THEORETICAL AND APPLIED MECHANICS
vol. 28-29, pp. 27-37, Belgrade 2002
Issues dedicated to memory of the Professor Rastko Stojanović

On a fractional derivative type of a viscoelastic body

T. M. Atanackovic B. N. Novakovic *

Abstract

We study a viscoelastic body, in a linear stress state with fractional derivative type of dissipation. The model was formulated in [1]. Here we derive restrictions on the model that follow from Clausius-Duhem inequality. Several known constitutive equations are derived as special cases of our model. Two examples are discussed.

1 A model of a visco-elastic body

We shall analyze a model of a viscoelastic body proposed in [1]. We first recall few results from [1]. Suppose that the stress $\sigma(t)$ and its derivatives of real (not necessarily integer) order at time instant t in a linear stress state depend on a strain $\varepsilon(t)$ and its derivatives of real order. Then, we may write

$$b_0\sigma + b_1\sigma^{(\alpha_1)} + ... + b_M\sigma^{(\alpha_M)} = a_0\varepsilon + a_1\varepsilon^{(\alpha_1)} + ... + a_N\varepsilon^{(\alpha_N)}, \tag{1}$$

where $a_0, ...a_N, b_0, ...b_M$ and $\alpha_1, ...\alpha_N$ are real constants and we used $\varepsilon^{(\alpha)}$ and $\sigma^{(\alpha)}$ to denote the $\alpha - th$ derivative of $\varepsilon(t)$ and $\sigma(t)$, respectively defined as (see [2])

$$\frac{d^\alpha}{dt^\alpha}\varepsilon(t) = \varepsilon^{(\alpha)} = \frac{d}{dt}\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\varepsilon(\xi)\,d\xi}{(t-\xi)^\alpha}, \tag{2}$$

*Faculty of Technical Sciences, University of Novi Sad, 21000 Novi Sad, Yugoslavia (e-mail: atanackovic@uns.ns.ac.yu)

where Γ is the Euler gamma function. For the case when $b_0 = 1, b_i = 0, i = 1, ...M, N = 1, \alpha_1 = 1$ we obtain Kelvin-Voigt model of viscoelastic body. Recently a distributed order differential equations are introduced, as a generalization of (1) (see [3], [4], [5] and references given there) that lead to the following type of $\sigma - \varepsilon$ relation proposed in ([1])

$$\int_0^1 \phi_\sigma(\gamma)\,\sigma^{(\gamma)} d\gamma = \int_0^1 \phi_\varepsilon(\gamma)\,\varepsilon^{(\gamma)} d\gamma, \tag{3}$$

where $\phi_\sigma(\gamma)$ and $\phi_\varepsilon(\gamma)$ are constitutive functions. To clarify notation in (3) we express (3) in another form. Let

$$F_\sigma(\gamma, t) \equiv \sigma^{(\gamma)} = \frac{d}{dt}\frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{\sigma(\tau)\,d\tau}{(t-\tau)^\gamma}, \tag{4}$$

$$F_\varepsilon(\gamma, t) \equiv \varepsilon^{(\gamma)} = \frac{d}{dt}\frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{\varepsilon(\tau)\,d\tau}{(t-\tau)^\gamma}, \tag{5}$$

and assume that the functions $F_\sigma(\gamma, t), F_\varepsilon(\gamma, t)$ are integrable with respect to γ for any $t \in [0, \infty)$ on the interval $[0, 1]$.Then (3) may be written as

$$\int_0^1 \phi_\sigma(\gamma, t)\, F_\sigma d\gamma = \int_0^1 \phi_\varepsilon(\gamma, t)\, F_\varepsilon d\gamma. \tag{6}$$

Note that in the special case when $\phi_\sigma(\gamma, t) = \delta(\gamma)$, where δ is the Dirac distribution, (6) becomes

$$\sigma(t) = \int_0^1 \phi_\varepsilon(\alpha)\,\varepsilon^{(\alpha)}(t)\,dt, \tag{7}$$

with $\phi_\varepsilon(\alpha)$ given. Of course the stress σ could depend on integer order derivatives, so that equation (7) may read

$$\sigma(t) = \sum_{i=0}^{m} a_i \varepsilon^{(i)} + \int_0^1 \phi(\alpha)\,\varepsilon^{(\alpha)}(t)\,dt, \tag{8}$$

where $m \geq 1$. Models (7) and (8) we shall use in the example that we present later.

The functions $\phi_\sigma(\gamma, t)$ and $\phi_\varepsilon(\gamma, t)$ in (3) characterize the material under consideration and must satisfy restrictions that follow from the entropy inequality. We proceed now to derive those restrictions.

2 The restriction following from the second law of thermodynamics

Constitutive equations must satisfy the restrictions that follow from the Clausisus-Duhem inequality. For fractional derivative type of constitutive equations, such an analysis is presented in [7] and [8], for example. The Second law of thermodynamics requires that the tangent of the mechanical loss angle is non-negative. We apply this criteria to the model (3)

$$\int_0^1 \phi_\sigma(\gamma)\,\sigma^{(\gamma)}d\gamma = \int_0^1 \phi_\varepsilon(\gamma)\,\varepsilon^{(\gamma)}d\gamma. \tag{9}$$

By applying Fourier transform to (9) we obtain

$$\hat{\sigma}(i\omega)\int_0^1 \phi_\sigma(\gamma)\,(i\omega)^\gamma\,d\gamma = \hat{\varepsilon}(i\omega)\int_0^1 \phi_\varepsilon(\gamma)\,(i\omega)^\gamma\,d\gamma, \tag{10}$$

where $\hat{\sigma}(i\omega) = \mathcal{F}(\sigma) = \int_{-\infty}^{\infty}\sigma(t)\,e^{-i\omega t}dt$ is the Fourier transform of $\sigma(t)$. From (10) we obtain the complex modulus as

$$E^*(i\omega) = \frac{\int_0^1 \phi_\varepsilon(\gamma)\,(i\omega)^\gamma\,d\gamma}{\int_0^1 \phi_\sigma(\gamma)\,(i\omega)^\gamma\,d\gamma}. \tag{11}$$

Writing $E^*(i\omega) = E' + iE"$ and using $(i\omega)^\gamma = \omega^\gamma\left(\cos\frac{\pi}{2}\gamma + i\sin\frac{\pi}{2}\gamma\right)$ in (11) we obtain

$$E' = \frac{C_1C_2 + S_1S_2}{(C_2)^2 + (S_2)^2}; \quad E" = \frac{S_1C_2 - C_1S_2}{(C_2)^2 + (S_2)^2}, \tag{12}$$

where

$$C_1 = \int_0^1 \phi_\varepsilon(\gamma)\,\omega^\gamma\cos\frac{\pi}{2}\gamma d\gamma; \quad C_2 = \int_0^1 \phi_\sigma(\gamma)\,\omega^\gamma\cos\frac{\pi}{2}\gamma d\gamma; \tag{13}$$

$$S_1 = \int_0^1 \phi_\varepsilon(\gamma)\,\omega^\gamma\sin\frac{\pi}{2}\gamma d\gamma; \quad S_2 = \int_0^1 \phi_\sigma(\gamma)\,\omega^\gamma\sin\frac{\pi}{2}\gamma d\gamma. \tag{14}$$

Therefore the tangent of the mechanical loss angle is

$$\tan\delta = \frac{E"}{E'} = \frac{S_1C_2 - C_1S_2}{C_1C_2 + S_1S_2} \tag{15}$$

The second law of thermodynamics requires that

$$\tan\delta \geq 0. \tag{16}$$

(see [6]) and that *both* E' and E" are positive for all values of ω (see [7]p.140). Thus we must have

$$C_1C_2 + S_1S_2 \geq 0; \quad S_1C_2 - C_1S_2 \geq 0; \quad \textit{for all } 0 \leq \omega \leq \infty. \tag{17}$$

We consider several special cases of the restrictions imposed by (17).

1. Suppose that

$$\phi_\sigma = \delta(\gamma) + a\delta(\gamma - \alpha)\,; \quad \phi_\varepsilon = \delta(\gamma) + b\delta(\gamma - \alpha)\,, \tag{18}$$

where a, b and $0 < \alpha < 1$ are constants. This choice correspond to the generalized Zener model $\sigma + a\sigma^{(\alpha)} = \varepsilon + b\varepsilon^{(\alpha)}$. By substituting (18) into (16) we obtain

$$\begin{gathered}
\left(b\omega^\alpha \sin\tfrac{\pi}{2}\alpha\right)\left(1 + a\omega^\alpha \cos\tfrac{\pi}{2}\alpha\right) - \\
\left(1 + b\omega^\alpha \cos\tfrac{\pi}{2}\alpha\right)\left(a\omega^\alpha \sin\tfrac{\pi}{2}\alpha\right) \geq 0; \\
\left(1 + b\omega^\alpha \cos\tfrac{\pi}{2}\alpha\right)\left(1 + a\omega^\alpha \cos\tfrac{\pi}{2}\alpha\right) + \\
\left(b\omega^\alpha \sin\tfrac{\pi}{2}\alpha\right)\left(a\omega^\alpha \sin\tfrac{\pi}{2}\alpha\right) \geq 0,
\end{gathered} \tag{19}$$

or

$$\begin{gathered}
b\omega^\alpha \sin\tfrac{\pi}{2}\alpha - a\omega^\alpha \sin\tfrac{\pi}{2}\alpha \geq 0; \\
\left(1 + b\omega^\alpha \cos\tfrac{\pi}{2}\alpha\right)\left(1 + a\omega^\alpha \cos\tfrac{\pi}{2}\alpha\right) + \\
\left(b\omega^\alpha \sin\tfrac{\pi}{2}\alpha\right)\left(a\omega^\alpha \sin\tfrac{\pi}{2}\alpha\right) \geq 0.
\end{gathered} \tag{20}$$

From (20) it follows that

$$b > a > 0, \tag{21}$$

a well known result (see [7]).

2. Suppose that

$$\phi_\varepsilon(\gamma) = c\phi_\sigma(\gamma)\,, \quad c > 0. \tag{22}$$

Then $\tan\delta = 0$ and we conclude that the body behaves as an elastic body.

3. Let us assume that

$$\phi_\sigma = \delta(\gamma); \quad \phi_\varepsilon = E(\tau_0)^\gamma, \tag{23}$$

where $E = const.$ and $\tau_0 = const.$ are known constants. Thus, the constitutive equation reads $\sigma = E\int_0^1 (\tau_0)^\gamma \varepsilon^{(\gamma)} d\gamma$ which is of the type (7). The condition (17) becomes

$$E\int_0^1 (\tau_0\omega)^\gamma \sin\frac{\pi}{2}\gamma d\gamma \geq 0; \quad E\int_0^1 (\tau_0\omega)^\gamma \cos\frac{\pi}{2}\gamma d\gamma \geq 0, \tag{24}$$

or

$$E > 0; \quad \tau_0 > 0. \tag{25}$$

4. Next we suppose that

$$\phi_\varepsilon = \delta(\gamma); \quad \phi_\sigma = C(\tau_1)^\gamma, \tag{26}$$

where C and τ_1 are constants. The constitutive equation corresponding to (26) reads

$$C\int_0^1 (\tau_1)^\gamma \sigma^{(\gamma)} d\gamma = \varepsilon. \tag{27}$$

The constants (14) are

$$C_1 = 1; \quad C_2 = C\int_0^1 (\tau_1\omega)^\gamma \cos\frac{\pi}{2}\gamma d\gamma; \tag{28}$$

$$S_1 = 0; \quad S_2 = C\int_0^1 (\tau_1\omega)^\gamma \sin\frac{\pi}{2}\gamma d\gamma, \tag{29}$$

so that (17) leads to

$$-C\int_0^1 (\tau_1\omega)^\gamma \sin\frac{\pi}{2}\gamma d\gamma \geq 0; \quad C\int_0^1 (\tau_1\omega)^\gamma \cos\frac{\pi}{2}\gamma d\gamma \geq 0. \tag{30}$$

From (30) we conclude that (30) violates the second law of thermodynamics for any value of $C \neq 0$.

In conclusion we write the conditions (17) in expanded form by using definitions of $S_1, ...C_2$, so that

$$\begin{aligned} &\left(\int_0^1 \phi_\varepsilon(\gamma)\,\omega^\gamma \cos\tfrac{\pi}{2}\gamma d\gamma\right)\left(\int_0^1 \phi_\sigma(\gamma)\,\omega^\gamma \cos\tfrac{\pi}{2}\gamma d\gamma\right) \\ &+\left(\int_0^1 \phi_\varepsilon(\gamma)\,\omega^\gamma \sin\tfrac{\pi}{2}\gamma d\gamma\right)\left(\int_0^1 \phi_\sigma(\gamma)\,\omega^\gamma \sin\tfrac{\pi}{2}\gamma d\gamma\right) \geq 0; \\ &\text{for all } 0 \leq \omega \leq \infty \end{aligned} \tag{31}$$

$$\begin{aligned} &\left(\int_0^1 \phi_\varepsilon(\gamma)\,\omega^\gamma \sin\tfrac{\pi}{2}\gamma d\gamma\right)\left(\int_0^1 \phi_\sigma(\gamma)\,\omega^\gamma \cos\tfrac{\pi}{2}\gamma d\gamma\right) \\ &-\left(\int_0^1 \phi_\varepsilon(\gamma)\,\omega^\gamma \cos\tfrac{\pi}{2}\gamma d\gamma\right)\left(\int_0^1 \phi_\sigma(\gamma)\,\omega^\gamma \sin\tfrac{\pi}{2}\gamma d\gamma\right) \geq 0. \\ &\text{for all } 0 \leq \omega \leq \infty. \end{aligned} \tag{32}$$

3 Examples

We use the constitutive equation (3) to formulate two concrete problems.

I. Consider a mechanical system shown in Fig. 1. The system consists of a body of mass m that moves translatory and is connected to one end of a viscoelastic rod. The other end of the rod is fixed to unmovable wall. Suppose that the initial (undeformed) length of the rod is $l_0 = 1$. In the deformed state the length is given as $l(t) = l_0 + y(t)$ where $y(t)$ is the change of the length. Therefore the strain is $\varepsilon = y/l_0$.

Suppose that the rod is made of a material described by (7). The equation of motion reads

$$my^{(2)}(t) + \frac{1}{l_0}\int_0^1 \phi(\alpha)\, y^{(\alpha)}(t)\, d\alpha = 0, \tag{33}$$

For $\phi(\alpha)$ we take (see [1])

$$\phi(\alpha) = E(\tau_\varepsilon)^\alpha \tag{34}$$

where $E > 0, \tau_\varepsilon > 0$ (see (25)) are constants.

By applying Laplace transform $\mathcal{L}(f) = \int_0^\infty e^{itz} f(t)\, dt = \bar{f}(z)$ to (33) we obtain (with $m = 1, l_0 = 1$)

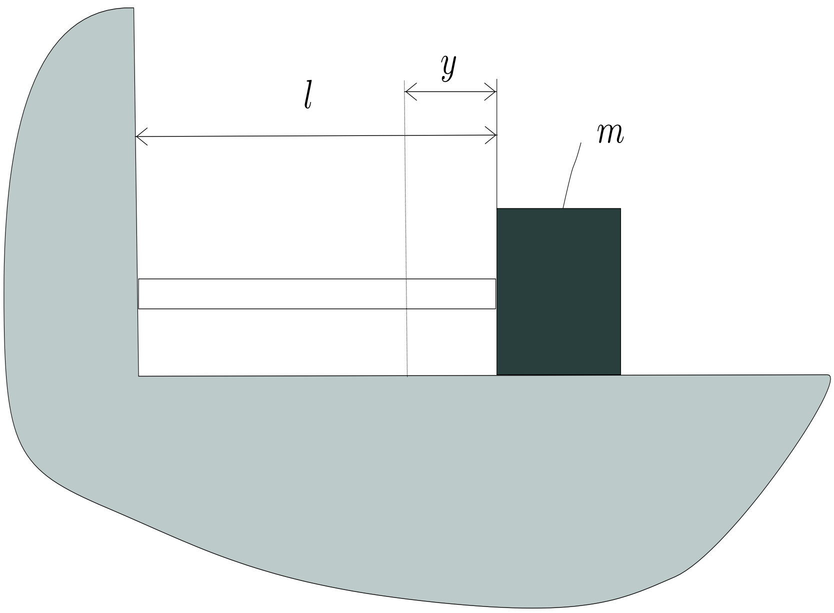

Figure 1: Mechanical system corresponding to (28)

$$
\begin{aligned}
&z^2\bar{y}\,(z) = \\
&\qquad \bar{h}\,(z) - E\int_0^1 (\tau_\varepsilon)^\alpha\, z^\alpha \bar{y}\,(z)\, d\alpha + \\
&\qquad + y^{(1)}\,(0) + zy\,(0) \\
&\qquad + \int_0^1 \left[E\,(\tau_\varepsilon)^\alpha \left(\tfrac{1}{\Gamma(1-\alpha)} \int_0^t \tfrac{y(\tau)d\tau}{(t-\tau)^\alpha} \right)_{t=0} \right] d\alpha,
\end{aligned} \tag{35}
$$

where we used the fact that

$$\mathcal{L}[y^{(\alpha)}] = p^\alpha \bar{f}\,(p) - \left(\frac{1}{\Gamma\,(1-\alpha)} \int_0^t \frac{y\,(\tau)\,d\tau}{(t-\tau)^\alpha} \right)_{t=0}. \tag{36}$$

The last term in (35) vanishes if $y\,(t)$ is bounded for $t \to +0$, so that from (35) we obtain

$$\left[z^2 + E\frac{\tau_\varepsilon z - 1}{\ln\,(\tau_\varepsilon z)} \right] \bar{y}\,(z) = y^{(1)}\,(0) + zy\,(0)\,, \tag{37}$$

or

$$\bar{y}(z) = \frac{y^{(1)}(0) + zy(0)}{z^2 + E\frac{\tau_\varepsilon z - 1}{\ln(\tau_\varepsilon z)}} \tag{38}$$

$$= \ln(\tau_\varepsilon z) \frac{y^{(1)}(0) + zy(0)}{z^2 \ln(\tau_\varepsilon z) + E(\tau_\varepsilon z - 1)} \tag{39}$$

$$= \frac{y^{(1)}(0) + zy(0)}{z^2 + \frac{E(\tau_\varepsilon z - 1)}{\ln(\tau_\varepsilon z)}}. \tag{40}$$

If $\lim\limits_{t\to 0} y(t)$ and $\lim\limits_{t\to\infty} y(t)$ exist, then they are given as

$$\lim_{t\to 0} y(t) = \lim_{z\to\infty} z\bar{y}(z) = \lim_{z\to\infty} z \frac{y^{(1)}(0) + zy(0)}{z^2 + \frac{E(\tau_\varepsilon z - 1)}{\ln(\tau_\varepsilon z)}} = y(0), \tag{41}$$

and

$$\lim_{t\to\infty} y(t) = \lim_{z\to 0} z\bar{y}(z) = \lim_{z\to 0} z \frac{y^{(1)}(0) + zy(0)}{z^2 + \frac{E(\tau_\varepsilon z - 1)}{\ln(\tau_\varepsilon z)}} = 0. \tag{42}$$

If the constitutive equation of the viscoelastic rod is given in the form (8),that is

$$\sigma(t) = a_1 \varepsilon^{(1)} + E \int_0^1 (\tau_\varepsilon)^\alpha \varepsilon^{(\alpha)}(t)\, dt, \tag{43}$$

where a_1, E and τ_ε are constants, the equation of motion reads

$$my^{(2)}(t) + \frac{a_1}{l_0} y^{(1)} + \frac{E}{l_0} \int_0^1 (\tau_\varepsilon)^\alpha y^{(\alpha)}(t)\, d\alpha = 0. \tag{44}$$

The asymptotic behavior of (44) could be obtained by the same procedure as in first case.

II. On the basis of constitutive equation (3) one can derive a moment curvature relation for a rod. Such relations may be used for the study of motion and stability of viscoelastic rods (see [9],[10] and [12]). Following the standard procedure (plane cross section hypothesis), as was described in [11] we obtain

$$\int_0^1 \phi_\sigma(\gamma, t)\, M^{(\gamma)} d\gamma = I \int_0^1 \phi_\varepsilon(\gamma, t) \left(\frac{1}{\rho}\right)^{(\gamma)} d\gamma, \tag{45}$$

where I is the moment of inertia of the rod's cross-section A, that is $I = \int_A y^2 dA$, where y is the distance from the neutral axis, and ρ is the radius of curvature of the rod axis.

Note that in the special case of elastic material $\phi_\sigma(\gamma, t) = \delta(\gamma), \phi_\varepsilon(\gamma, t) = E\delta(\gamma)$ equation (45) becomes

$$M = EI\left(\frac{1}{\rho}\right), \tag{46}$$

i.e., the moment curvature relation of classical Bernoulli-Euler rod theory.

For the case $\phi_\sigma = \delta(\gamma) + a\delta(\gamma - \alpha); \quad \phi_\varepsilon = \delta(\gamma) + b\delta(\gamma - \alpha)$ (see (18)) we obtain

$$M + aM^{(\alpha)} = I\left(\frac{1}{\rho}\right) + bI\left(\frac{1}{\rho}\right)^{(\alpha)}. \tag{47}$$

The constitutive equation of the type (47) was used in [9], [10] and [11].

If we choose $\phi_\sigma = \delta(\gamma), \phi_\varepsilon = E\delta(\gamma) + b\delta(\gamma - \alpha)$ we obtain

$$M = EI\left(\frac{1}{\rho}\right) + bI\left(\frac{1}{\rho}\right)^{(\alpha)}. \tag{48}$$

Equation (48), as a matter of fact its linearized version, was used in [12].

4 Conclusion

In this note we studied the viscoelastic body that in linear, isothermal stress state has constitutive equation of the form (3). The restrictions on the constitutive functions ϕ_σ and ϕ_ε following from the second law of thermodynamics, are given by (32). We showed that several known constitutive aerations and corresponding restrictions on coefficients follow from (3),(32). An example is formulated and asymptotic behavioral of the solution is examined. Also the moment curvature relation for the rod made of material described by (3) is derived in the form (45).

Acknowledgements

This research was supported by Serbian Ministry of Science and Technologies Grant No 1402.

References

[1] Atanackovic, T. M., 2002, A model for the uniaxial isothermal deformation of a Viscoelastic body. Acta Mechanica (in press).

[2] Samko, S. G., Kilbas, A. A., and Marichev, O. I., 1993, *Fractional integrals and derivatives*, Gordon and Breach, Amsterdam.

[3] Bagley, R. L., and Torvik, P. J., 2000, On the existence of the order domain and the solution of Distributed order equations-part I. *Int. Journal of Appl. Mathematics,* **2**, 865-882.

[4] Bagley, R. L., and Torvik, P. J., 2000, On the existence of the order domain and the solution of Distributed order equations-part II. *Int. Journal of Appl. Mathematics,* **2**, 965-987.

[5] Caputo, M., 2001, Distributed order differential equations modelling dielectric induction and diffusion. *Fractional Calculus&Applied Analysis,* **4**, 421-442.

[6] Rossikhin, Yu., and Shitikova, M. V., 2001, Analysis of Dynamic Behavior of Viscoelastic rods Whose Rheological Models Contain Fractional Derivatives of Two Different Orders. *ZAMM*, **81**, 363-376.

[7] Bagley, R. L., and Torvik, P. J., 1986, On the Fractional Calculus Model of Viscoelastic Behavior. Journal of Rheology*,* **30**, 133-155.

[8] Atanackovic, T. M., 2002 , A modified Zener model of a viscoelastic body. *Continuum Mech. Thermodyn.,* **14**, 137-148.

[9] Stankovic, B., and Atanackovic, T. M., 2001, On a model of a viscoelastic rod. *Fractional Calculus and Appl. Analysis*, **4**, 501-522.

[10] Stankovic, B., and Atanackovic, T. M., 2002, Dynamics of a Rod made of Generalized Kelvin-Voigt Visco-Elastic Material. *Journal of Math. Analysis and Appl.,* **268**, 55–563

[11] Atanackovic, T. M., and Stankovic, B., 2002, Dynamics of a Viscoelastic Rod of Fractional Derivative Type. *Z. Angew. Math. Mech.(ZAMM)*, **82**, 377-386.

[12] Li Gen-guo, Zhu Zheng-you, and Cheng Chang-jun, 2001, Dynamical Stability of viscoelastic column with fractional Derivative constitutive relation. *Applied Mathematics and Mechanics,* **22**, 294-303.

O jednom tipu frakcionog izvoda za neko viskoelastično telo UDK 531.01, 539.374

Posmatra se neko viskoelastično telo, u linearnom naponskom stanju sa disipacijom tipa frakciog izvoda. Ovaj model je formulisan u radu [1]. Ovde izvodimo ograničenja koja na model propisuje Clausius-Duhemova nejednakost. Nekoliko poznatih konstitutivnih jednačina se izvode kao specijalni slučajevi našeg modela. Diskutujemo takodje i dva primera.