THEORETICAL AND APPLIED MECHANICS
vol. 28-29, pp. 337-377, Belgrade 2002
Issues dedicated to memory of the Professor Rastko Stojanović

Stochastic homogenization and macroscopic modelling of composites and flow through porous media

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Abstract

The aim of this contribution is mainly twofold. First, the stochastic two-scale convergence in the mean developed by Bourgeat et al. [13] is used to derive the macroscopic models of: (i) diffusion in random porous medium, (ii) nonstationary flow of Stokesian fluid through random linear elastic porous medium. Second, the multi-scale convergence method developed by Allaire and Briane [7] for the case of several microperiodic scales is extended to random distribution of heterogeneities characterized by separated scales (stochastic reiterated homogenization).

1 Introduction

Since the pioneering papers on G- and Γ-convergence by the Italian School of the Calculus of Variations (see the relevant references in the book [19]) and on H-convergence by Murat and Tartar [32], one can distinguish, grosso modo, two domains of extensive research linked with

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homogenization. Prior to a brief characterization of those domains we should also mentioned Russian contributions largely summarized in the book by Jikov et al. [29].

The first domain of homogenization pertains to the elaboration of mathematical methods of convergence of sequences of operators and functionals depending on a small parameter $\varepsilon > 0$ intended to tend to zero [10, 29, 41]. It is worth noting that though significant progress has been achieved in the study of homogenization of nonconvex functionals [14, 19] yet the problem of homogenization of nonconvex functionals typical for finite elasticity remains open.

The second domain, already quite vast, pertains to various applications of the homogenization methods to linear and nonlinear problems of micromechanics, porous media and structural mechanics, cf. [6, 15, 16, 25, 29, 30, 45]. We observe that biological tissues are in fact porous materials [17, 18, 22, 26].

It is often believed that from the point of view of applications the homogenization methods are applicable only to media with periodic microstructures. Such a conviction is evidently false, cf. [6, 19, 32]. In fact, periodicity is important since one may say that microperiodic composites are "dense" in the class of composites characterized by a small parameter, cf. [30, 37]. In practice, two approaches can be distinguished in the study of nonperiodic composites. The first approach relies on finding lower and upper bounds on effective quantities. Recall here the famous Hashin-Strikman bounds [6]. Obviously, bounding techniques are also useful in the periodic case since except the case layering, i.e., dependence of say elastic moduli on only one spatial variable, one cannot analytically solve the so-called basic or local problem involving partial derivatives.

Composites and porous media often exhibit random microstructures. To construct macroscopic models of such materials one can use stochastic homogenization methods. The comprehensive book by Torquato [45] summarizes the results of finding effective quantities for media with random microstructures, including various bounding techniques. We also observe that in our paper [43] various approaches to micro-macro passage for random porous media were reviewed.

Torquato [45] does not exploit the possibilities offered by mathematically rigorous stochastic homogenization methods. An overview of those methods which cover stochastic G- and H-convergence, stochastic Γ-convergence and stochastic two-scale convergence in the mean has been performed by the first author [42]. The emphasis in the review article [42] is put on applications to linear and nonlinear composites with random distribution of microinhomogeneities.

In the present contribution the stochastic two-scale convergence method in the mean developed by Bourgeat et al. [13] is applied to: (i) the derivation of effective stationary transport model in the case of random porous media, (ii) the study of nonstationary flow of Stokesian fluid through random porous linear elastic medium. Our next aim is to extend the multi-scale convergence method [7], being a convenient mathematical tool of reiterated homogenization, to the case of media with several separated random (hierarchical) microstructures. The results due to Allaire and Briane [7] are confined to the case of multiple separated scales of periodic oscillations.

We observe that the general approach developed by Bourgeat et al. [13] is an extension of the so-called periodic nonuniform homogenization to media with random distribution of microinhomogeneities. It means that after, this time stochastic, homogenization the effective moduli depend on the macroscopic variable, say $\boldsymbol{x}$.

To make the paper accessible to a wider audience, in Section 2 we provide some useful data on description of porous media as well as on fundamental aspects of the method of stochastic two-scale convergence in the mean.

2 Mathematical preliminaries

An essential problem in the study of random porous media constitutes the description of porous microstructure. Therefore in Section 2.1 we gathered some useful relevant data. Section 2.2 introduces indispensable concepts of the stochastic convergence in the mean.

2.1 Description of random porous media

Natural and man-made porous materials usually possess formidably complicated architecture. Obviously, in modelling, one has to assume reasonable idealizations, cf. Telega and Bielski [43] and the relevant references cited therein. Random porous media constitute an important class of such idealizations. We pass to a brief description of general characterization of random porous media. We observe that Adler and Thovert [4] described random objects such as fractures and fracture networks using the methods of the geometry of random fields, cf. [2].

The first general approach is due to Beliaev and Kozlov [9], cf. Jikov et al. [29]. Let $(\Omega, \mathcal{F}, \mu)$ denote a probability space, where $\mathcal{F}$ is a complete σ-algebra and μ is the probability measure. Assume that Ω is acted on by an *n-dimensional dynamical system* $T(x) : \Omega \to \Omega$, such that for each $\boldsymbol{x} \in \mathbb{R}^n$, both $T(\boldsymbol{x})$ and $T(\boldsymbol{x})^{-1}$ are measurable, and such that the following conditions are satisfied:

(a) $T(\mathbf{0})$ is the identity map on Ω and for $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$, $T(\boldsymbol{x}_1+\boldsymbol{x}_2) = T(\boldsymbol{x}_1)T(\boldsymbol{x}_2)$;

(b) for each $\boldsymbol{x} \in \mathbb{R}^n$ and measurable set $F \in \mathcal{F}$, $\mu(T(\boldsymbol{x})^{-1}F) = \mu(F)$, i.e., μ is an invariant measure for T;

(c) for each $F \in \mathcal{F}$, the set $\{(\boldsymbol{x}, \omega) \in \mathbb{R}^n \times \Omega \,|\, T(\boldsymbol{x})\omega \in F\}$ is a $d\boldsymbol{x} \times d\mu$ measurable subset of $\mathbb{R}^n \times \Omega$, where $d\boldsymbol{x}$ denotes the Lebesgue measure on $\mathbb{R}^n$.

We observe that $T(\boldsymbol{x})^{-1} = T(-\boldsymbol{x})$. The dynamical system satisfying (a)–(c) is also called a *measure preserving flow*, cf. Sab [40].

Consequently, we can introduce random homogeneous fields, starting from the random variable

$$f \in L^1(\Omega); \;\; \tilde{f}(\boldsymbol{x}, \omega) \equiv f(T(\boldsymbol{x})\omega). \tag{2.1}$$

We observe that $\tilde{f}$ is also called the statistically homogeneous (i.e. stationary) random process. Statistical homogeneity means that two geometric points of the space are statistically undistinguishable. In other words, the statistical properties of the medium are invariant under the action of translations. In this way we have a group $\{U(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{R}^n\}$ of isometries on $L^2(\Omega) = L^2(\Omega, \mathcal{F}, \mu)$ defined by $(U(\boldsymbol{x})f)(\omega) = f(T(\boldsymbol{x})\omega)$, $\boldsymbol{x} \in \mathbb{R}^n, \omega \in \Omega$, $f \in L^2(\Omega)$. The function $\boldsymbol{x} \to U(\boldsymbol{x})$ is continuous in the strong topology, i.e., for each $f \in L^2(\Omega)$, $U(\boldsymbol{x})f \to f$

strongly in $L^2(\Omega)$ as $\boldsymbol{x} \to 0$. The strong convergence holds provided that the probability space is separable, what we tacitly assume throughout this paper.

A dynamical system is said to be *ergodic*, if every invariant function, i.e. satisfying $f(T(\boldsymbol{x})\omega) = f(\omega)$, is *constant* almost everywhere in Ω.

Example 2.1.

(i) *Spherical inclusions or voids in a matrix* (Sab [39, 40]). Consider an infinite composite medium consisting of a matrix with identical spherical particles (or voids) randomly embedded in the matrix. Then a realization $\omega \in \Omega$ is identified with the set $\omega = \{\boldsymbol{a}^m : m \in \mathbb{N}\}$ of the centers $\boldsymbol{a}^m$ of the particles (or voids). Here $\mathbb{N}$ denotes the set of natural numbers. Let $Nb(\omega, Q)$ denote the number of centers that fall in the open domain $Q \subset \mathbb{R}^n$. $\mathcal{F}$ is defined as the smallest σ-algebra containing the subsets of Ω of the form:

$$\{\omega \in \Omega : Nb(\omega, Q_1) = k_1, \ldots, Nb(\omega, Q_i) = k_i\},$$

where $Q_1, \ldots, Q_i$ are disjoint domains and $k_1, \ldots, k_i$ are positive integers. The probability measure μ is uniquely defined on $\mathcal{F}$ by its values on these subsets; the translations act on Ω as follows:

$$\forall \boldsymbol{x}, \forall \omega = \{\boldsymbol{a}^m : m \in \mathbb{N}\},\ T(\boldsymbol{x})\omega = \{\boldsymbol{a}^m + \boldsymbol{x} \,|\, m \in \mathbb{N}\}.$$

To guarantee the statistical homogeneity of the composite, μ must be invariant under the action of $T(\boldsymbol{x})$, for all $\boldsymbol{x}$. This will be the case if, for instance, μ obeys the Poisson distribution:

$$\mu[Nb(\omega, Q_1) = k_1, \ldots, Nb(\omega, Q_i) = k_i] =$$

$$\mu[Nb(\omega, Q_1) = k_1] \times \ldots \times \mu[Nb(\omega, Q_i) = k_i],$$

with

$$\mu(Nb(\omega, Q) = k) = \frac{(a|Q|)^k}{k!} \exp(-a|Q|).$$

Here $a > 0$ is a constant and $|Q|$ denotes the measure of Q (its volume).

It is known that the Poisson distribution is ergodic, cf. Papanicolaou [35]. Unfortunately, it is not always a good model for high concentration of particles (voids) because of possible overlapping.

(ii) *Statistically periodic media* are modelled by a probability space $(\Omega', \mathcal{F}', \mu')$ on which acts a measure preserving $\mathbb{Z}^n$ - group $T'(\boldsymbol{z})$, $\boldsymbol{z} \in \mathbb{Z}^n$, which is a set of bijective maps from Ω into itself satisfying the group property in $\mathbb{Z}^n$ and the following invariance property (Sab [40]):

$$\forall \boldsymbol{z} \in \mathbb{Z}^n, \forall F \in \mathcal{F}', \; T'(-\boldsymbol{z})F = \{\omega : T'(-\boldsymbol{z})\omega \in F\} \in \mathcal{F}'$$

and $\mu'(T'(\boldsymbol{z})F) = \mu'(F)$.

We recall that $\mathbb{Z}$ denotes the set of integers. A statistically periodic medium is ergodic if constants are the only real random variables such that:

$$\forall \boldsymbol{z} \in \mathbb{Z}^n, \; f \circ T'(\boldsymbol{z}) = f \text{ a. s. (almost surely).}$$

For examples the reader is referred to Dal Maso and Modica [20], and Sab [39, 40].

We observe that statistically periodic media are a special case of statistically homogeneous media. Indeed, it suffices to take $\Omega = \Omega' \times Y$ with $Y = [0, 1)^n$, $\mathcal{F}$ to be $\mathcal{F}' \otimes \mathcal{L}(Y)$, the completion with respect to $\mu = \mu' \otimes d\boldsymbol{y}$ of the σ-algebra product of $\mathcal{F}'$ and Borel σ-algebra on Y. Now $T(\boldsymbol{x})$ defined by:

$$\forall \boldsymbol{x} \in \mathbb{R}^n, \; \forall \, \omega' \in \Omega', \forall \boldsymbol{y} \in Y, \; T(\boldsymbol{x})(\omega', \boldsymbol{y}) = (T'[\boldsymbol{x}+\boldsymbol{y}]\omega', \boldsymbol{x}+\boldsymbol{y}-[\boldsymbol{x}+\boldsymbol{y}])$$

is a measure preserving flow on $(\Omega, \mathcal{F}, \mu)$; moreover, the medium is ergodic if and only if T' is ergodic.

(iii) *Periodic media* are also a special case of statistically homogeneous ergodic media (Sab [40]). Now Y is the set $Y = [0, 1)^n$ (the unit basic cell), whilst $\mathcal{F}$ and μ are the Borel σ-algebra and Lebesgue measure on Y, respectively. Then $T(\boldsymbol{x})$ acts on Y as follows

$$\forall \boldsymbol{x} \in \mathbb{R}^n, \; \boldsymbol{y} \in Y, \;\; T(\boldsymbol{x})\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{y} - [\boldsymbol{x} + \boldsymbol{y}].$$

We recall that $[\boldsymbol{x}]$ denotes the integer part of $\boldsymbol{x}$. In this particular case, a random variable is identified with a measurable map on Y, $\tilde{f}$ is identified with the periodic continuation of f to the whole space whilst the expectation is identified to the volume average on Y and $d\mu = d\boldsymbol{y}$.

(iv) *Quasiperiodic media.* Let $\Omega = \mathbf{T}^m$, $m > n$, be an m-dimensional torus endowed with the Lebesgue measure. To define the dynamical system $T(\boldsymbol{x})$ we fix an $(m \times n)$-matrix $\boldsymbol{\Lambda} = (\Lambda_{ij})$ and set, cf. Pankov [34],

$$T(\boldsymbol{x})\omega = \omega + \boldsymbol{\Lambda x} \mod \mathbb{Z}^n.$$

The map $T(\boldsymbol{x})$ preserves the measure μ. For $T(\boldsymbol{x})$ to be ergodic, it is necessary and sufficient that $\boldsymbol{\Lambda k} \neq \boldsymbol{0}$ for any $\boldsymbol{k} \in \mathbb{Z}^n$, $\boldsymbol{k} \neq \boldsymbol{0}$. Any measurable function f on Ω may be identified with a unique measurable $1-$periodic function on $\mathbb{R}^n$. However, in this case we have a lot of essentially different realizations $f(\omega + \boldsymbol{\Lambda x})$. Realizations of this type are called *quasiperiodic functions*, if $f(\omega)$ is continuous. □

Let us fix a set $G \in \mathcal{F}$. The *random domain* $G(\omega)$ is the set

$$G(\omega) = \{\boldsymbol{x} \in \mathbb{R}^n \,|\, T(\boldsymbol{x})\omega \in G\}.$$

It is usually assumed that $G(\omega)$ is almost surely (a. s.) an open domain in $\mathbb{R}^n$. We also introduce the characteristic function $\chi(\omega)$ of the domain G and the realization $\chi(T(\boldsymbol{x})\omega)$ of this random variable. Then $\boldsymbol{x} \to \chi(T(\boldsymbol{x})\omega)$ for fixed $\omega \in \Omega$ is the characteristic function of the domain $G(\omega)$.

To formulate the ergodic theorem we introduce the notion of the mean value for functions defined in $\mathbb{R}^n$. Let $g \in L^1_{\text{loc}}(\mathbb{R}^n)$, i.e. the function f is integrable on every measurable bounded set $K \subset \mathbb{R}^n$. A number $M\{g\}$ is called the *mean value* of g if

$$\lim_{\varepsilon \to 0} \int_K g(\varepsilon^{-1}\boldsymbol{x})\, d\boldsymbol{x} = |K| M\{g\}. \tag{2.2}$$

Here $|K|$ denotes the Lebesgue measure of K.

Of crucial importance is the following famous result, see Jikov et al. [46].

Birkhoff ergodic theorem. Let $f \in L^\alpha(\Omega)$, $\alpha \geq 1$. Then for almost all $\omega \in \Omega$ the realization $f(T(\boldsymbol{x})\omega)$ possesses a mean value in the following sense:

$$f(T(\frac{\boldsymbol{x}}{\varepsilon})\omega) \rightharpoonup M\{f(T(\boldsymbol{x})\omega)\} \quad \text{weakly in } \; L^\alpha_{\text{loc}}. \tag{2.3}$$

Moreover, the mean value $M\{f(T(\boldsymbol{x})\omega)\}$, considered as a function of $\omega \in \Omega$, is invariant, and

$$\langle f \rangle \stackrel{df}{=} \int_\Omega f(\omega)\, d\mu = \int_\Omega M\{f(T(\boldsymbol{x})\omega)\}\, d\mu.$$

In particular, if the system $T(\boldsymbol{x})$ is ergodic, then

$$M\{f(T(\boldsymbol{x})\omega)\} = \langle f\rangle \quad \text{for almost all } \omega \in \Omega. \qquad \square$$

Beliaev and Kozlov [9] introduced the following definition:
The random domain $G(\omega)$ is called porous if there exists a random variable $h(\omega) > 0$ such that

$$\langle h^{-1}\rangle < \infty, \tag{2.4}$$

and for any $\varphi \in C_0^\infty(G(\omega))$

$$\int_{\mathbb{R}^n} h(T(\boldsymbol{x})\omega)\varphi^2(\boldsymbol{x})\, d\boldsymbol{x} \le \int_{\mathbb{R}^n} |\nabla\varphi(\boldsymbol{x})|^2\, d\boldsymbol{x}. \tag{2.5}$$

We observe that for a periodic domain $G(\omega)$ inequality (2.5) holds true for h equal to λ_0, the principal eigenvalue of the periodic domain for the Laplacian with Dirichlet boundary conditions on $\partial G(\omega)$.

Beliaev and Kozlov [9] provided examples oriented towards checkerboard porous media.

Let Q be a given deterministic, bounded domain in $\mathbb{R}^n$ and $G(\omega)$ the statistically homogeneous random domain introduced above. Media with a microstructure, including porous media, are characterized by a small parameter $\varepsilon = l/L$, where l is the characteristic microscopic dimension whilst L denotes the characteristic macroscopic dimension. Obviously, for a real porous material $\varepsilon = \varepsilon_0 > 0$. Homogenization is nothing else as smearing out heterogeneities, for instance the pores. Mathematically it means that we have to perform the limit passage $\varepsilon \to 0$. Beliaev and Kozlov [9] introduced the random domain $Q_\varepsilon(\omega) = Q \cap \varepsilon G(\omega)$. On the other hand, Wright [47, 48] sets

$$Q_\varepsilon(\omega) = Q \setminus \mathcal{G}_\varepsilon(\omega),\ \text{where } \mathcal{G}_\varepsilon(\omega) = \{\boldsymbol{x} \in \mathbb{R}^n \,|\, \varepsilon^{-1}\boldsymbol{x} \in G(\omega)\}.$$

Such definitions of the random domain $Q_\varepsilon(\omega)$ are suitable for theoretical considerations. In practice, we have to describe $G(\omega)$ or $\mathcal{G}_\varepsilon(\omega)$ more precisely. Many papers have been written on this subject. The reader is advised to consult an excellent review paper by Adler and Thovert [3], cf. also Roberts [38]. In essence, the methods of the geometry of random fields are exploited, see Adler [2]. More precisely, the approach reviewed in Adler and Thovert [3] consists in an approximation of the characteristic function $\chi(T(\boldsymbol{x})\omega)$, denoted simply by $Z(\boldsymbol{x})$:

$$Z(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x} \text{ belongs to the pore space,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

The function $Z(\boldsymbol{x})$ is called a phase function. The determination of the pore space is equivalent to the determination of the function $Z(\boldsymbol{x})$ and of some of its properties. Adler and Thovert [3] reviewed computational and experimental methods enabling to determine this function. Among various available techniques let us mention high resolution microtomography, very suitable for systematic studies of porous architectures of geomaterials and trabecular bones, cf. also Müller and Rüegsegger [31].

To reconstruct homogeneous random fields one may use Gaussian or non-Gaussian (like Markov) random fields. Let us briefly discuss the most frequently used case of GRFs (Gaussian random fields), cf. also Roberts [38]. Once $Z(\boldsymbol{x})$ is determined, the porosity is calculated according to

$$\Phi = \langle Z(\boldsymbol{x}) \rangle. \tag{2.7}$$

The other important quantity is the second moment of the phase function which is also called the correlation function

$$R_{\boldsymbol{z}}(u) = \langle [Z(\boldsymbol{x}) - \Phi][Z(\boldsymbol{x} + \boldsymbol{u}) - \Phi] \rangle / (\Phi - \Phi^2). \tag{2.8}$$

Here $u = |\boldsymbol{u}|$ denotes the modulus of the translation vector $\boldsymbol{u}$. For the sake of simplicity macroscopic isotropy has been assumed, hence the modulus u of $|\boldsymbol{u}|$. For more details the reader is referred to Adler and Thovert [3] and Roberts [38].

Unfortunately, a porous medium is incompletely, i.e., in a nonunique manner, defined by only its two first moments: Φ and $R_{\boldsymbol{z}}$. Rigorously, it is precisely defined by an infinite series of centered moments (or correlation functions R_k):

$$R_k(\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}) = (\Phi - \Phi^2)^{-k/2} \langle [Z(\boldsymbol{x}) - \Phi] \prod_{i=1}^{k-1} [Z(\boldsymbol{x} + \boldsymbol{u}_i) - \Phi] \rangle \tag{2.9}$$

Porous media are often not statistically homogeneous and their local properties may depend on the location. To classify various types of heterogeneities one may use the classification proposed for fractals. For a review, the reader is referred to Adler and Thovert [3]. It seems that an alternative possibility is offered by stochastic non-uniform homogenization, considered in the present paper.

Here it suffices to recall that homogeneous random fields are invariant under the action of an arbitrary translation. Heterogeneous random fields lack this property.

2.2 Stochastic two-scale convergence in the mean

In the case of periodic homogenization a fundamental role in the derivation of macroscopic constitutive equations is played by the so-called local or cell problem posed on a basic cell, usually denoted by Y, cf. Bensoussan et al. [10], Bielski and Telega [11], Sanchez-Palencia [41]. The periodic homogenization obviously involves the differentiation with respect to $\boldsymbol{y} \in Y$. Now, in the case of stochastic homogenization the role of Y is played by Ω. We assume that Ω is a separable as a measure space. To carry out stochastic homogenization we need to introduce elements of local stochastic calculus. For more details the reader is referred to Andrews and Wright [8] and Bourgeat et al. [13].

Let f be a measurable function defined on Ω, the function $(\boldsymbol{x}, \omega) \to f(T(\boldsymbol{x})\omega)$ is $d\boldsymbol{x} \times d\mu$ measurable on $\mathbb{R}^n \times \Omega$ and consequently we can define a group $\{U(\boldsymbol{x}) \,|\, \boldsymbol{x} \in \mathbb{R}^n\}$ of isometries on $L^2(\Omega) = L^2(\Omega, \mathcal{F}, \mu)$ by $(U(\boldsymbol{x})f)(\omega) = f(T(\boldsymbol{x})\omega)$, $\boldsymbol{x} \in \mathbb{R}^n$, $\omega \in \Omega$, $f \in L^2(\Omega)$. Then $U(\boldsymbol{x})f \to f$ strongly in $L^2(\Omega)$ as $\boldsymbol{x} \to \mathbf{0}$.

A fixed n-dimensional dynamical system T on Ω enables us to define a stochastic differential calculus on $L^2(\Omega)$ which comes from the individual coordinate actions arising from the isometry group $\{U(\boldsymbol{x}) \,|\, \boldsymbol{x} \in \mathbb{R}^n\}$. To this end, we observe that when each coordinate of $\boldsymbol{x} = (x_1, \ldots . x_n)$ varies over $\mathbb{R}$ with the other coordinates held equal to zero in $U(\boldsymbol{x})$, we obtain n one-parameter, strongly continuous group of isometries U_i on $L^2(\Omega)$ which pairwise commute. Thus we may write

$$U_i(x_i)f = U(0, 0, \ldots, x_i, 0, \ldots, 0) = f(T(0, 0, \ldots, x_i, 0, \ldots, 0)).$$

Let $D_1, \ldots, D_n$ denote the *infinitesimal generators* in $L^2(\Omega)$ of these one-parameter groups and let $\mathcal{D}_1, \ldots, \mathcal{D}_n$ denote their respective domains. Consequently $f \in \mathcal{D}_i$ if and only if $f \in L^2(\Omega)$ and

$$D_i(f) = \lim_{\xi \to 0} \frac{U_i(\xi)f - f}{\xi} \tag{2.10}$$

exists strongly as an element of $L^2(\Omega)$. We set $\mathcal{D}(\Omega) = \bigcap_{i=1}^{n} \mathcal{D}_i$. For each multi-index $\alpha = (\alpha_1 \ldots \alpha_n)$, let $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$, and set

$$\mathcal{D}^\infty(\Omega) = \{f \in L^2(\Omega) \,|\, D^\alpha f \in \mathcal{D}(\Omega), \text{ for all multi-indices } \alpha\}.$$

Let us now denote by $C^\infty(\Omega)$ the set of all functions $f \in L^\infty(\Omega)$ such that for each multi-index α, $D^\alpha f \in L^\infty(\Omega)$ and $\xi^{-1}(U_i(\xi)D^\alpha f - D^\alpha f)$ converges strongly in $L^\infty(\Omega)$ as $\xi \to 0$, for $i = 1, \ldots, n$.

Let $f \in L^1(\Omega)$ and α be a multi-index. The *stochastic derivative* $D^\alpha f$ is the linear functional on $C^\infty(\Omega)$ defined by

$$(D^\alpha f)(\varphi) = (-1)^{|\alpha|} \int_\Omega f D^\alpha \varphi \, d\mu, \tag{2.11}$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_n$ The T-invariance of the probability measure μ yields

$$\int_\Omega f D_i g \, d\mu = -\int_\Omega g D_i f \, d\mu, \tag{2.12}$$

where $f \in \mathcal{D}_i,\ g \in \mathcal{D}_i$. If we endow $\mathcal{D}(\Omega)$ with the natural norm

$$\|f\|^2_{\mathcal{D}(\Omega)} = \|f\|^2_{L^2(\Omega)} + \sum_i \|D_i g\|^2_{L^2(\Omega)}, \tag{2.13}$$

then the resulting Banach space can be viewed as a stochastic generalization of the Sobolev space $H^1(\mathbb{R}^n)$. With a slight abuse of notation, we will let $D_i f$ denote the stochastic weak derivative $D^{\alpha_i} f$ for $f \in L^2(\Omega)$.

For $f \in L^2(\Omega)$ (resp. $\boldsymbol{v} = (v_1, \ldots, v_n) \in [L^2(\Omega)]^n = L^2(\Omega)^n$), we define the *stochastic gradient* $\nabla_\omega f$*, stochastic divergence* $\operatorname{div}_\omega \boldsymbol{v}$*, stochastic Laplacian* $\Delta_\omega f$*, and curl* $\operatorname{curl}_\omega \boldsymbol{v}$ by

$$\nabla_\omega f = (D_1 f, \ldots, D_n f), \ \operatorname{div}_\omega \boldsymbol{v} = \sum_i D_i v_i,$$

$$\Delta_\omega f = \sum_i D_i^2 f,\ (\operatorname{curl}_\omega \boldsymbol{v})_{ij} = D_i v_j - D_j v_i.$$

Let us pass to the presentation of some properties of stochastically differentiable functions. We recall that a function $f \in L^2(\Omega)$ is *invariant* for T if $f(T(\boldsymbol{x})) = f$, $\mu -$ a. e. on Ω, for all $\boldsymbol{x} \in \mathbb{R}^n$ ($\mu-$ a. e. = almost everywhere with respect to the probability measure μ). The set of all functions in $L^2(\Omega)$ invariant for T is a closed subspace of $L^2(\Omega)$ and denoted by $I^2(\Omega)$. We set $M^2(\Omega) = [I^2(\Omega)]^\perp$, i.e. $M^2(\Omega)$ is the orthogonal complement of $I^2(\Omega)$ in $L^2(\Omega)$.

Another characterization of the subspace $I^2(\Omega)$ of invariant functions is that it is complemented in $L^2(\Omega)$ by a projection $E : L^2(\Omega) \to L^2(\Omega)$ of norm 1 which can be calculated as follows

$$(Ef)(\omega) = \lim_{\lambda\to\infty} \frac{1}{(2\lambda)^n} \int\limits_{[-\lambda,\lambda]^n} f(T(\boldsymbol{x})\omega)\, d\boldsymbol{x},\ f \in L^2(\Omega),\ \text{for } \mu\text{-a.e. } \omega \in \Omega. \tag{2.14}$$

The last limit exists strongly in $L^2(\Omega)$ and also pointwise μ-a.e.; we have $M^2(\Omega) = \ker E$. Let us specify some of the properties of stochastically differentiable functions.

Lemma 2.1.

(i) If $f \in L^2(\Omega)$ then $f \in I^2(\Omega)$ if and only if $\nabla_\omega f = 0$.

(ii) If $\varphi \in \mathcal{D}^\infty(\Omega)$, then for μ- a. e. $\omega \in \Omega$, the function $\boldsymbol{x} \to \varphi(T(\boldsymbol{x}))$ is in $C^\infty(\mathbb{R}^n)$, and for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$,

$$\partial^\alpha \varphi(T(\boldsymbol{x})\omega) = (D^\alpha \varphi)(T(\boldsymbol{x})\omega),$$

where $\partial^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$.

(iii) $C^\infty(\Omega)$ is strongly dense in $L^2(\Omega)$.

(iv) Let $\boldsymbol{u} \in L^2(\Omega)^n,\ \boldsymbol{v} \in L^2(\Omega)^n, \operatorname{curl}_\omega \boldsymbol{u} = \boldsymbol{0},$ and $\operatorname{div}_\omega \boldsymbol{v} = 0.$ Then

$$\int_\Omega u_i v_i \, d\mu = \int_\Omega (E\boldsymbol{u}) \cdot (E\boldsymbol{v})\, d\mu. \tag{2.15}$$

Furthermore, if T is ergodic then (2.15) reads

$$\int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\, d\mu = \int_\Omega \boldsymbol{u}\, d\mu \cdot \int_\Omega \boldsymbol{v}\, d\mu. \qquad \square$$

The last equation extends the well-known Hill-type relation [30].

If F is a μ-measurable subset of Ω, we denote by $C^\infty(F)$ the set of all functions in $C^\infty(\Omega)$ vanishing on $\Omega \setminus F$. If $f \in L^1(\Omega)$, we will say that $D_i f$ vanishes on F if $(D_i f)\varphi = 0$ for all $\phi \in C^\infty(F)$.
We observe that (i) implies that if $f \in L^2(\Omega)$ agrees on F with an element of $I^2(\Omega)$, then $\nabla_\omega f = 0$ on F. In order to obtain a converse statement, we say that F is *T-open* in Ω if $C^\infty(F)$ is strongly dense in the set $L^2(F)$ of all elements of $L^2(\Omega)$ vanishing on $\Omega \setminus F$ and that F is *T-connected* in Ω if whenever $f \in C^\infty(\Omega)$ has a stochastic gradient which vanishes on F, then f agrees on F with an element of $L^2(\Omega)$.

We have

Lemma 2.2. If F is a μ-measurable subset of Ω that is T-open and T-connected, then $f \in L^2(\Omega)$ has a stochastic gradient which vanishes on F if and only if f agrees on F with an element of $I^2(\Omega)$. □

In the analysis of the Stokes equations in the deterministic case a fundamental result states that if Q is a bounded Lipschitz domain in $\mathbb{R}^n$ and if d is a distribution in $H^{-1}(Q)^n$ which vanishes on the kernel of the divergence in $H^1_0(Q)^n$, then d is the gradient of a function $f \in L^2(Q)$. In the stochastic case we have a similar result.

Theorem 2.1. Suppose that d is a bounded linear functional on $H^1(\Omega)^n$ which vanishes on the kernel of the stochastic divergence. Then there exists $f \in L^2(\Omega)$, uniquely determined modulo $I^2(\Omega)$, such that $d = \nabla_\omega f$, i.e.,

$$d(\boldsymbol{v}) = -\int_\Omega f \operatorname{div}_\omega \boldsymbol{v}\, d\mu, \qquad \forall \boldsymbol{v} \in H^1(\Omega)^n. \qquad \square$$

Let us now introduce stochastic counterparts of the classical function spaces of fluid mechanics.

Proposition 2.1. Let

$$J(\Omega) = L^2(\Omega)^n - \text{norm closure of the set } \{\boldsymbol{\varphi} \in C^\infty(\Omega)^n \,|\, \operatorname{div}_\omega \boldsymbol{\varphi} = 0\},$$

$$X(\Omega) = \text{range of } \nabla_\omega \text{ restricted to } H^1(\Omega).$$

Then

$$L^2(\Omega)^n = X(\Omega) \oplus J(\Omega). \qquad \square$$

Now we are in a position to formulate the fundamental definition.

Definition 2.1. A sequence $\{u^\varepsilon\}_{\varepsilon>0}$ in $L^2(Q\times\Omega)$ is said to *stochastically two-scale converge in the mean* to $u \in L^2(Q\times\Omega)$ if for all $\psi \in L^2(Q\times\Omega)$

$$\lim_{\varepsilon\to 0}\int_{Q\times\Omega} u^\varepsilon(\boldsymbol{x},\omega)\psi(\boldsymbol{x},T(\varepsilon^{-1}\boldsymbol{x})\omega)\,d\boldsymbol{x}\,d\mu = \int_{Q\times\Omega} u(\boldsymbol{x},\omega)\psi(\boldsymbol{x},\omega)\,d\boldsymbol{x}\,d\mu. \tag{2.16}$$

The notion of two-scale convergence in the case of periodic homogenization was introduced by Nguetseng [33] and developed by Allaire [5], cf. also Bielski and Telega [11]. Bielski et al. [12] exploited this notion to justify the effective model of nonstationary flow of Stokesian fluid through a linear elastic porous medium.

Let us pass to providing some basic facts concerning stochastic two-scale convergence in the mean.

Theorem 2.2. Suppose $\{u^\varepsilon\}_{\varepsilon>0}$ is a bounded sequence in $L^2(Q\times\Omega)$. Then there exists a subsequence of $\{u^\varepsilon\}_{\varepsilon>0}$ which stochastically two-scale converges in the mean to $u \in L^2(Q\times\Omega)$. □

Theorem 2.3.

(a) Suppose that $\{u^\varepsilon\}$ and $\{\varepsilon\nabla_x u^\varepsilon\}$ are bounded sequences in $L^2(Q\times\Omega)$. Then there exist $u \in L^2(Q,H^1(\Omega))$ and a subsequence, still denoted by $\{u^\varepsilon\}$, such that $\{u^\varepsilon\}$ (resp. $\{\varepsilon\nabla_x u^\varepsilon\}$) stochastically two-scale converges in the mean to u (resp. $\nabla_\omega u$).

(b) Let $\mathcal{X}$ be a norm-closed, convex subset of $H^1(\Omega)$ (for instance a ball). Suppose $\{u^\varepsilon\}$ is a sequence in $L^2(Q\times\Omega)$ which satisfies the following conditions: for $\varepsilon > 0$,

(i) $u^\varepsilon(\cdot,\omega) \in \mathcal{X}$, for μ- a. e. $\omega \in \Omega$, and

(ii) there exists an absolute constant $C > 0$ such that

$$\int_\Omega \|u^\varepsilon(\cdot,\omega)\|^2_{H^1(Q)}\,d\mu \le C.$$

Then there exist $u \in H^1(Q,L^2(\Omega))$, $\boldsymbol{v} \in L^2(Q\times\Omega)^n$, and subsequence, still denoted by $\{u^\varepsilon\}$, which satisfy the following conditions:

(iii) for a.e. $\boldsymbol{x} \in Q,\ u(\boldsymbol{x}, \cdot) \in I^2(\Omega)$ and for μ- a. e. $\omega \in \Omega,\ u(\cdot, \omega) \in \mathcal{X}$;

(iv) $\boldsymbol{v}$ is contained in the $L^2(Q \times \Omega)^n$ - norm closure of $L^2(Q) \otimes$ (range of ∇_ω), and for a.e. $\boldsymbol{x} \in Q,\ \boldsymbol{v}(\boldsymbol{x}, \cdot) \in M^2(\Omega)^n,\ \mathrm{curl}_\omega \boldsymbol{v}(\boldsymbol{x}, \cdot) = \mathbf{0}$;

(v) $\{u^\varepsilon\}$ (resp. $\{\nabla_x u^\varepsilon\}$) stochastically two-scale converges in the mean to u (resp. $\nabla_{\boldsymbol{x}} u + \boldsymbol{v}$);

(vi) $(\nabla_{\boldsymbol{x}} u(\boldsymbol{x}, \cdot) \in I^2(\Omega)^n,$ for a. e. $\boldsymbol{x} \in Q$. □

Remark 2.1. Let us extend the mapping E defined by Eq. (2.14). To this end for each $\boldsymbol{y} \in \mathbb{R}^n$ we define the mapping $\tilde{T}(\boldsymbol{y}) : Q \times \Omega \to Q \times \Omega$ by $\tilde{T}(\boldsymbol{y})(\boldsymbol{x}, \omega) = (\boldsymbol{x}, T(\boldsymbol{y})\omega)$.
Then $\{\tilde{T}(\boldsymbol{y}) \,|\, \boldsymbol{y} \in \mathbb{R}^n\}$ is an n-dimensional dynamical system on $Q \times \Omega$ with an invariant measure $d\boldsymbol{x} \times d\mu$. Let $I^2(Q \times \Omega)$ denote the set of all functions in $L^2(Q \times \Omega)$ which are invariant for $\tilde{T}$. We have $I^2(Q \times \Omega) = L^2(Q, I^2(\Omega))$. When (Ω, T) is replaced by $(Q \times \Omega, \tilde{T})$ then

$$\tilde{E}g(\boldsymbol{x}, \omega) = E[g(\boldsymbol{x}, \cdot)](\omega), \tag{2.17}$$

where $g \in L^2(Q \times \Omega)$. Formula (2.14) yields

$$\tilde{E}g(\boldsymbol{x}, \omega) = \lim_{\lambda \to 0} \frac{1}{(2\lambda)^n} \int_{[-\lambda, \lambda]^n} g(\boldsymbol{x}, T(\boldsymbol{y})\omega)\, d\boldsymbol{y}. \tag{2.18}$$

We observe that $\tilde{E}g$ does not depend on $\omega \in \Omega$ provided that μ is ergodic for T.

Useful properties of $\tilde{E}$

(a) $\tilde{E}$ defines a projection of norm 1 of $L^2(Q \times \Omega)$ onto $I^2(Q \times \Omega)$.

(b) If $g \in L^2(Q \times \Omega)$ then

$$E\Big(\int_Q g(\boldsymbol{x}, \cdot)\, d\boldsymbol{x}\Big) = \int_Q \tilde{E}g(\boldsymbol{x}, \cdot)\, d\boldsymbol{x}.$$

3 Stationary diffusion in random porous medium made of nonhomogeneous material

Many authors studied already the problem of stochastic homogenization of the diffusion equation, sometimes called the transport equations, cf [8, 10, 13, 24, 29, 36] and the relevant references therein. For the case of periodic homogenization the reader is referred to [10,15,29]. General, nonperiodic case is considered in [15, 32]. The essential novelty of the present section lies in taking into account both the random distribution of micropores and material inhomogeneity.

The domains Q and $Q_\varepsilon(\omega)$ have been introduced in Section 2.1. We set

$$Q^s_\varepsilon(\omega) = Q \backslash \bar{Q}_\varepsilon(\omega), \tag{3.1}$$

where the bar over a set denotes its closure. We employ the superscript s for the solid phase. Let $\Gamma^\varepsilon(\omega) = Q \cap \partial Q^s_\varepsilon(\omega)$, $S^\varepsilon(\omega) = \partial Q \cap \partial Q^s_\varepsilon(\omega)$, and $F = \Omega \backslash G$; moreover $F(\omega) = \{\boldsymbol{x} \in \mathbb{R}^n \,|\, T(\boldsymbol{x})\,\omega \in F\}$. We assume that the set $\mathbb{R}^n \backslash F(\omega)$ is open and connected for each $\omega \in \Omega$.

Consider the following transport equation

$$\begin{aligned} -\mathrm{div}_{\boldsymbol{x}}\,[\boldsymbol{A}(\boldsymbol{x}, T(\varepsilon^{-1}(\boldsymbol{x})\omega)\nabla_{\boldsymbol{x}} u^\varepsilon(\boldsymbol{x},\omega)] = f(\boldsymbol{x}) \quad & \text{in } Q^s_\varepsilon(\omega), \\ u^\varepsilon(\boldsymbol{x},\omega) = 0 \quad & \text{on } S^\varepsilon(\omega), \\ [\boldsymbol{A}(\boldsymbol{x}, T(\varepsilon^{-1}\boldsymbol{x})\omega)\nabla u^\varepsilon] \cdot \boldsymbol{n} = 0 \quad & \text{on } \Gamma^\varepsilon(\omega). \end{aligned} \tag{3.2}$$

Here $\boldsymbol{n}$ is the outward unit normal vector to $\partial Q^s_\varepsilon(\omega)$ and $f \in L^2(Q)$. The moduli A_{ij} $(i, j = 1, \ldots, n)$ are assumed to satisfy the following conditions:

(i) $A_{ij} \in L^\infty(Q \times G)$;

(ii) $A_{ij} = A_{ji}$;

(iii) there exist constants $\beta \geq \alpha > 0$ such that for $dx \times d\mu$-a. e. $(\boldsymbol{x}, \omega) \in Q \times G$,

$$\alpha |\boldsymbol{\xi}|^2 \leq A_{ij}(\boldsymbol{x},\omega)\xi_i\xi_j \leq \beta|\boldsymbol{\xi}|^2 \qquad \forall \boldsymbol{\xi} \in \mathbb{R}^n.$$

Here $|\boldsymbol{\xi}|^2 = \xi_i \xi_i$.

We observe that u^ε may be, for instance, a temperature field or an electric potential.

A function u^ε satisfying the following variational equation

$$\int_{Q^s_\varepsilon(\omega)} A_{ij}(\boldsymbol{x}, T(\varepsilon^{-1}\boldsymbol{x})\omega) u^\varepsilon_{,j}(\boldsymbol{x},\omega) v_{,i}(\boldsymbol{x})\, d\boldsymbol{x} = \int_{Q^s_\varepsilon(\omega)} f\, v\, d\boldsymbol{x}, \tag{3.3}$$

for each $v \in H^1(Q^s_\varepsilon)$ with trace 0 on $S^\varepsilon(\omega)$, is called a weak solution of (3.2) provided that $u^\varepsilon(\cdot,\omega) \in H^1(Q^s_\varepsilon)$ and $(3.2)_2$ holds. Under the above assumptions u^ε exists and is unique. We recall that the trace of a function on the boundary (or the interface) practically means its value on this boundary.

From (iii) and (3.3) we conclude that there exists $C > 0$ such that, for all $\varepsilon > 0$

$$\int_\Omega \|u^\varepsilon(\cdot,\omega)\|^2_{H^1(Q^s_\varepsilon)}\, d\mu \le C. \tag{3.4}$$

Similarly to the case of periodic homogenization [16], the function u^ε has to be extended to the whole domain Q. After Andrews and Wright [8], we say that a $\mu-$measurable set $F \in \mathcal{F}$ has the *extension property with respect* to Q if the following conditions are satisfied: there exists $\varepsilon_0 > 0$ and $C_1 > 0$ such that for each $u(\cdot,\omega) \in H^1(Q^s_\varepsilon(\omega))$ with trace 0 on $S^\varepsilon(\omega)$, there exists $\tilde{\tilde{u}}^\varepsilon(\cdot,\omega) \in H^1_0(Q)$ such that

(a) $\quad (\boldsymbol{x},\omega) \to \tilde{\tilde{u}}^\varepsilon(\boldsymbol{x},\omega)$ is $d\boldsymbol{x} \times d\mu-$measurable on $Q \times \Omega$;

(b) $\quad \tilde{\tilde{u}}^\varepsilon(\cdot,\omega)\Big|_{Q^s_\varepsilon(\omega)} = u(\cdot,\omega), \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall \omega \in \Omega;$

(c) $\quad \|\tilde{\tilde{u}}^\varepsilon(\cdot,\omega)\|_{H^1(\Omega)} \le C_1 \|u(\cdot,\omega)\|_{H^1(Q^s_\varepsilon(\omega))}, \quad \forall \varepsilon \in (0,\varepsilon_0], \quad \forall \omega \in \Omega.$

Remark 3.1.

(i) Andrews and Wright [8] consider a more general extension, namely these authors additionally assume that there exists $p > 1$ and in condition (c) the space $H^1(Q)$ is to be replaced by $W^{1,p}(Q)$. Then obviously $\tilde{\tilde{u}}^\varepsilon(\cdot,\omega) \in W^{1,p}_0(Q)$ and p is called the exponent of smoothness of the extension.

(ii) The construction of extension operators is a delicate problem even in the periodic case, cf. [16,30] and the relevant references cited therein. For a discussion and illustrative examples in the case of randomly perforated media the reader is referred to the papers by Jikov [27,28] and the book by Jikov et al. [29] □

Let $f_\varepsilon(\cdot,\omega)$ be a function defined on $Q^s_\varepsilon(\omega)$. Similarly to the periodic case [16,29,30], we want to find its limit when $\varepsilon \to 0$. We set

$$f^0_\varepsilon(\boldsymbol{x},\omega) = \begin{cases} f_\varepsilon(\boldsymbol{x},\omega), & \text{if } \boldsymbol{x} \in Q^s_\varepsilon(\omega), \\ 0, & \text{if } \boldsymbol{x} \in Q\backslash Q^s_\varepsilon(\omega). \end{cases}$$

Hence we conclude that f^0_ε is the function obtained by extending $f_\varepsilon(\cdot,\omega)$ by 0 into the perforations $F_\varepsilon(\omega)$ of Q.

The following lemma is a specific case of Lemma 3.5 formulated in [8].

Lemma 3.1. Let $F \in \mathcal{F}$ be a μ-measurable subset of Ω. Suppose that for each $\omega \in \Omega$, $\{v_\varepsilon(\cdot,\omega)\}_{\varepsilon>0}$ is a sequence of functions with $v_\varepsilon(\cdot,\omega)$ defined on $Q^s_\varepsilon(\omega)$, and assume that

(a) $\{\tilde{\tilde{v}}_\varepsilon\}_{\varepsilon>0}$ is a sequence of functions defined on $Q \times \Omega$ for which $\tilde{\tilde{v}}_\varepsilon(\cdot,\omega)\Big|_{Q^s_\varepsilon(\omega)} = v_\varepsilon(\cdot,\omega),\ \forall\omega \in \Omega$;

(b) $\{\tilde{\tilde{v}}_\varepsilon\}$ stochastically two-scale converges in the mean to v in $L^2(Q \times \Omega)$.

Then $\{v^0_\varepsilon\}$ stochastically two-scale converges in the mean to $\chi_{\Omega\backslash F} v$ in $L^2(Q \times \Omega)$, where $\chi_{\Omega\backslash F} v$ denotes the characteristic function of $\Omega\backslash F$.

Proof. From (a) for all $(\boldsymbol{x},\omega) \in Q \times \Omega$, we have

$$v^0_\varepsilon(\boldsymbol{x},\omega) = \chi_{Q\backslash F_\varepsilon(\omega)}(\boldsymbol{x})\tilde{\tilde{v}}_\varepsilon(\boldsymbol{x},\omega) = \chi_{\Omega\backslash F}(T(\varepsilon^{-1}\boldsymbol{x})\omega)\tilde{\tilde{v}}_\varepsilon(\boldsymbol{x},\omega).$$

Thus $v^0_\varepsilon \in L^2(Q \times \Omega)$, for each ε. Using now (b) we readily get

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_{Q\times\Omega} v^0_\varepsilon(\boldsymbol{x},\omega)\psi(\boldsymbol{x},T(\varepsilon^{-1}\boldsymbol{x})\omega)\, d\boldsymbol{x}d\mu \\ &= \lim_{\varepsilon\to 0} \int_{Q\times\Omega} \tilde{\tilde{v}}_\varepsilon(\boldsymbol{x},\omega)(\chi_{\Omega\backslash F}\psi)(\boldsymbol{x},T(\varepsilon^{-1}\boldsymbol{x})\omega)\, d\boldsymbol{x}d\mu \\ &= \int_{Q\times\Omega} (\chi_{\Omega\backslash F} v\psi)(\boldsymbol{x},\omega)\, d\boldsymbol{x}d\mu \end{aligned}$$

for all $\psi \in L^2(Q \times \Omega)$. □

Now we are in a position to pass with $\varepsilon \to 0$ in (3.2) in the sense of stochastic two-scale convergence in the mean.

Theorem 3.1. Let $F \in \mathcal{F}$ have the extension property with respect to Q. Then there exists a subsequence of the sequence $\{u^\varepsilon\}_{\varepsilon>0}$, still denoted by $\{u^\varepsilon\}$, such that there exists $u \in H^1(Q, I^2(\Omega))$, and $\boldsymbol{\zeta} \in L^2(Q, M^2(\Omega))^n$, with $\boldsymbol{\zeta}$ contained in $L^2(Q \times \Omega)^n$ – norm closure of $L^2(Q) \otimes (\text{range}\, \nabla_\omega)$, for which $\{(u^\varepsilon)^0, (\nabla_{\boldsymbol{x}} u^\varepsilon)^0\}_{\varepsilon>0}$ stochastically two-scale converges in the mean to $(\chi_{\Omega\backslash F} u, \chi_{\Omega\backslash F}(\boldsymbol{\zeta} + \nabla_{\boldsymbol{x}} u))$, and such that $(u, \boldsymbol{\zeta})$ is a weak solution of

$$\operatorname{div}_\omega\{\chi_{\Omega\backslash F}(\omega)\boldsymbol{A}(\boldsymbol{x},\omega)(\boldsymbol{\zeta}(\boldsymbol{x},\omega) + \nabla_{\boldsymbol{x}} u(\boldsymbol{x},\omega)]\} = 0, \quad (3.5)$$

$$-\operatorname{div}_{\boldsymbol{x}}\{\tilde{E}[\chi_{\Omega\backslash F}(\omega)\boldsymbol{A}(\boldsymbol{x},\omega)(\boldsymbol{\zeta}(\boldsymbol{x},\omega) + \nabla_{\boldsymbol{x}} u(\boldsymbol{x},\omega))]\} = \mu(\Omega\backslash F)f, \quad (3.6)$$

$$u(\boldsymbol{x},\omega) = 0 \text{ on } \partial Q, \quad (3.7)$$

where (3.5) holds almost everywhere on Q whilst (3.6) and (3.7) hold μ–a.e. on Ω.

Proof. Using the extension property of F with respect to Q, we find $\varepsilon_0 > 0$, $C > 0$ and a measurable function $\tilde{\tilde{u}}^\varepsilon$ defined on $Q \times \Omega$ such that for all $0 < \varepsilon \le \varepsilon_0$ we have

$$\tilde{\tilde{u}}^\varepsilon(\cdot,\omega) \in H_0^1(Q) \quad (3.8)$$

$$\tilde{\tilde{u}}^\varepsilon(\cdot,\omega)|_{Q_\varepsilon^s} = u^\varepsilon(\cdot,\omega), \quad (3.9)$$

$$\|\tilde{\tilde{u}}^\varepsilon(\cdot,\omega)\|_{H^1(Q)} \le C\|u^\varepsilon(\cdot,\omega)\|_{H^1(Q_\varepsilon^s(\omega))}, \quad (3.10)$$

for all $\omega \in \Omega$. From (3.4) and (3.10) we conclude that the sequence $\{\tilde{\tilde{u}}^\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ is a bounded set in $L^2(\Omega, H^1(Q))$. Hence, by (3.8), (3.9), Theorem 2.3 and Lemma 3.1 there exists a subsequence $\{(u^\varepsilon)^0\}$ and functions $(u, \zeta) \in H_0^1(Q, I^2(\Omega)) \times L^2(Q, M^2(\Omega))^n$ with $\boldsymbol{\zeta}$ contained in $L^2(Q\times\Omega)^n$– norm closure of $L^2(Q)\otimes(\text{range}\nabla_\omega)$ such that $\{(u^\varepsilon)^0, \nabla_{\boldsymbol{x}}(u^\varepsilon)\}$ stochastically two-scale converges in the mean to $(\chi_{\Omega\backslash F} u, \chi_{\Omega\backslash F}(\boldsymbol{\zeta} + \nabla_{\boldsymbol{x}} u))$.

Let us now verify that $(u, \boldsymbol{\zeta})$ is a weak solution of Eqs. (3.5) and (3.6). First we observe that if $v \in H_0^1(Q)$, then on account of the T–invariance of μ, (3.3) and the fact that $\chi_{Q_\varepsilon(\omega)}(\boldsymbol{x}) = \chi_f(T(\varepsilon^{-1}\boldsymbol{x})\omega), (\boldsymbol{x}, \omega) \in$

$Q \times \Omega$, we get

$$\int\limits_{Q\times\Omega} A_{ij}(\boldsymbol{x}, T(\varepsilon^{-1}\boldsymbol{x})\omega)\tilde{\tilde{u}}^{\varepsilon}_{,j}(\boldsymbol{x},\omega)v_{,i}(\boldsymbol{x})\,d\boldsymbol{x}d\mu$$
$$= \int_\Omega \int_{Q_\varepsilon(\omega)} A_{ij}(\boldsymbol{x}, T(\varepsilon^{-1}\boldsymbol{x})\omega)\tilde{\tilde{u}}^{\varepsilon}_{,j}(\boldsymbol{x},\omega)v_{,i}(\boldsymbol{x})\,d\boldsymbol{x}d\mu$$
$$+ \int_\Omega \int_{Q^s_\varepsilon(\omega)} A_{ij}(\boldsymbol{x}, T(\varepsilon^{-1}\boldsymbol{x})\omega)u^{\varepsilon}_{,j}(\boldsymbol{x},\omega)v_{,i}(\boldsymbol{x})\,d\boldsymbol{x}d\mu \tag{3.11}$$
$$= \int\limits_{Q\times\Omega} \chi_F(T(\varepsilon^{-1}\boldsymbol{x})\omega)A_{ij}(\boldsymbol{x}, T(\varepsilon^{-1}\boldsymbol{x})\omega)\tilde{\tilde{u^\varepsilon}}_{,j}(\boldsymbol{x},\omega)v_{,i}(\boldsymbol{x})\,d\boldsymbol{x}d\mu$$
$$+\mu(\Omega\backslash F)\int_Q fv\,d\boldsymbol{x}.$$

Hence

$$\int\limits_{Q\times\Omega} \chi_{\Omega\backslash F}(T(\varepsilon^{-1}\boldsymbol{x})\omega)A_{ij}(\boldsymbol{x}, T(\varepsilon^{-1}\boldsymbol{x})\omega)\tilde{\tilde{u^\varepsilon}}_{,j}(\boldsymbol{x},\omega)v_{,i}(\boldsymbol{x})\,d\boldsymbol{x}d\mu$$
$$= \mu(\Omega\backslash F)\int_Q fv\,d\boldsymbol{x}. \tag{3.12}$$

In virtue of Theorem 2.3 we conclude that $\{\tilde{\tilde{u^\varepsilon}}, \nabla_{\boldsymbol{x}}\tilde{\tilde{u^\varepsilon}}\}$ stochastically two-scale converges in the mean to $(u, \boldsymbol{\zeta} + \nabla_{\boldsymbol{x}}u)$. We will now prove that $(u, \boldsymbol{\zeta})$ satisfies Eqs. (3.5) and (3.6). To this end we apply Lemma 2.1(b) of Andrews and Wright [8] which states that there exists a countable subset $\mathcal{A}$ of $C^\infty(\Omega)$,strongly dense in $W^{1,p}(\Omega)$ for $1 \le p < \infty$. Let g, $h \in C_0^\infty(Q)$, $b \in I^2(\Omega)$ and $k \in \mathcal{A}$. For $0 < \varepsilon \le \varepsilon_0$ the function $\boldsymbol{x} \mapsto g(\boldsymbol{x})b(\omega) + \varepsilon h(\boldsymbol{x})k(T(\varepsilon^{-1}\boldsymbol{x})\omega)$, $\boldsymbol{x} \in Q$, is an element of $C_0^\infty(Q)$. We use this as a test function in (3.12) and obtain

$$\int\limits_{Q\times\Omega} \chi_{\Omega\backslash F}(T(\varepsilon^{-1}\boldsymbol{x})\omega)A_{ij}(\boldsymbol{x}, T(\varepsilon^{-1}\boldsymbol{x})\omega)\tilde{\tilde{u}}^{\varepsilon}_{,j}(\boldsymbol{x},\omega)[b(\omega)g_{,i}(\boldsymbol{x})$$
$$+\varepsilon(h(\boldsymbol{x})k(T(\varepsilon^{-1}\boldsymbol{x})\omega))_{,i}\,d\boldsymbol{x}d\mu \tag{3.13}$$
$$= \mu(\Omega\backslash F)\int_Q f(\boldsymbol{x})[g(\boldsymbol{x})b(\omega) + \varepsilon h(\boldsymbol{x})k(T(\varepsilon^{-1}\boldsymbol{x})\omega)]\,d\boldsymbol{x}d\mu.$$

We have

$$\varepsilon(h(\boldsymbol{x})k(T(\varepsilon^{-1}\boldsymbol{x})\omega)_{,i} = \varepsilon k(T(\varepsilon^{-1}\boldsymbol{x})\omega)h_{,i}(\boldsymbol{x}) + h(\boldsymbol{x})D_i[k(T(\varepsilon^{-1}\boldsymbol{x})\omega)].$$

Substituting the last relation into (3.13) and letting ε tend to zero, we conclude from the stochastic two-scale convergence in the mean that

$$\int\limits_{Q\times\Omega} \chi_{\Omega\backslash F}(\omega)\boldsymbol{A}(\boldsymbol{x},\omega)[\boldsymbol{\zeta} + \nabla_{\boldsymbol{x}}u(\boldsymbol{x},\omega)]\cdot[b(\omega)\nabla_{\boldsymbol{x}}g(\boldsymbol{x}) + h(\boldsymbol{x})\nabla_\omega k(\omega)]\,d\boldsymbol{x}d\mu$$
$$= \mu(\Omega\backslash F)\int_Q f(\boldsymbol{x})g(\boldsymbol{x})b(\omega)\,d\boldsymbol{x}. \tag{3.14}$$

We now set $g \equiv 0$ in (3.14) and conclude from the fact that $h \in C_0^\infty(Q)$ is arbitrary that, for all $k \in \mathcal{A}$ and for a. e. $\boldsymbol{x} \in Q$,

$$\int_\Omega \chi_{\Omega\setminus F}(\omega)[\boldsymbol{\zeta}(\boldsymbol{x},\omega) + \nabla_{\boldsymbol{x}} u(\boldsymbol{x},\omega)] \cdot \nabla_\omega k(\omega)\, d\mu = 0.$$

Since $\mathcal{A}$ is countable, we conclude that $(u, \boldsymbol{\zeta})$ is a weak solution of Eq. (3.5) for a. e. $\boldsymbol{x} \in Q$.

In order to verify that $(u, \boldsymbol{\zeta})$ is a weak solution of Eq. (3.6) we apply the property (b) of the mapping $\tilde{E}$, given in Section 2.2. Indeed, let $h \equiv 0$ in Eq. (3.14). Since $b \in I^2(\Omega)$ is otherwise arbitrary, it follows that for all $\varphi \in L^2(\Omega)$ we have

$$\int_Q \chi_{\Omega\setminus F}(\omega)\Big(\int_Q \boldsymbol{A}(\boldsymbol{x},\omega)[\boldsymbol{\zeta}(\boldsymbol{x},\omega) + \nabla_{\boldsymbol{x}}(\boldsymbol{x},\omega)] \cdot \nabla_{\boldsymbol{x}} g(\boldsymbol{x})\, d\boldsymbol{x} -$$

$$\int_Q f(\boldsymbol{x}) g(\boldsymbol{x}) d\boldsymbol{x}\Big) E(\varphi)\, d\mu = 0.$$

Hence

$$E\Big(\int_Q \boldsymbol{A}(\boldsymbol{x},\omega)[\boldsymbol{\zeta}(\boldsymbol{x},\omega) + \nabla_{\boldsymbol{x}} u(\boldsymbol{x},\omega)] \cdot \nabla_{\boldsymbol{x}} g(\boldsymbol{x})\, d\boldsymbol{x}\Big)$$

$$= \mu(\Omega\setminus F) \int_Q f(\boldsymbol{x}) g(\boldsymbol{x})\, d\boldsymbol{x}.$$

By the aforementioned property (b) of E, for μ– a. e. $\omega \in \Omega$, we get

$$\int_Q \tilde{E}(\chi_{\Omega\setminus F}(\omega)\boldsymbol{A}(\boldsymbol{x},\omega)[\boldsymbol{\zeta}(\boldsymbol{x},\omega) + \nabla_{\boldsymbol{x}} u(\boldsymbol{x},\omega)]) \cdot \nabla_{\boldsymbol{x}} g(\boldsymbol{x})\, dx$$

$$= \mu(\Omega\setminus F) \int_Q f(\boldsymbol{x}) g(\boldsymbol{x})\, d\boldsymbol{x}.$$

Since $g \in C_0^\infty(Q)$ is arbitrary, from the last relation we deduce Eq. (3.6).

In order to prove that $(u, \boldsymbol{\zeta})$ is uniquely determined by (3.5)–(3.7) it suffices to follow the corresponding part of the proof of Theorem 4.1.1. of Bourgeat et al. [13] or the proof of Theorem 3.1 of Andrews and Wright [8]. The proof of Theorem 3.1 is thus complete. □

Remark 3.2. Consider the case where T is ergodic. Then u does not depend on ω and Eq. (3.6) is written as follows

$$-\mathrm{div}_{\boldsymbol{x}} \int_\Omega \chi_{\Omega\setminus F}(\omega)\boldsymbol{A}(\boldsymbol{x},\omega)[\nabla_{\boldsymbol{x}} u(\boldsymbol{x}) + \boldsymbol{\zeta}(\boldsymbol{x},\omega)]\, d\mu = f(\boldsymbol{x}), \quad \text{in } Q.$$

We may set $\boldsymbol{a} = \nabla_{\boldsymbol{x}} u(\boldsymbol{x}) \in \mathbb{R}^n$ and treat $\boldsymbol{x}$ as a parameter. The effective (homogenized) moduli are given by

$$\boldsymbol{A}^h(\boldsymbol{x})\boldsymbol{a} = \int_\Omega \chi_{\Omega\setminus F}(\omega)\boldsymbol{A}(\boldsymbol{x},\omega)(\boldsymbol{a} + \boldsymbol{\zeta}(\boldsymbol{x},\omega))\, d\mu, \tag{3.15}$$

for each $\boldsymbol{a} \in \mathbb{R}^n$. On account of linearity we may write $\boldsymbol{\zeta} = a_i \nabla_\omega \psi^i$, where $\psi^i \in M^2(\Omega)$.
Finally we get

$$A^h_{ij}(\boldsymbol{x}) = \int_\Omega \chi_{\Omega\setminus F}(\omega) A_{ik}(\boldsymbol{x},\omega)(\delta_{jk} + D_k\psi^j)\, d\mu. \tag{3.16}$$

The local functions ψ^i $(i = 1,\ldots,n)$ are solutions to the following problem easily obtained from Eq. (3.5)

$$\psi^i \in M^2(\Omega), \quad D_i[A_{ik}(\boldsymbol{x},\omega)(\delta_{jk} + D_k\psi^k)] = 0.$$

4 Flow of Stokesian fluid through random linear elastic porous medium

In the present section nonstationary flow of a Stokesian fluid through random elastic porous medium is considered. Comments on the passage to stationary flow are also provided. For earlier results on the Stokesian flow through undeformed random porous medium the reader is referred to the papers by Wright [47,48]. This section extends the results obtained by Bielski et al. [12] for the case of microperiodic medium to the case of random porous elastic medium.

4.1 Basic equations

For a fixed $\varepsilon > 0$ all the relevant quantities are denoted by the superscript ε. Similarly to the previous case, the small parameter ε characterizes the microstructure. By $\boldsymbol{u}^\varepsilon(t,\boldsymbol{x},\omega)$ and $\boldsymbol{v}^\varepsilon(t,\boldsymbol{x},\omega)$ we denote the fields of displacement in the elastic skeleton $Q^s_\varepsilon(\omega)$ and the velocity field in $Q^l_\varepsilon(\omega) = Q\setminus\overline{Q}^s_\varepsilon(\omega)$, respectively. In what follows we assume that the sets $Q^l_\varepsilon(\omega)$ are connected, cf. [47,48]. The pressure field in the liquid phase $Q^l_\varepsilon(\omega)$ is denoted by $p^\varepsilon(t,\boldsymbol{x},\omega)$. The small strain tensor $\boldsymbol{e}(\boldsymbol{u})$ is defined by

$$e_{ij}(\boldsymbol{u}) = u_{i,j} = (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})/2. \tag{4.1}$$

The matrix of elastic moduli is denoted by

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x},\omega) = (a_{ijkl}(\boldsymbol{x},\omega)), \quad i,j,k,l = 1,\dots,n.$$

Obviously, $n = 3$ in the three–dimensional case. The elastic moduli a_{ijkl} are defined on $Q \times G$ and satisfy the usual symmetry condition:

$$a_{ijkl} = a_{jikl} = a_{klij}. \tag{4.2}$$

We make the following assumptions:

(A$_1$) for each (i,j,k,l), $a_{ijkl} \in L^\infty(Q \times G)$;

(A$_2$) there exist constants $c_1 \geq c_0 > c$ such that

$$c_0|\boldsymbol{\epsilon}|^2 \leq a_{ijkl}(\boldsymbol{x},\omega)\epsilon_{ij}\epsilon_{kl} \leq c_1|\boldsymbol{\epsilon}|^2 \quad \forall \boldsymbol{\epsilon} \in \mathbb{E}^n_s.$$

Here $\mathbb{E}^n_s$ stands for the space of symmetric $n \times n$ matrices.

The fields $\boldsymbol{u}^\varepsilon$, $\boldsymbol{v}^\varepsilon$ and p^ε satisfy the following equations

$$\varrho^s \ddot{\boldsymbol{u}}^\varepsilon = \operatorname{div}_{\boldsymbol{x}}[\boldsymbol{a}^\varepsilon \boldsymbol{e}(\boldsymbol{u}^\varepsilon)] + \boldsymbol{F}^s(t,\boldsymbol{x}) \quad \text{in } (0,\tau) \times Q^s_\varepsilon(\omega), \tag{4.3}$$

$$\varrho^l \dot{\boldsymbol{v}}^\varepsilon = \varepsilon^2 \eta \Delta \boldsymbol{v}^\varepsilon - \nabla p^\varepsilon + \boldsymbol{F}^l(t,\boldsymbol{x}) \quad \text{in } (0,\tau) \times Q^l_\varepsilon(\omega), \tag{4.4}$$

$$\operatorname{div}_{\boldsymbol{x}} \boldsymbol{v}^\varepsilon = 0 \quad \text{in} \quad (0,\tau) \times Q^l_\varepsilon(\omega). \tag{4.5}$$

Here $\boldsymbol{a}^\varepsilon(\boldsymbol{x},\omega) = \boldsymbol{a}(\boldsymbol{x}, T(\varepsilon^{-1}\boldsymbol{x})\omega)$, $\dot{\boldsymbol{u}}^\varepsilon = \dfrac{\partial \boldsymbol{u}^\varepsilon}{\partial t}$, etc. Obviously, $(0,\tau)$, $\tau > 0$, denotes the time interval, and ϱ^s, ϱ^l are the solid and fluid density, respectively. The scaling $\varepsilon^2\eta$ of the fluid viscosity is typical for the flow of Stokesian fluids through porous media, cf. [11,12,41] and the relevant references cited therein.

We assume that $\boldsymbol{u}^\varepsilon(t,\boldsymbol{x},\omega)$ and $\boldsymbol{v}^\varepsilon(t,\boldsymbol{x},\omega)$ vanish on the exterior boundary. The conditions on the solid–liquid interface $\Gamma^\varepsilon(\omega)$ are specified by

$$\begin{aligned} (\boldsymbol{a}^\varepsilon \boldsymbol{e}(\boldsymbol{u}^\varepsilon))\boldsymbol{n} &= (-p^\varepsilon \boldsymbol{I} + \varepsilon^2 \eta \boldsymbol{e}(\boldsymbol{v}^\varepsilon))\boldsymbol{n} \quad \text{on } (0,\tau) \times \Gamma^\varepsilon(\omega), \\ \boldsymbol{v}^\varepsilon(t,\boldsymbol{x},\omega) &= \dot{\boldsymbol{u}}^\varepsilon(t,\boldsymbol{x},\omega) \quad \text{on } (0,\tau) \times \Gamma^\varepsilon(\omega). \end{aligned} \tag{4.6}$$

Here $\boldsymbol{n} = (n_i)$ denotes the outward unit normal vector to $\Gamma^\varepsilon(\omega)$, directed from $Q^s_\varepsilon(\omega)$ to $Q^l_\varepsilon(\omega)$.

For the sake of simplicity we assume homogeneous initial conditions:

$$\boldsymbol{u}^\varepsilon(0,\boldsymbol{x},\omega) = \boldsymbol{0}, \quad \dot{\boldsymbol{u}}^\varepsilon(0,\boldsymbol{x},\omega) = \boldsymbol{0} \quad \text{in } Q^s_\varepsilon(\omega), \tag{4.7}$$

$$\boldsymbol{v}^\varepsilon(0,\boldsymbol{x},\omega) = \boldsymbol{0} \quad \text{in } Q^l_\varepsilon(\omega). \tag{4.8}$$

The case of nonhomogeneous initial condition can be investigated similarly to Wright [48].

4.2 Passage with ε to zero: effective model

Passing with $\varepsilon \to 0$ in the sense of the stochastic two-scale convergence in the mean we arrive at the homogenized equations describing the macroscopic (effective) behaviour. The general homogenization theorem is formulated without the assumption of ergodicity of the dynamical system T on Ω.

We recall that $F = \Omega\backslash G$; we set $\Phi = \mu(F)$, $Q_\tau = (0,\tau)\times Q$.

Theorem 4.1. Under the assumptions listed in Sec. 4.1, for a fixed $\varepsilon > 0$ a weak solution $(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon, p^\varepsilon)$ solving (4.3)–(4.8) exists and is unique. The pressure field is unique in the space $L^2(Q^s_\varepsilon(\omega))/\mathbb{R}$. There exist $\boldsymbol{u}$, $\boldsymbol{\xi} = (\xi_{ij})$, $\boldsymbol{v}$, p and $p^{(1)}$, functions of $(t,\boldsymbol{x},\omega)$, such that $\boldsymbol{u} \in H^1(Q, I^2(\Omega))^n$, $\boldsymbol{\xi} \in L^2(Q, M^2(\Omega))^{n^2}$ and:

(i) $\{\boldsymbol{u}^\varepsilon, \nabla_{\boldsymbol{x}}\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon\}_{\varepsilon>0}$ stochastically two-scale converges in the mean to

$$(\chi_{\Omega\backslash F}\boldsymbol{u}, \chi_{\Omega\backslash F}(\boldsymbol{\xi} + \nabla_{\boldsymbol{x}}\boldsymbol{u}), \chi_F\boldsymbol{v});$$

(ii) p agrees on $Q_\tau \times F$ with an element of $L^2(Q_\tau, I^2(\Omega))$ and p^ε stochastically two-scale converges in the mean to p along $L^2(Q_\tau \times F)$;

(iii) the homogenized equation of motion is given by

$$\begin{aligned}
&\tilde{E}[\chi_{\Omega\backslash F}(\omega)\varrho^s(\boldsymbol{x})\ddot{\boldsymbol{u}}(t,\boldsymbol{x},\omega)] + \tilde{E}[\chi_F(\omega)\varrho^l(\boldsymbol{x})\dot{\boldsymbol{v}}(t,\boldsymbol{x},\omega)] \\
&= \operatorname{div}_{\boldsymbol{x}}\tilde{E}[\chi_{\Omega\backslash F}(\omega)\boldsymbol{a}(\boldsymbol{x},\omega)(\boldsymbol{\xi}(t,\boldsymbol{x},\omega) + \nabla_{\boldsymbol{x}}\boldsymbol{u}(t,\boldsymbol{x},\omega))] \\
&-\tilde{E}(\chi_F(\omega)\nabla_{\boldsymbol{x}}p(t,\boldsymbol{x},\omega)) - \tilde{E}(\chi_F(\omega)\nabla_\omega p^{(1)}(t,\boldsymbol{x},\omega)) \\
&+\tilde{E}(\chi_{\Omega\backslash F}(\omega)\boldsymbol{F}^s(t,\boldsymbol{x})) + \tilde{E}(\chi_F(\omega)\boldsymbol{F}^l(t,\boldsymbol{x})) \quad \text{in } Q_\tau \times \Omega,
\end{aligned} \tag{4.9}$$

where the operator $\tilde{E}$ has been introduced in Remark 2.1;

(iv) the local elastic cell problem takes the form

$$\operatorname{div}_\omega[\chi_{\Omega\setminus F}(\omega)\boldsymbol{a}(\boldsymbol{x},\omega)(\boldsymbol{\xi}(t,\boldsymbol{x},\omega)+\nabla_{\boldsymbol{x}}\boldsymbol{u}(t,\boldsymbol{x},\omega))]=0 \quad \text{in } Q_\tau\times\Omega, \tag{4.10}$$

$$\boldsymbol{a}(\boldsymbol{x},\omega)[\nabla_{\boldsymbol{x}}\boldsymbol{u}(t,\boldsymbol{x},\omega)+\boldsymbol{\xi}(t,\boldsymbol{x},\omega)]\boldsymbol{N}=-p(t,\boldsymbol{x},\omega)\boldsymbol{N} \quad \text{on } Q_\tau\times\Gamma(\omega), \tag{4.11}$$

where $\Gamma(\omega)=\partial G(\omega)$ and $\boldsymbol{N}$ the exterior unit vector normal to $\Gamma(\omega)$, directed the solid phase to the fluid phase;

(v) $(\boldsymbol{v},p,p^{(1)})$ is a solution on $Q_\tau\times F$ of the time-dependent, two-pressure Stokes system:

$$\varrho^l\dot{\boldsymbol{v}}+\nabla_{\boldsymbol{x}}p+\nabla_\omega p^{(1)}-\eta\Delta_\omega\boldsymbol{v}=\boldsymbol{F}^l(t,\boldsymbol{x});$$

(vi) $\operatorname{div}_\omega\boldsymbol{v}=0$ on $\Omega,\ dt\times d\boldsymbol{x}-$ a. e. on Q_τ ;

(vii) $\exists\ \boldsymbol{v}^1(t,\boldsymbol{x},\omega)$ such that

$$\operatorname{div}_{\boldsymbol{x}}\boldsymbol{v}(t,\boldsymbol{x},\omega)+\operatorname{div}_\omega\boldsymbol{v}^{(1)}(t,\boldsymbol{x},\omega)=0 \quad \text{on} \quad Q_\tau\times F;$$

(viii) the functions $\boldsymbol{u}(t,\boldsymbol{x},\omega),\ \dot{\boldsymbol{u}}(t,\boldsymbol{x},\omega)$ and $\boldsymbol{v}(t,\boldsymbol{x},\omega)$ satisfy homogeneous initial conditions;

(ix) the Darcy law is nonlocal in time:

$$\tilde{E}(\chi_F(\omega)(\boldsymbol{v}-\dot{\boldsymbol{u}}))= \frac{1}{\varrho^l}\int_0^t\boldsymbol{\kappa}(t-s,\omega)[\boldsymbol{F}^l(s,\boldsymbol{x})-\varrho^l\ddot{\boldsymbol{u}}(s,\boldsymbol{x},\omega)-\nabla_{\boldsymbol{x}}p(s,\boldsymbol{x},\omega)]\,ds, \tag{4.12}$$

where the permeability matrix $\boldsymbol{\kappa}=(\kappa_{ij})$ is defined by

$$\kappa_{ij}(t,\omega)=E(\chi_F(\omega)\dot{\boldsymbol{w}}^{(i)}(t,\omega)\cdot\boldsymbol{e}_j),\ i,j=1\ldots,n. \tag{4.13}$$

Here $\boldsymbol{e}_j$ stands for the jth standard basis vector in $\mathbb{R}^n$. The permeability matrix $\boldsymbol{A}$ is symmetric and positive definite, cf. also

Wright [48]. The function $\boldsymbol{w}^i$, $i = 1, \ldots, n$, is a solution to the following *flow cell problem*:

$$\begin{aligned}
&\varrho^l \frac{\partial \boldsymbol{w}^{(i)}}{\partial t} + \nabla_\omega q^{(i)} - \eta \Delta_\omega \boldsymbol{w}^{(i)} = \boldsymbol{e}_i, && \text{on } (0,\tau) \times F, \\
&\operatorname{div}_\omega \boldsymbol{w}^{(i)}(t,\omega) = 0, && \text{on } (0,\tau) \times F, \\
&\boldsymbol{w}^{(i)}(t,\omega) = 0 && \text{on } (0,\tau) \times \Gamma(\omega).
\end{aligned} \tag{4.14}$$

We note that the local field $q^{(i)}$ plays the role of the local pressure generated by $\boldsymbol{e}_i$. □

The proof of the last theorem is rather lengthly and we omit it here. Anyway, it can be performed by randomization of the proof devised in [12] for the periodic homogenization, cf. also the proof of Theorem 3.1. In the case of nonhomogeneous initial condition an additiona telrm appears in Eq. (4.12), cf. [48].

4.3 Ergodicity of dynamical system and comments

If T is ergodic on Ω then the displacement $\boldsymbol{u}$ and pressure p do not depend on ω. In such a case the Birkhoff theorem formulated in Sec. 2.1 implies that the operations E, $\tilde{E}$ can be replaced by ensemble averaging. For instance, Darcy law (4.12) takes the form

$$\begin{aligned}
&\langle \chi_F(\omega)(\boldsymbol{v}(t,\boldsymbol{x},\omega) - \dot{\boldsymbol{u}}(t,\boldsymbol{x})\rangle = \\
&\tfrac{1}{\varrho^l} \textstyle\int_0^t \boldsymbol{\kappa}(t-s)[\boldsymbol{F}^l(s,\boldsymbol{x}) - \varrho^l \ddot{\boldsymbol{u}}(s,\boldsymbol{x}) - \nabla_{\boldsymbol{x}} p(s,\boldsymbol{x})]\, dx,
\end{aligned} \tag{4.15}$$

where

$$\kappa_{ij}(t) = \langle \chi_F(\omega) \dot{\boldsymbol{w}}^{(i)}(t,\omega) \cdot \boldsymbol{e}_j \rangle. \tag{4.16}$$

The equation of motion (4.9) is then written as follows

$$\begin{aligned}
&(1-\Phi)\varrho^s(\boldsymbol{x})\ddot{\boldsymbol{u}}(t,\boldsymbol{x}) + \varrho^l \langle \chi_F(\omega) \dot{\boldsymbol{v}}(t,\boldsymbol{x},\omega) \rangle \\
&= \operatorname{div}_{\boldsymbol{x}} \langle \chi_{\Omega \setminus F}(\omega) \boldsymbol{a}(\boldsymbol{x},\omega)(\boldsymbol{\xi}(t,\boldsymbol{x},\omega) + \nabla_{\boldsymbol{x}} \boldsymbol{u}(t,\boldsymbol{x})\rangle \\
&- \langle \chi_F(\omega) \nabla_{\boldsymbol{x}} p(t,\boldsymbol{x}) \rangle - \langle \chi_F(\omega) \nabla_\omega p^{(1)}(t,\boldsymbol{x},\omega) \rangle \\
&+ (1-\Phi)\boldsymbol{F}^s(t,\boldsymbol{x}) + \Phi \boldsymbol{F}^l(t,\boldsymbol{x}).
\end{aligned} \tag{4.17}$$

We recall that $\Phi = \mu(F)$. In the periodic case Φ denotes the volume fraction of pores.

If T is ergodic, the macroscopic elastic moduli $\boldsymbol{a}^h(\boldsymbol{x})$ and the macroscopic stress tensor $\langle\boldsymbol{\sigma}^0\rangle(t,\boldsymbol{x})$, $\boldsymbol{x}\in Q$, are determined by

$$a^h_{ijkl}(\boldsymbol{x}) = \langle \chi_{\Omega\backslash F}(\omega) a_{ijkl}(\boldsymbol{x},\omega) + a_{ijmn}(\boldsymbol{x},\omega) e^\omega_{mn}(\boldsymbol{\Delta}^{(pq)})\rangle, \tag{4.18}$$

$$\begin{aligned}\langle\boldsymbol{\sigma}^0\rangle(t,\boldsymbol{x}) &= \langle \boldsymbol{\chi}_{\Omega\backslash F}(\omega)\boldsymbol{\sigma}^{s0}(t,\boldsymbol{x},\omega)\rangle + \langle \boldsymbol{\chi}_F(\omega)\boldsymbol{\sigma}^{l0}(t,\boldsymbol{x},\omega)\rangle \\ &= \langle \boldsymbol{\chi}_{\Omega\backslash F}(\omega)\boldsymbol{\sigma}^{s0}(t,\boldsymbol{x},\omega)\rangle - \Phi p(t,\boldsymbol{x})\boldsymbol{I}.\end{aligned} \tag{4.19}$$

Here $\boldsymbol{I} = (\delta_{ij})$ and

$$\begin{aligned}&\boldsymbol{\sigma}^{s0}(t,\boldsymbol{x},\omega) = \\ &\boldsymbol{a}(\boldsymbol{x},\omega)[\nabla_{\boldsymbol{x}}\boldsymbol{u}(t,\boldsymbol{x}) + \nabla_\omega \boldsymbol{u}^1(t,\boldsymbol{x},\omega)] \quad \text{in} \quad Q_\tau\times(\Omega\backslash F),\end{aligned} \tag{4.20}$$

$$\boldsymbol{\sigma}^{l0}(t,\boldsymbol{x},\omega) = \boldsymbol{\sigma}^{l0}(t,\boldsymbol{x}) = -p(t,\boldsymbol{x})\boldsymbol{I}, \quad \text{in} \quad Q_\tau\times F, \tag{4.21}$$

$$u^{(1)}_m(t,\boldsymbol{x},\omega) = \Delta^{(pq)}_m(t,\boldsymbol{x},\omega)e_{pq}(\boldsymbol{u}(t,\boldsymbol{x})) + P^{(m)}(t,\boldsymbol{x},\omega)p(t,\boldsymbol{x}).$$

The local strains $e^\omega_{ij}(\boldsymbol{\Delta}^{(pq)})$ are calculated as follows

$$e^\omega_{ij}(\boldsymbol{\Delta}^{(pq)}) = (D_i\Delta^{(pq)}_j + D_j\Delta^{(pq)}_i)/2.$$

The functions $\Delta^{(pq)}_m$ and $P^{(m)}$ are solutions to the following local equations posed on $Q_\tau\times\Omega$ (in fact $Q_\tau\times(\Omega\backslash F)$):

$$D_j[\chi_{\Omega\backslash F}(\omega)(a_{ijpq}(\boldsymbol{x},\omega) + a_{ijmn}D_m\Delta^{(pq)}_n(t,\boldsymbol{x},\omega))] = 0, \tag{4.22}$$

$$D_j[\chi_{\Omega\backslash F}(\omega)(a_{ijmn}(\boldsymbol{x},\omega)D_nP^{(m)}(t,\boldsymbol{x},\omega) + \delta_{ij})] = 0. \tag{4.23}$$

Hence we get

$$\begin{aligned}&[a_{ijpq}(\boldsymbol{x},\omega) + a_{ijmn}(\boldsymbol{x},\omega)D_m\Delta^{(pq)}_n(t,\boldsymbol{x},\omega)]N_j = 0, \\ &\qquad\qquad\qquad\qquad\qquad \text{on} \quad (0,\tau)\times\Gamma(\omega),\end{aligned} \tag{4.24}$$

and

$$(a_{ijmn}D_nP^{(m)}(t,\boldsymbol{x},\omega) + \delta_{ij})N_j = 0, \quad \text{on} \quad (0,\tau)\times\Gamma(\omega). \tag{4.25}$$

We observe that in Eqs. (4.22)–(4.25) the variables $t, \boldsymbol{x}$ are treated as parameters. We also notice that the local problems are in fact posed on infinite domains: the elastic problem on $G(\omega)$ and the local flow problem on $F(\omega)$ cf. Dal Maso and Modica [20,21], Gambin et al. [23], Golden and Papanicolaou [24], Telega and Bielski [43], Torquato [45]. In practice the local problems are posed on a sufficiently large domain called a representative volume element (RVE), for instance a cube. In general, the effective moduli depend on the choice of RVE. Rigorously, the effective moduli are obtained by letting the dimensions of the RVE tend to infinity. Unfortunately, there are no practical procedures to carry out such a theoretically lucid approach. Hence the need for finding bounds on the effective moduli like permeability coefficients and elastic coefficients, cf. Torquato [45].

The reader probably noticed that within the framework adopted in the present paper the effective moduli A^h_{ij} and a^h_{ijkl} depend on the macroscopic variable $\boldsymbol{x} \in Q$. It means that the stochastic homogenization procedure, primary developed by Bourgeat et al. [13], may be qualified as *nonuniform*. Such a case is practically very important since many materials possess macroscopically inhomogeneous properties, cf. for instance Adler and Thovert [3,4], Cowin[17,18]. What does it mean mathematically? In fact, the operator $\tilde{E}$ involves the dynamical system $\tilde{T}(\boldsymbol{y}) : Q \times \Omega \to Q \times \Omega$ defined by, cf. [8,13,47,48]

$$\tilde{T}(\boldsymbol{y})(\boldsymbol{x}, \omega) = (\boldsymbol{x}, T(\boldsymbol{y})\omega), \quad (\boldsymbol{x}, \omega) \in Q \times \Omega.$$

The last comment pertains the passage to the stationary case. This is performed by letting t tend to infinity, cf. [12]. The Darcy law assumes then a more familiar form:

$$\langle \chi_F(\omega)\boldsymbol{v}(\boldsymbol{x}, \omega)\rangle = \boldsymbol{K}[\boldsymbol{F}(\boldsymbol{x}) - \nabla_{\boldsymbol{x}} p(\boldsymbol{x})], \tag{4.26}$$

where

$$K_{ij} = \frac{1}{\varrho^l}\langle \chi_F(\omega)\eta(\nabla_\omega \boldsymbol{w}^{(i)}_\infty) : \nabla_\omega \boldsymbol{w}^{(j)}_\infty\rangle. \tag{4.27}$$

Now the local velocity $\boldsymbol{w}^{(i)}$ is a solution to

$$\begin{aligned} \nabla_\omega q^{(i)}_\infty - \eta\Delta_\omega \boldsymbol{w}^{(i)}_\infty &= \boldsymbol{e}_i \quad \text{in} \quad F, \\ \operatorname{div}_\omega \boldsymbol{w}^{(i)}_\infty &= 0 \quad \text{in} \quad F. \end{aligned} \tag{4.28}$$

Obviously, $q^{(i)}_\infty$ is the local pressure generated by $\boldsymbol{e}_i$.

5 Stochastic multiscale convergence in the mean

Materials with hierarchical microstructures are common in nature. For instance, fractured reservoirs are modelled as materials possessing two separated scales. cf. [4,46]. From the point of view of periodic homogenization such materials as called materials with double periodicity (or double porosity), cf. [11,25,46]. Another striking examples are biological tissues like bone and soft tissues, [17,18,22,26]. Architecture of those biological materials covers several separated scales and ranges from nanometers to milimeters or even centimeters. Obviously, in the case of geomaterials with hierarchical structure scales are on the orders of magnitude larger. Another interesting example is provided by optimal design for minimum compliance and optimal shape design [6,30]. It was shown that optimal structures are made of so-called laminates of appropriate orders [6,30], i.e., hierarchical microstructures.

Periodic multiscale convergence was introduced by Allaire and Briane [7] as an extension of two-scale convergence. The latter was introduced by Nguetseng [33] and developed by Allaire [5].

We observe that the multiscale convergence provides a rigorous mathematical foundation for the so-called reiterated homogenization in the case of not necessarily stationary problems. On the other hand, the method of Γ-convergence can also be used to the determination of materials with hierarchical architecture (microstructure), cf. [14, 44]. Unfortunately, this case of the reiterated homogenization is confined to sequences of functionals with suitable mathematical properties.

The aim of the present section is to extend the stochastic two-scale convergence in the mean to several microscopic scales. We note that the intermediate scales are sometimes called mesoscales. In other words, we intend to perform randomization of the periodic multiscale convergence studied by Allaire and Briane [7].

Let $\varepsilon_1, \ldots, \varepsilon_N$ be positive functions of $\varepsilon > 0$ which converge to 0 as ε does. We make the following assumption:

(H) The functions $\varepsilon_1, \ldots, \varepsilon_N$ are assumed to be separated, i.e. they satisfy $\displaystyle\lim_{\varepsilon\to 0} \frac{\varepsilon_{k+1}}{\varepsilon_k} = 0 \quad \forall k \in \{1, \ldots, N-1\}.$

For instance, the scales $\varepsilon_k,\ 1 \le k \le N$ may be successive powers of the parameters of the parameter ε. Then we have

$$\varepsilon_k = \varepsilon^k \quad \text{for} \quad 1 \le k \le N$$

and the assumption (H) is satisfied.

Each scales is now governed by a dynamical system $T_k,\ k = 1, \dots, N$, acting on Ω_k. Obviously, we now have to introduce N measure spaces $(\Omega_k, \mathcal{F}_k, \mu_k)$. We assume that the spaces $L^2(\Omega_k),\ k = 1, \dots, N$ are separable, cf. [13,29,47,48]. The dynamical systems $T_k,\ k = 1 \dots, N$, satisfy the conditions specified in Sec. 2.1.

Ergodicity of T_k is defined similarly as ergodicity of T in Sec. 2.1. In considerations which follow we assume that each dynamical system T_k on Ω_k is n-dimensional. However, from practical point of view, it may be of importance to consider also lower dimensional dynamic systems. For instance, it may happen that at a, say, scale i the dynamical system is one-dimensional. Then T_i defines a one-dimensional dynamical system on Ω_i as a family $\{T_i(x) \,|\, x \in \mathbb{R}\}$ of invertible maps $T_i(x) : \Omega_i \to \Omega_i$ satisfying the usual properties.

Thus we come to the central definition of this section.

Definition 5.1. A sequence $\{u^\varepsilon\}_{\varepsilon>0}$ in $L^2(Q \times \Omega_1 \times \dots \times \Omega_N)$ is said to $(N+1)$-scale converge in the mean to $u \in L^2(Q \times \Omega_1 \times \dots \times \Omega_N)$ if for every $\psi \in L^2(Q \times \Omega_1 \times \dots \times \Omega_N)$,

$$\lim_{\varepsilon \to 0} \int\limits_{Q \times \Omega_1 \times \dots \times \Omega_N} u^\varepsilon(\boldsymbol{x}, \omega_1, \dots, \omega_N) \psi(\boldsymbol{x}, T_1(\varepsilon_1^{-1}\boldsymbol{x})\omega_1, \dots, T_N(\varepsilon_N^{-1}\boldsymbol{x})\omega_N) \times d\boldsymbol{x} d\omega_1 \dots d\omega_N$$

$$= \int\limits_{Q \times \Omega_1 \times \dots \times \Omega_N} u(\boldsymbol{x}, \omega_1, \dots, \omega_N) \psi(\boldsymbol{x}, \omega_1, \dots \omega_N)\, d\boldsymbol{x} d\omega_1 \dots d\omega_N.$$

As we already know, periodic homogenization is a specific case of stochastic one. It means that Definition 2.3 of Allaire and Briane [7] is a specific case of definition just introduced. Moreover, we can consider "mixed" homogenization, namely stochastic–periodic multi–scale convergence. For instance one or several scales may be periodic and the remaining ones still stochastic. To illustrate the notion of mixed stochastic–periodic homogenization consider the following 3–scale mixed stochastic–periodic convergence.

Example 5.1. Let $N = 2$, $\Omega_1 = Y$ with $d\mu_1 = d\boldsymbol{y}$ and $\Omega_2 = \Omega$, $T_2 = T$. As usual Y denotes the so-called basic cell [10,11,41]. Then a sequence u^ε in $L^2(Q\times\Omega)$ 3–scale converges to $u(\boldsymbol{x},\boldsymbol{y},\omega) \in L^2(Q\times Y\times\Omega)$ if for every $\psi \in L^2(Q \times Y \times \Omega)$ with $\psi(\boldsymbol{x},\cdot,\omega) \in C_\sharp(Y)$ for every $(\boldsymbol{x},\omega) \in Q \times \Omega$,

$$\lim_{\varepsilon\to 0} \int_{Q\times\Omega} u^\varepsilon(\boldsymbol{x},\omega)\psi(\boldsymbol{x},\varepsilon_1^{-1}\boldsymbol{x},T(\varepsilon_2^{-1}\boldsymbol{x})\omega)\, d\boldsymbol{x}d\boldsymbol{y}d\mu$$
$$= \int_Q \int_Y \int_\Omega u(\boldsymbol{x},\boldsymbol{y},\omega)\, d\boldsymbol{x}d\boldsymbol{y}d\mu.$$

Here $C_\sharp(Y)$ is the space of continuous periodic functions. The reader can now easily consider more complex mixed stochastic–periodic multiscale convergence.

The next result establishes a compactness property of stochastic $(N+1)$-scale convergence in the mean.

Theorem 5.1. Let $\{u^\varepsilon\}_{\varepsilon>0}$ be a bounded sequence in $L^2(Q \times \Omega_1 \times \cdots \times \Omega_N)$. Under assumption (H), one can extract a subsequence of $\{u^\varepsilon\}$ which stochastically $(N+1)$-scale converges to $u \in L^2(Q \times \Omega_1 \times \cdots \times \Omega_N)$.

Proof. By Cauchy-Schwartz's inequality and the invariance of T_k with respect to μ_k, $k = 1,\ldots,N$, for all $\varepsilon > 0$ and $\psi \in L^2(Q\times\Omega_1\times\cdots\times\Omega_N)$ we have

$$\Big| \int_{Q\times\Omega_1\times\cdots\times\Omega_N} u^\varepsilon(\boldsymbol{x},\omega_1\ldots,\omega_N)\psi(\boldsymbol{x},T_1(\varepsilon_1^{-1}\boldsymbol{x})\omega_1,\ldots,T_N(\varepsilon_N^{-1}\boldsymbol{x})\omega_N) \times d\boldsymbol{x}d\mu_1\cdots d\mu_N \Big|$$
$$\le C\|\psi\|_{L^2(Q\times\Omega_1\times\cdots\times\Omega_N)},$$

where $C = \sup_\varepsilon \|u^\varepsilon\|_{L^2(Q\times\Omega_1\times\cdots\times\Omega_N)}$. The integral $\int_{Q\times\Omega_1\times\cdots\times\Omega_N} u^\varepsilon\psi\, d\boldsymbol{x}d\mu_1\cdots d\mu_N$ is a linear form on the space of test functions ψ, i.e. on $L^2(Q \times \Omega_1 \times \cdots \times \Omega_N)$. According to our assumptions this space is separable. This form can be identified with a duality product $\langle f^\varepsilon,\psi\rangle$ where $\{f^\varepsilon\}_{\varepsilon>0}$ is a bounded sequence of functionals still contained in $L^2(Q \times \Omega_1 \times \cdots \times \Omega_N)$. One can extract a subsequence weakly convergent to a limit functional f. By the Riesz representation theorem, there is a $u \in L^2(Q \times \Omega_1 \times \cdots \times \Omega_N)$ such that

$$\lim_{\varepsilon\to 0}\int_{Q\times\Omega_1\times\cdots\times\Omega_N} u^\varepsilon(\boldsymbol{x},\omega_1,\dots,\omega_N)\psi(\boldsymbol{x},T_1(\varepsilon_1^{-1}\boldsymbol{x})\omega_1,\dots,T_N(\varepsilon_N^{-1}\boldsymbol{x})\omega_N)\times d\boldsymbol{x}d\mu_1\dots d\mu_N$$

$$=\langle f,\psi\rangle = \int\limits_{Q\times\Omega_1\times\cdots\times\Omega_N} u(\boldsymbol{x},\omega_1,\dots,\omega_N)\psi(\boldsymbol{x},\omega_1,\dots,\omega_N)\times d\boldsymbol{x}d\mu_1\dots d\mu_N$$

for each $\psi \in L^2(Q \times \Omega_1 \times \cdots \times \Omega_N)$. □

The next theorem is of paramount importance for applications to homogenization problems in the case of materials with random hierarchical microstructure.

Theorem 5.2. Let X be a norm-closed convex subset of $H^1(Q)$. Suppose $\{u^\varepsilon\}_{\varepsilon>0}$ is a sequence in $L^2(Q \times \Omega_1 \times \cdots \times \Omega_N)$ such that

(i) $\forall\varepsilon > 0$, $u^\varepsilon(\cdot,\omega_1,\dots,\omega_N) \in X$ for $\mu_1 \times \mu_2 \times \dots \times \mu_N$- a. e $(\omega_1,\dots,\omega_N) \in \Omega_1,\dots,\Omega_N$;

(ii) $\{u^\varepsilon\}$ is bounded in $L^2(\Omega, H^1(\Omega_1 \times \dots \times \Omega_N))$. Then there exists $u \in H^1(Q, I^2(\Omega_1 \times \dots \times \Omega_N))$, $\boldsymbol{\zeta}^{(k)} \in L^2(Q \times \Omega_1 \times \dots \times \Omega_{k-1}, M^2(\Omega_k))^N$ such that, up to a subsequence, the following conditions are satisfied:

(iii) for $\mu_1 \times \mu_2 \times \cdots \times \mu_N -$a. e.$(\omega_1 \dots, \mu_N)$, $u(\cdot,\omega_1 \dots,\mu_N) \in X$;

(iv) $\boldsymbol{\zeta}^{(k)}$ is contained in $L^2(Q \times \Omega_1 \times \Omega_2 \times \cdots \times \Omega_k)$ - norm closure of
$L^2(Q) \otimes (\text{range of } \nabla_{\omega_1}) \otimes \cdots \otimes (\text{range of } \nabla_{\omega_k})$

(v) $\{u^\varepsilon\}$ (respectively, $\{\nabla_{\boldsymbol{x}} u^\varepsilon\}$) stochastically $(N+1)$–converges in the mean to $u(\boldsymbol{x},\omega_1,\dots,\omega_N)$ (respectively, $\nabla_{\boldsymbol{x}} u+$ $\sum_{k=1}^N \boldsymbol{\zeta}^{(k)}(\boldsymbol{x},\omega_1\dots,\omega_k)$). □

The proof of the above theorem is lengthly and will not be given here. Anyway, it is an extension of Theorem 3.7 of Bourgeat et al. [13] and of Allaire and Briane [7].

To better grasp condition (ii) we have to define the space $I^2(\Omega_1 \times \cdots \times \Omega_k)$, $k = 1,\dots,N$.

A function $f \in L^2(\Omega_1 \times \cdots \times \Omega_k)$ is said to be invariant for $T = (T_1, \dots, T_k)$ (relative to $\mu_1 \times \mu_2 \times \cdots \times \mu_k$) if for every $\boldsymbol{x} \in \mathbb{R}^n$, $f \circ T(\boldsymbol{x}) = f \circ (T_1(\boldsymbol{x}), \dots, T_k(\boldsymbol{x})) = f$, $\mu_1 \times \mu_2 \times \cdots \times \mu_k$-a.e. on $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_k$. By $I^2(\Omega_1 \times \Omega_2 \times \cdots \times \Omega_k)$ we denote the subspace of functions in $L^2(\Omega_1 \times \Omega_2 \times \cdots \times \Omega_k)$ which are invariant for $T = (T_1, T_2, \dots, T_k)$. Similarly to Andrews and Wright [8], who considered only the case $k = 1$, we could introduce the space $I^p(\Omega_1 \times \Omega_2 \times \cdots \times \Omega_k)$, $1 \le p \le \infty$.

The operator $E_{(1,\dots,k)} : L^2(\Omega_1 \times \Omega_2 \times \cdots \times \Omega_k) \to L^2(\Omega_1 \times \Omega_2 \times \cdots \times \Omega_k)$ is now defined as follows

$$\begin{aligned}
&(E_{(1\dots,k)} f)(\omega_1, \dots, \omega_k) \\
&= \lim_{\lambda_1 \to \infty} (2\lambda_1)^{-n} \cdots \lim_{\lambda_k \to \infty} (2\lambda_k)^{-n} \int\limits_{[-\lambda_1, \lambda_1]^n} \cdots \\
&\cdots \int\limits_{[-\lambda_k, \lambda_k]^n} f(T_1(\boldsymbol{x})\omega_1, \dots, T_k(\boldsymbol{x})\omega_k)\, d\boldsymbol{x}
\end{aligned} \tag{5.1}$$

where the limits exist strongly in $L^2(\Omega_1 \times \Omega_2 \times \cdots \times \Omega_k)$ and also pointwise $(\mu_1 \times \mu_2 \times \cdots \times \mu_k)$–a.e. on $\Omega = (\Omega_1 \times \Omega_2 \times \cdots \times \Omega_k)$.

We observe that formula (5.1) is valid for $f \in L^2(\Omega_1 \times \Omega_2 \times \cdots \times \Omega_k)$ and $(\mu_1 \times \mu_2 \times \cdots \times \mu_k)$– a.e. in $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_k$. If $k = 1$, we recover formula (2.14) where $E_1 = E$.

We also introduce the space

$$M^2(\Omega_1 \times \Omega_2 \times \cdots \times \Omega_k) = \ker E_{(1,\dots,k)}. \tag{5.2}$$

Obviously, $M^2(\Omega_1 \times \Omega_2 \times \cdots \times \Omega_k) = [I^2(\Omega_1 \times \Omega_2 \times \cdots \times \Omega_k)]^\perp$ or $L^2(\Omega_1 \times \Omega_2 \times \cdots \times \Omega_k) = M^2(\Omega_1 \times \Omega_2 \times \cdots \times \Omega_k) \oplus I^2(\Omega_1 \times \Omega_2 \times \cdots \times \Omega_k)$.

The operator $\tilde{E}_{(1,\dots,k)}$ is introduced similarly to the operator $\tilde{E}$ defined by (2.18). More precisely, we define the mapping

$$\tilde{T}_{(1,\dots,k)}(\boldsymbol{y}) : Q \times (\Omega_1 \times \Omega_2 \times \cdots \times \Omega_k) \to Q \times (\Omega_1 \times \Omega_2 \times \cdots \times \Omega_k)$$

by $\tilde{T}_{(1,\dots,k)}(\boldsymbol{y})(\boldsymbol{x}, \omega_1, \dots, \omega_k) = (\boldsymbol{x}, T_1(\boldsymbol{y})\omega_1, \dots, T_k(\boldsymbol{y})\omega_k)$. Then

$$\tilde{E}_{(1,\dots,k)} g(\boldsymbol{x}, \omega_1, \dots, \omega_k) = E_{(1,\dots,k)}[g(\boldsymbol{x}, \cdot, \dots, \cdot)](\omega_1, \dots, \omega_k), \tag{5.3}$$

where $g \in L^2(Q \times (\Omega_1 \times \Omega_2 \times \cdots \times \Omega_k))$. Formula (5.1) gives

$$\tilde{E}_{(1,\ldots,k)} g(\boldsymbol{x}, \omega_1, \ldots, \omega_k) = \lim_{\lambda_1 \to \infty} \ldots \lim_{\lambda_k \to \infty} (2\lambda_1)^{-n} \cdot \ldots \cdot (2\lambda_k)^{-n} \int\limits_{[-\lambda_1, \lambda_1]^n} \cdots \int\limits_{[-\lambda_k, \lambda_k]^n} g(\boldsymbol{x}, T_1(\boldsymbol{y})\omega_1, \ldots, T_k(\boldsymbol{y})\omega_k)\, d\boldsymbol{y}. \tag{5.4}$$

We observe that $\tilde{E}_{(1,\ldots,k)} g$ does not depend on $(\omega_1, \ldots, \omega_k) \in \Omega_1 \times \cdots \times \Omega_k$ provided that $\mu_1 \times \mu_2 \times \ldots \times \mu_k$ is ergodic for $T = (T_1, \ldots, T_k)$.

Application to reiterated stochastic homogenization of stationary diffusion equation

To illustrate the general scheme of multiscale stochastic homogenization we consider the following diffusion or transport problem with $f \in L^2(Q)$ and $\varepsilon > 0$:

$$-\operatorname{div}_{\boldsymbol{x}} [\boldsymbol{A}(\boldsymbol{x}, T_1(\varepsilon_1^{-1}\boldsymbol{x})\omega_1, \ldots, T_N(\varepsilon_N^{-1}\boldsymbol{x})\omega_N) \nabla_{\boldsymbol{x}} u^\varepsilon] = f \quad \text{in } Q, \tag{5.5}$$

$$u = 0 \quad \text{on} \quad \partial Q. \tag{5.6}$$

Here Q is a bounded Lipschitz domain of $\mathbb{R}^n$ whilst $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{x}, \omega_1, \ldots, \omega_N)$ is a symmetric $n \times n$ matrix of functions defined on $Q \times \Omega_1 \times \ldots \times \Omega_N$ which satisfies the following conditions:

(i) $A_{ij} \in L^\infty(Q \times \Omega_1 \times \ldots \times \Omega_N),\ i, j = 1, \ldots, n$;

(ii) there exist constants $\beta \geq \alpha > 0$ such that for $d\boldsymbol{x} \times d\mu_1 \times \cdots \times d\mu_N$– a.e. $(\boldsymbol{x}, \omega_1, \ldots, \omega_N) \in Q \times \Omega_1 \times \cdots \times \Omega_N$,

$$\alpha |\boldsymbol{\xi}|^2 \leq A_{ij}(\boldsymbol{x}, \omega_1, \ldots, \omega_N) \xi_i \xi_j \leq \beta |\boldsymbol{\xi}|^2, \qquad \forall \boldsymbol{\xi} \in \mathbb{R}^n.$$

We now wish to determine the limit behaviour as $\varepsilon \to 0$ (homogenization). Our subsequent theorem provides a procedure for the reiterated stochastic homogenization.

Theorem 5.3. Under assumption (H) on the separation of scales and assuming further that the matrix $\boldsymbol{A}$ satisfies conditions (i), (ii), then there exist $u \in H_0^1(Q, I^2(\Omega_1 \times \cdots \times \Omega_N))$ and $\boldsymbol{\zeta}^{(k)} \in L^2(Q \times \Omega_1 \times \cdots \times \Omega_{k-1}, M^2(\Omega_k))^n,\ k = 1, \ldots, N$, with $\boldsymbol{\zeta}^{(k)}$ contained in the $L^2(Q \times \Omega_1 \times$

$\cdots \times \Omega_k$)–closure of $L^2(Q) \otimes$(range of ∇_{ω_1})$\otimes \cdots \otimes$(range of ∇_{ω_k}), such that $\{u^\varepsilon\}$ and $\{\nabla_{\boldsymbol{x}} u^\varepsilon\}$ stochastically two-scale converge in the mean as $\varepsilon \to 0$ to u and

$$\nabla_{\boldsymbol{x}} u + \sum_{k=1}^{N} \boldsymbol{\zeta}^{(k)}(\boldsymbol{x}, \omega_1, \ldots, \omega_k),$$

respectively. Moreover, $(u, u_1, \ldots, u_N)$ is the unique solution to

$$-\mathrm{div}_{\,\omega_N} \boldsymbol{A}(\nabla_{\boldsymbol{x}} u + \sum_{k=1}^{N} \boldsymbol{\zeta}^{(k)}) = 0, \tag{5.7}$$

$$-\mathrm{div}_{\,\omega_k} \{\tilde{E}_{k+1,\ldots,N}[\boldsymbol{A}(\nabla_{\boldsymbol{x}} u + \sum_{j=1}^{N} \boldsymbol{\zeta}^{(j)})]\} = 0, \quad 1 \le k \le N-1, \tag{5.8}$$

$$-\mathrm{div}_{\,\boldsymbol{x}} \{\tilde{E}_{1,\ldots,N}[\boldsymbol{A}(\nabla_{\boldsymbol{x}} u + \sum_{j=1}^{N} \boldsymbol{\zeta}^{(j)})]\} = f. \tag{5.9}$$

We recall that $\boldsymbol{A}$ and u depend on $(\boldsymbol{x}, \omega_1 \ldots \omega_N)$ whilst $\boldsymbol{\zeta}^{(j)}$ on $(\boldsymbol{x}, \omega_1 \ldots \omega_j)$. □

The proof of this theorem is lengthy and is omitted here.

Consider now the specific case when $T = (T_1, \ldots, T_N)$ is ergodic. Then u does not depend on $(\omega_1 \ldots \omega_N)$; *i.e.*, $u = u(\boldsymbol{x})$, $\boldsymbol{x} \in Q$, and is the unique solution of the following homogenized equation

$$u \in H_0^1(Q), \quad -\mathrm{div}\,(\boldsymbol{A}^h(\boldsymbol{x})\nabla u) = f \quad \text{in } Q. \tag{5.10}$$

The matrix of homogenized coefficients $(A_{ij}(\boldsymbol{x}))$ is defined by the inductive homogenization formulae

$$\left.\begin{array}{l} \boldsymbol{A}_N^h = \boldsymbol{A}(\boldsymbol{x}, \omega_1, \ldots, \omega_N), \\ \boldsymbol{A}_k^h = \boldsymbol{A}_k(\boldsymbol{x}, \omega_1, \ldots, \omega_k), \\ \vdots \\ \boldsymbol{A}^h = \boldsymbol{A}^h(\boldsymbol{x}) \end{array}\right\} \tag{5.11}$$

The matrix $\boldsymbol{A}_k^h(\boldsymbol{x}, \omega_1, \ldots, \omega_k)$ is obtained by the stochastic two-scale convergence in the mean of $\boldsymbol{A}_{k+1}^h(\boldsymbol{x}, \omega_1, \ldots, \omega_k, T_{k+1}(\varepsilon_{k+1}^{-1}\boldsymbol{x}))$; here $\omega_1, \ldots, \omega_k$

play the role of parameters. Finally $\boldsymbol{A}^h(\boldsymbol{x})$ is obtained by the stochastic two-scale convergence in the mean of $\boldsymbol{A}_1^h(\boldsymbol{x}, T_1(\varepsilon_1^{-1}\boldsymbol{x}))$. In other words, recalling that the dynamical systems $T_1, \dots, T_N$ are now ergodic, we have

$$\begin{aligned} &\boldsymbol{A}_k^h(\boldsymbol{x}, \omega_1, \dots, \omega_k)\boldsymbol{a} = \\ &\int\limits_{\Omega_{k+1}} \boldsymbol{A}_{k+1}^H(\boldsymbol{a} + \boldsymbol{w}^{(k+1)}(\boldsymbol{x}, \omega_1, \dots, \omega_k, \omega_{k+1}))\, d\mu_{k+1} \end{aligned} \tag{5.12}$$

for all $\boldsymbol{a} \in \mathbb{R}^n$, with $\boldsymbol{w}^{(k+1)} \in L^2(Q \times \Omega_1 \times \dots \times \Omega_k, M^2(\Omega_{k+1}))^n$ solution of

$$\operatorname{div}_{\omega_{k+1}}[\boldsymbol{A}_{k+1}^h(\boldsymbol{a} + \boldsymbol{w}^{(k+1)})] = 0, \text{ in } \Omega_{k+1}. \tag{5.13}$$

Obviously in Eqs. (5.12) and (5.13) there is no summation over k.

The reader is advised to proceed further and find formulae similar to Eq. (3.16).

6 Final remarks

We used the method of stochastic two-scale convergence in the mean developed by Bourgeat et al. [13] to two physical problems of randomly porous media. This method offers many other possibilities of applications to not necessarily linear problems. From the theoretical point of view, it would be interesting to know what are the relations between the stochastic two-scale convergence in the mean and the stochastic Γ-convergence developed in [1,20,21,40]. In the last two papers the authors studied nonconvex problems, yet no precisely those arising in finite elasticity.

In the last section of our paper we have only briefly outlined an extension of periodic reiterated homogenization to the stochastic case. To this end we have generalized the stochastic two-scale convergence in the mean and introduced the notion of stochastic multiscale convergence in the mean. An alternative approach would consist in a randomization of the iterated homogenization developed in the book by Braides and Defranceschi [14]. These authors use the method of Γ-convergence in the periodic case. The advantage of multiscale convergence (periodic and stochastic) is that it applies to evolution problems whilst Γ-convergence is confined to sequences of functionals with suitable mathematical properties. We know of no application of Γ-convergence to evolution, say dynamic, problems.

Acknowledgements

The authors were partially supported by the State Committee for Scientific Research (KBN, Poland) through the grant No 8 T07A 052 21. The first author was also partially supported through the grant No 7 T07A 043 18.

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Stohastička homogenizacija i makroscopsko modeliranje kompozita i tečenje kroz poroznu sredinu

UDK 531.01, 532.528

Svrha ovog rada je uglavnom dvostruka. Prvo, stohastička dvoskalna konvergencija u srednjem razvijena od Bourgeat i saradnika [13] se koristi za izvodjenje sledećih makroskopskih modela: (i) difuzije u slučajnoj poroznoj sredini, (ii) nenstacionarno tečenje Stokes-ovog fluida kroz slučajnu linearno elastičnu poroznu sredinu. Drugo, metoda višeskalne konvergencije razvijena od Allaire-a i Briane-a [7] za slučaj više mikroperiodičnih skala je ovde proširen na slučajni raspored heterogenosti okarakterisan razdvojenim skalama (stohastička homogenizacija sa reiteracijom).