}}(\mathbf{R}^e)^T\mathbf{u}) - (\nabla_{\mathcal{K}}(\mathbf{R}^e)^T\mathbf{u})(\nabla_{\mathcal{K}}(\mathbf{R}^e)^T\mathbf{v}) \\
&+(\mathbf{R}^e)^T[((\nabla^2_{\mathcal{K}}\mathbf{R}^e)\mathbf{v})\mathbf{u} - ((\nabla^2_{\mathcal{K}}\mathbf{R}^e)\mathbf{u})\mathbf{v}] \\
&+((\mathbf{R}^e)^T(\nabla_{\mathcal{K}}\mathbf{R}^e)\mathbf{v}) \circ (\mathbf{R}^e)^T(\nabla_{\mathcal{K}}\mathbf{R}^e)\mathbf{u} \\
&-((\mathbf{R}^e)^T(\nabla_{\mathcal{K}}\mathbf{R}^e)\mathbf{u}) \circ (\mathbf{R}^e)^T(\nabla_{\mathcal{K}}\mathbf{R}^e)\mathbf{v}
\end{aligned}
\tag{36}
$$

written for all $\mathbf{u}, \mathbf{v} \in \mathcal{V}_k$.

2. $(\overset{(R)}{\mathcal{R}}\ \mathbf{u})\mathbf{v} = 0$, for all $\mathbf{u}, \mathbf{v}$.

Proof. We put into evidence the formula for the differential of the connection (26), taking into account the second order in magnitude

$$
\begin{aligned}
&(d\ \overset{(e)}{\mathbf{\Gamma}}_{\mathcal{K}}\ \mathbf{v})\mathbf{u} = \\
&(\mathbf{R}^e)^T((\nabla_{\mathcal{K}}\mathbf{R}^e)\mathbf{u})(\nabla_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{v} - (\nabla_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{v}(\mathbf{R}^e)^T((\nabla_{\mathcal{K}}\mathbf{R}^e)\mathbf{u}) \\
&+((\nabla^2_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{u})\mathbf{v} - (\nabla_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{v}(\nabla_{\mathcal{K}}\boldsymbol{\epsilon}^e)\mathbf{u} + \\
&+(\nabla_{\mathcal{K}}(\mathbf{R}^e)^T\mathbf{v})(\nabla_{\mathcal{K}}(\mathbf{R}^e)^T\mathbf{u}) + ((\mathbf{R}^e)^T(\nabla^2_{\mathcal{K}}\mathbf{R}^e)\mathbf{v})\mathbf{u}) + \mathcal{O}(\delta^3).
\end{aligned}
\tag{37}
$$

By the direct calculus we can prove the result written in (37).

The curvature tensor $\overset{(R)}{\mathcal{R}}$ is attached to an orthogonal mapping. The equality with zero follows from the property of orthogonal tensor field.

4 Constitutive and evolution equations elasto-plastic materials with small elastic strains

In [8] the set of variables to describe the behaviour of elasto-plastic materials as elastic with respect to the non-holonomic configuration are chosen based on the energetic arguments presented in [5]. The evolution equations for the irreversible behaviour complete the mathematical model.

The Piola-Kirchhoff stress tensor and the stress momentum as pulled back fields to the non-holonomic configuration and Mandel's non-symmetric stress measure, all of them referred to $\mathcal{K}$ are defined by

$$\begin{aligned} &\mathbf{\Pi}_{\mathcal{K}} \equiv \mathbf{\Pi} = \det(\mathbf{F}^e)(\mathbf{F}^e)^{-1}\mathbf{T}^s(\mathbf{F}^e)^{-T}, \quad \det\mathbf{F}^e = \frac{\tilde{\rho}}{\rho} \\ &\boldsymbol{\mu}_{\mathcal{K}} = (\det\mathbf{F}^e)(\mathbf{F}^e)^T\boldsymbol{\mu}[(\mathbf{F}^e)^{-T}, (\mathbf{F}^e)^{-T}], \\ &\mathbf{\Sigma}_{\mathcal{K}} \equiv \mathbf{\Sigma} = \mathbf{C}^e\frac{\mathbf{\Pi}_{\mathcal{K}}}{\tilde{\rho}}, \qquad \mathbf{C}^e = (\mathbf{F}^e)^T\mathbf{F}^e. \end{aligned} \tag{38}$$

Here $\mathbf{T}^s$ is the symmetric part of Cauchy stress tensor and $\boldsymbol{\mu}$ denotes stress momentum, a non-symmetric third order tensor field with respect to the actual configuration. The notation introduced in (10) is also used.

In the case of small elastic strains

$$\begin{aligned} &\mathbf{\Pi}_{\mathcal{K}} \equiv \mathbf{\Pi} = (\mathbf{R}^e)^T\mathbf{T}^s\mathbf{R}^e, \quad \mathbf{\Sigma}_{\mathcal{K}} \equiv \mathbf{\Sigma} = \frac{\mathbf{\Pi}_{\mathcal{K}}}{\rho}, \\ &\boldsymbol{\mu}_{\mathcal{K}} = (\mathbf{R}^e)^T\boldsymbol{\mu}[\mathbf{R}^e, \mathbf{R}^e]. \end{aligned} \tag{39}$$

First we recall that

A6. The **elastic like behaviour of a material element** is described in terms of the Cauchy stress tensor and the stress momentum with the deformation represented by the pair $(\mathbf{F}_{\mathcal{K}}, \mathbf{G}_{\mathcal{K}}) \in Lin^+ \times Lin(\mathcal{V}, Lin)$

$$\begin{aligned} &\mathbf{T}^s(\mathbf{X}, t) = \mathbf{f}_{\mathcal{K}(t)}(\mathbf{F}_{\mathcal{K}(t)},\ \{\mathbf{G}_{\mathcal{K}(t)}\}^a), \\ &\boldsymbol{\mu}(\mathbf{X}, t) = \mathbf{g}_{\mathcal{K}(t)}(\mathbf{F}_{\mathcal{K}(t)},\ \{\mathbf{G}_{\mathcal{K}(t)}\}^a). \end{aligned} \tag{40}$$

Here $\mathbf{F}_{\mathcal{K}}-$ is the deformation gradient and $\mathbf{G}_{\mathcal{K}}-$ denotes the second order gradient of the deformation, both of them from $\mathcal{K}$ to the actual configuration

$$\begin{aligned} &\mathbf{F}_{\mathcal{K}} \equiv \mathbf{F}^e, \quad \mathbf{G}_{\mathcal{K}} \equiv \nabla_{\mathcal{K}}\mathbf{F}^e, \\ &(\{\mathbf{G}_{\mathcal{K}}\}^a\tilde{\mathbf{u}})\tilde{\mathbf{v}} = \frac{1}{2}[((\nabla_{\mathcal{K}}\mathbf{F}^e)\tilde{\mathbf{u}})\tilde{\mathbf{v}} - ((\nabla_{\mathcal{K}}\mathbf{F}^e)\tilde{\mathbf{v}})\tilde{\mathbf{u}}] = \\ &\frac{1}{2}\mathbf{F}^e(\mathbf{S}_{\mathcal{K}}\tilde{\mathbf{u}})\tilde{\mathbf{v}}, \forall\ \tilde{\mathbf{u}}, \tilde{\mathbf{v}} \in \mathcal{V}_{\mathcal{K}}. \end{aligned} \tag{41}$$

These relations contain the symmetric part of Cauchy stress only and stress momentum. The skew-symmetric part of the Cauchy stress is obtained from the momentum balance equation (in agreement with [10], for instance).

As a consequence of **The objectivity assumption** it can be proved that

Theorem- Objectivity restrictions. *The elastic type behaviour of the material element satisfies the frame indifference principle if and only if it can be equivalently represented under the invariant (with respect to a change of frame) form:*

$$\mathbf{\Pi}_{\mathcal{K}} = \overline{f}_{\mathcal{K}}(\mathbf{C}^e,\ \mathbf{S}_{\mathcal{K}}), \quad \text{or} \quad \mathbf{\Sigma}_{\mathcal{K}} = \overline{h}_{\mathcal{K}}(\mathbf{C}^e,\ \mathbf{S}_{\mathcal{K}}), \quad \boldsymbol{\mu}_{\mathcal{K}} = \overline{g}_{\mathcal{K}}(\mathbf{C}^e,\ \mathbf{S}_{\mathcal{K}}). \tag{42}$$

In the case of small elastic strains

$$\mathbf{\Pi}_{\mathcal{K}} = \overline{f}_{\mathcal{K}}(\boldsymbol{\epsilon}^e,\ \mathbf{S}_{\mathcal{K}}), \quad \boldsymbol{\mu}_{\mathcal{K}} = \overline{g}_{\mathcal{K}}(\boldsymbol{\epsilon}^e,\ \mathbf{S}_{\mathcal{K}}), \tag{43}$$

with the torsion $\mathbf{S}_{\mathcal{K}}$ expressed through the formula (29).

In [8] we mentioned the evolution equations for irreversible variables with respect to non-holonomic configuration. This means evolution equations for $\mathbf{F}^p$ and $\nabla\mathbf{F}^p$.

Remark 3. We proved that $\overset{(e)}{\mathcal{R}} = 0$ in the case of small elastic strains. Thus $\mathbf{F}^p$ is a differentiable field. Consequently it is sufficient to define the evolution equation only for the plastic distortion. We assume that:
A7. The evolution equations for $\mathbf{F}^p$ with respect to $\mathcal{K}$ is described by

$$\mathbf{L}^p \equiv \dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1} = \mathcal{B}_{\mathcal{K}}(\mathbf{\Pi}_{\mathcal{K}}, \boldsymbol{\mu}_{\mathcal{K}}, \kappa_1, \kappa_2, \dot{\mathbf{F}}, \frac{d}{dt}(\nabla_k\mathbf{F}), \mathbf{F}, \nabla_k\mathbf{F}) \tag{44}$$

related as usual to the yield surface defined in the appropriate stress space, say the space of stress and stress momentum.

Let the yield condition be defined with respect to the set of variables $(\mathbf{\Sigma}_{\mathcal{K}}, \boldsymbol{\mu}_{\mathcal{K}}, \kappa_1, \kappa_2)$, under the form

$$\begin{gathered}\mathcal{F}_{\mathcal{K}}(\mathbf{\Pi}_{\mathcal{K}}, \boldsymbol{\mu}_{\mathcal{K}}, \mathbf{S}_{\mathcal{K}}, \kappa_1, \kappa_2) \equiv \\ \mathcal{F}_{\mathcal{K}}((\mathbf{R}^e)^T\mathbf{T}^s\mathbf{R}^e,\ (\mathbf{R}^e)^T\boldsymbol{\mu}[\mathbf{R}^e, \mathbf{R}^e],\ (\mathbf{R}^e)^T\mathbf{S}_{\mathcal{K}}[\mathbf{R}^e, \mathbf{R}^e], \kappa_1, \kappa_2).\end{gathered} \tag{45}$$

$\mathbf{S}_\mathcal{K}$ may be introduced among the variables in all constitutive and evolution function, since the torsion tensor can be identify with an internal variable, with the evolution equation derived from (28) in term of $\mathbf{L}^p$ and $\nabla_\mathcal{K}\mathbf{L}^p$.

Remark 4. The frame indifference requirement has to be imposed on the evolution equations (45) for the plastic distorsion $\mathbf{F}^p$ and for the gradient of the plastic distorsion (if it is also involved in the model as for instance in [8]). This **objectivity restriction** ensures the consistency of **A4.** and **A7.**

The motion connection $\mathbf{\Gamma}$ is related to $\nabla_k\mathbf{F}$ by the definition (11). Moreover $\mathbf{\Gamma}$ is invariant with respect o a change of frame in the actual configuration and so it is $\dot{\mathbf{\Gamma}}$ its material derivative.

First of all we remark that any function containing the set of variables $\dot{\mathbf{F}}, \dfrac{d}{dt}(\nabla_k\mathbf{F})), \mathbf{F}, \nabla_k\mathbf{F}$ can be written as dependent on

$$\dot{\mathbf{F}}\mathbf{F}^{-1} \equiv \mathbf{L} = \mathbf{D} + \mathbf{W}, \quad \mathbf{D}^* = \mathbf{Q}\mathbf{D}\mathbf{Q}^T, \quad \mathbf{W}^* = \mathbf{Q}\mathbf{W}\mathbf{Q}^T + \dot{\mathbf{Q}}\mathbf{Q}^T,$$
$$\frac{d}{dt}(\nabla_k\mathbf{F})\mathbf{F}^{-1}\mathbf{u} = ((\nabla_\chi\mathbf{L})\mathbf{u})\mathbf{F} + \mathbf{L}(\nabla_k\mathbf{F})\mathbf{F}^{-1}\mathbf{u}, \tag{46}$$

due to the rule of the calculus put into evidence in (9). Here $\mathbf{D} = \mathbf{L}^s$ is the rate of strain and $\mathbf{W} = \mathbf{L}^a$ is the spin motion.

We take into account that for all $\mathbf{u} \in \mathcal{V}_k$ the material derivative of the motion connection and the second order gradient of the velocity field are related by

$$\dot{\mathbf{\Gamma}}\mathbf{u} = \mathbf{F}^{-1}((\nabla_\chi\mathbf{L})\mathbf{F}\mathbf{u})\mathbf{F}. \tag{47}$$

We pass from $\mathbf{D}$ an objective field to $\dot{\mathbf{C}}-$ the material derivative of Cauchy-Green total strain tensor, via the formulae

$$\dot{\mathbf{C}} = 2\mathbf{F}^T\mathbf{D}\mathbf{F}, \quad \text{with} \quad \mathbf{C} = \mathbf{F}^T\mathbf{F}. \tag{48}$$

Finaly we proved that

Proposition 5. The evolution equation for plastic distortion is invariant with respect to the change of frame in the actual configuration if and only if (44) can be expressed under the form

$$\mathbf{L}^p \equiv \dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1} = \mathcal{B}_\mathcal{K}(\mathbf{\Pi}_\mathcal{K}, \boldsymbol{\mu}_\mathcal{K}, \kappa_1, \kappa_2, \mathbf{C}, \mathbf{\Gamma}, \dot{\mathbf{C}}, \dot{\mathbf{\Gamma}}), \tag{49}$$

where $\mathbf{\Pi}_{\mathcal{K}}, \boldsymbol{\mu}_{\mathcal{K}}, \kappa_1, \kappa_2$ are invariant tensor and scalar, respectively, tensor fields.

A similar representation follows for the rate of the gradient of plastic distortion, i.e. for $\dfrac{d}{dt}(\nabla_k \mathbf{F})(\mathbf{F}^p)^{-1}$.

Note the complete set of the constitutive and evolution equations is developed with respect to the non-holonomic reference configuration, in an invariant form. The behaviour of elasto-plastic material with continuously distributed dislocations can be described with respect to the actual configuration by pushing forward procedure, see for instance [5].

We take into account the multiplicative decomposition of the deformation gradient (3) and the following kinematic relationships are to be considered

$$\mathbf{L} = \dot{\mathbf{F}}^e(\mathbf{F}^e)^{-1} + \mathbf{F}^e\mathbf{L}^p(\mathbf{F}^e)^{-1}, \quad \mathbf{L}^p = \dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1}, \quad \mathbf{L} = \nabla_\chi \mathbf{v}. \tag{50}$$

Proposition 5. In the *actual configuration* kinematic relationships between the appropriate strain rates

$$\begin{aligned} &\mathbf{D} = \overset{\circ}{\overline{\boldsymbol{\epsilon}}} + \hat{\mathbf{D}}^p, \quad \hat{\mathbf{D}}^p := \mathbf{R}^e\mathbf{D}^p(\mathbf{R}^e)^T, \quad \mathbf{D} = \mathbf{L}^s, \\ &\overset{\circ}{\overline{\boldsymbol{\epsilon}}} = \dot{\overline{\boldsymbol{\epsilon}}} - \boldsymbol{\Omega}\overline{\boldsymbol{\epsilon}} + \overline{\boldsymbol{\epsilon}}\boldsymbol{\Omega}, \quad \text{with} \quad \overline{\boldsymbol{\epsilon}} = \mathbf{R}^e\boldsymbol{\epsilon}^e(\mathbf{R}^e)^T \\ &\mathbf{R}^e\dot{\boldsymbol{\epsilon}}^e(\mathbf{R}^e)^T = \overset{\circ}{\overline{\boldsymbol{\epsilon}}} \quad \text{where} \quad \boldsymbol{\Omega} = \dot{\mathbf{R}}^e(\mathbf{R}^e)^T \end{aligned} \tag{51}$$

and between their *associated spins*

$$\{\mathbf{L}\}^a = \boldsymbol{\Omega} + \hat{\mathbf{W}}^p, \quad \text{with} \quad \hat{\mathbf{W}}^p = \mathbf{R}^e\{\mathbf{L}^p\}^a(\mathbf{R}^e)^T, \tag{52}$$

hold. The corotational rate of the fields taken in the actual configuration with respect to the elastic spin $\boldsymbol{\Omega}$ replace the material derivatives of the fields defined in the non-holonomic configuration $\mathcal{K}$.

The rate form of the elastic type constitutive equations is derived by standard procedure, see for instance [5], [7]. For instance the objective rates of the symmetric part of the Cauchy stress and of the stress momentum are involved in the actual description of the material behaviour:

$$\frac{\overset{\circ}{\mathbf{T}}}{\rho} = \mathbf{R}^e \frac{d}{dt}\left(\frac{\mathbf{\Pi}}{\tilde{\rho}}\right)(\mathbf{R}^e)^T \equiv \frac{d}{dt}\left(\frac{\mathbf{T}}{\rho}\right) - \mathbf{\Omega}\frac{\mathbf{T}}{\rho} + \frac{\mathbf{T}}{\rho}\mathbf{\Omega}$$

$$\overset{\circ}{\boldsymbol{\mu}} = \mathbf{R}^e \frac{d}{dt}(\boldsymbol{\mu}_{\mathcal{K}})[(\mathbf{R}^e)^T, (\mathbf{R}^e)^T], \text{ with} \tag{53}$$

$$\overset{\circ}{\boldsymbol{\mu}} \equiv -\mathbf{\Omega}\boldsymbol{\mu} + \frac{d}{dt}\boldsymbol{\mu} + \boldsymbol{\mu}[\mathbf{\Omega}, \mathbf{I}] + \boldsymbol{\mu}[\mathbf{I}, \mathbf{\Omega}].$$

We pointed out that the elastic spin $\mathbf{\Omega}$ is replaced by the spin of the motion $\mathbf{L}^a$ via the formula (52) by

$$\mathbf{\Omega} = \{\mathbf{L}\}^a - \mathbf{R}^e\{\mathcal{B}_{\mathcal{K}}\}^a(\mathbf{R}^e)^T, \tag{54}$$

where $\mathcal{B}_{\mathcal{K}}$ represents the evolution equation from (49).

The objective rate of the elastic strain measure is expressed from (51) by the rate of strain

$$\overset{\circ}{\bar{\boldsymbol{\epsilon}}} = \mathbf{D} - \mathbf{R}^e\{\mathcal{B}_{\mathcal{K}}\}^s(\mathbf{R}^e)^T, \quad \mathbf{D} = \mathbf{L}^s, \tag{55}$$

in terms of the evolution function which characterized the rate of plastic distorsion.

Remark. If the constitutive and evolution functions are isotropic with respect to their arguments the remarkable simplified form can be derived in the actual configuration. That is the elastic rotations disappears from the constitutive representations.

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Male elastične deformacije konačnih elastoplastičnih materijala sa neprekidno rasporedjenim dislokacijama

UDK 514.8, 531.01

U ovom radu predlažemo jedan makroscopski model elastoplastičnih materiala sa neprekidno rasporedjenim dislokacijama, sa malim elastičnim deformacijama, ali konačnim elastičnim rotacijama i plastičnim distorzijama. Materijal nije homogen i ponaša se kao neki elastični materijalni element u odnosu na neholonomnu konfiguraciju. Tenzor elastične krivine isčezava. Dakle, za kompletiranje definicije mathematičkog modela dovoljno je propisati evolucionu jednačinu za plastičnu distorziju (t.j. za vremenski promenljivu neholonomnu konfiguraciju), kao i za unutrašnje promenljive.