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Contact interactions, stress, and material symmetry, for nonsimple elastic materials

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Abstract

An argument is given intimating that, for nonsimple materials, the concepts of contact interaction and material symmetry, as well as the bridging concept of stress, should be carefully revised and generalized.

1 Introduction

The purpose of this paper is to give an argument intimating that, for *nonsimple* materials, the concepts of contact interaction and material symmetry should be carefully revised and generalized, as well as the concept of stress, which bridges between the first two.

After the pioneering work of Toupin [1, 2] (summarized in Section 98 of [3]), the subject has received scarce attention until very recently [4, 5, 6, 7, 8], with only a few important earlier contributions, by Forte & Vianello [9] and Noll & Virga [10] on the foundational side and by Mindlin [11] and Wu [12] on the applicative side. Other relevant work, which may go under the general heading of "asymmetric elasticity" [13], is [14, 15]; see also the books by Grioli [16] and Stojanovic [17].

To set the stage, we begin by recalling a few facts, most of which are well-known (Section 2). Using elastic solids as an example, we point

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out that, for simple materials, the notion of material symmetry can be made precise in terms of the stress vector, that is to say, in terms of the contact interaction between body parts; and that the Stress Theorem allows us to use, alternatively, the stress constitutive relation. Crucial for the alternative to hold is the nature of the contact interaction. For simple material bodies, such interaction consists in one vector measure being absolutely continuous with respect to area. There are however continuum theories of nonsimple materials, such as second- and third-*gradient theories* of elasticity, where not only additional area-bounded contact interactions but also edge and vertex contact interactions are in order. Two questions arise: (i) whether a Cauchy-like construction of "stresses" associated to these nonstandard contact interactions is possible; (ii) what concept of material symmetry is appropriate. At this moment, we are unable to answer these questions in general. But, within the easier variational format of gradient hyperelasticity (Section 3), we can at least indicate what issues we are confronted with by studying the equilibrium of a cubic body part (Section 4; for convenience, we collect in a final Appendix a number of preparatory developments). Finally, in Section 5, we briefly discuss how our findings relate to the cited foundational [8, 9, 10] and applicative [11, 12] papers, and to the work of Fosdick & Virga [18].

2 The response symmetry of simple materials

Contact interactions are typical of continuum physics. Both a body and its environment and two body parts are presumed to have distance and contact interactions, which is standard to assume partwise balanced whatever the deformation f of a given reference shape. In the classical case of Cauchy continua, the contact interaction is represented by the stress vector, a field $\boldsymbol{s}(\cdot,\cdot;f)$ that, when evaluated at a point p of the common boundary surface oriented by the unit normal $\boldsymbol{n}$, delivers the force $\boldsymbol{s}(p,\boldsymbol{n};f)$ per unit area exerted by the environment over the body or by the part lying on the positive side of the boundary surface over the adjacent part. By virtue of the Cauchy's Stress Theorem, there is a tensor field $\boldsymbol{S}(\cdot; f)$, whose linear action over the normal gives the stress vector:

$$\boldsymbol{s}(p, \boldsymbol{n}; f) = \boldsymbol{S}(p; f)\boldsymbol{n}. \tag{2.1}$$

In general, the material response of a certain class of Cauchy continua is specified by the assignment of a constitutive relation between the history ${}^t\boldsymbol{F}$ up to time t of the deformation gradient

$$\boldsymbol{F} = \nabla f \tag{2.2}$$

and the stress $\boldsymbol{S}$; these material bodies are collectively called *simple*. For the *elastic* class, only the present value of $\boldsymbol{F}$ matters, so that, at an interior point p of the chosen reference shape, the constitutive relation can be written as

$$\boldsymbol{S} = \widehat{\boldsymbol{S}}(p, \boldsymbol{F}). \tag{2.3}$$

A material symmetry transformation of the reference shape is a rigid rotation of gradient $\boldsymbol{R}$ that cannot be detected by any further straining. To make this concept precise, we can choose either the stress vector or the stress tensor as the relevant feature of the material's response to strain. The customary choice is the latter: one says that $\boldsymbol{R}$ determines a material symmetry transformation at p if

$$\widehat{\boldsymbol{S}}(p, \boldsymbol{F}\boldsymbol{R}) = \widehat{\boldsymbol{S}}(p, \boldsymbol{F})\boldsymbol{R} \quad \text{for all invertible } \boldsymbol{F}, \tag{2.4}$$

or rather, in view of the frame-indifference requirement,[1] if

$$\widehat{\boldsymbol{S}}(p, \boldsymbol{R}\boldsymbol{F}\boldsymbol{R}^T) = \boldsymbol{R}\widehat{\boldsymbol{S}}(p, \boldsymbol{F})\boldsymbol{R}^T \quad \text{for all invertible } \boldsymbol{F}. \tag{2.5}$$

The former choice is less familiar, so we pause to describe it briefly (see [19], Section 14, and [20]). For q a point in a neighborhood of p, we denote by

$$q^+ = r(q) = p + \boldsymbol{R}(q - p) \quad \text{and} \quad \boldsymbol{n}^+ = \boldsymbol{R}\boldsymbol{n}, \tag{2.6}$$

[1] Since all constitutive prescriptions must be frame indifferent, the mapping $\widehat{\boldsymbol{S}}(p, \cdot)$ has to satisfy

$$\widehat{\boldsymbol{S}}(p, \boldsymbol{Q}\boldsymbol{F}) = \boldsymbol{Q}\widehat{\boldsymbol{S}}(p, \boldsymbol{F}) \quad \text{for all orthogonal } \boldsymbol{Q} \text{ and invertible } \boldsymbol{F}.$$

respectively, the point q^+ where a point q in a neighborhood of p is taken by the rotation r and the normal $\boldsymbol{n}^+$ to a material plane of normal $\boldsymbol{n}$ after the same rotation. For $\boldsymbol{u}(q) = f(q) - q$ the displacement field associated to a deformation f, we define

$$\boldsymbol{u}^+(q^+) = \boldsymbol{R}\boldsymbol{u}(q), \quad f^+(q^+) = q^+ + \boldsymbol{u}^+(q^+), \tag{2.7}$$

and note that (2.7) imply that $f^+(q^+) = p + \boldsymbol{R}(f(q) - p)$, or rather that

$$f^+ = r \circ f \circ r^{-1}; \tag{2.8}$$

by the chain rule, (2.2) and (2.8) yield

$$\boldsymbol{F}^+ = \boldsymbol{R}\boldsymbol{F}\boldsymbol{R}^T. \tag{2.9}$$

With these definitions, we model a situation where the experiment inducing the displacement field $\boldsymbol{u}$ in a neighborhood of p in the chosen reference configuration, when performed after the rotation r of the reference configuration about p, induces the same displacement field in the rotated neighborhood. In such a situation, we stipulate to classify r as a material symmetry transformation if

$$\boldsymbol{s}^+(p, \boldsymbol{n}^+; f^+) = \boldsymbol{R}\, \boldsymbol{s}(p, \boldsymbol{n}; f) \quad \text{for all } \boldsymbol{n}, f. \tag{2.10}$$

With the use of (2.1), it is easy to check that $\boldsymbol{R}$ satisfies (2.10) if and only if it satisfies (2.5).

3 Gradient hyperelasticity

A *hyperelastic* class of simple Cauchy materials is specified by choosing a *stored-energy mapping* $\widehat{\sigma}(p, \cdot)$, with $\widehat{\sigma}(p, \boldsymbol{F})$ interpreted as the elastic energy per unit referential volume that is stored at point p due to a deformation of gradient $\boldsymbol{F}$. In this case, the stress concept is *derived*, not primary, and such is the stress constitutive relation:

$$\widehat{\boldsymbol{S}}(\boldsymbol{F}) = \partial_{\boldsymbol{F}}\widehat{\sigma}(\boldsymbol{F}).^2 \tag{3.1}$$

[2] From now on, we leave the dependence on p tacit.

Likewise, the contact interaction between two adjacent body parts with common boundary of normal $\boldsymbol{n}$ is *defined* in terms of the stress:

$$\boldsymbol{s}(\boldsymbol{n}; f) := \widehat{\boldsymbol{S}}(\nabla f)\boldsymbol{n}. \tag{3.2}$$

Material symmetry is expressed by a requirement involving the one basic constitutive object, the stored energy: we say that $\boldsymbol{R}$ determines a material symmetry transformation at p if

$$\widehat{\sigma}(\boldsymbol{F}\boldsymbol{R}) = \widehat{\sigma}(\boldsymbol{F}) \quad \text{for all invertible } \boldsymbol{F}; \tag{3.3}$$

a frame-indifferent version of this notion is

$$\widehat{\sigma}(\boldsymbol{F}^{+}) = \widehat{\sigma}(\boldsymbol{F}) \quad \text{for all invertible } \boldsymbol{F}.^{3} \tag{3.4}$$

We now explore the case of nonsimple hyperelastic materials. We let the stored energy mapping depend on deformation gradients of order higher than the first:

$$\widehat{\sigma}({}_1\boldsymbol{F}, {}_2\boldsymbol{F}, {}_3\boldsymbol{F}), \quad {}_1\boldsymbol{F} \equiv \boldsymbol{F}, \;\; {}_2\boldsymbol{F} = \nabla {}_1\boldsymbol{F}, \;\; {}_3\boldsymbol{F} = \nabla {}_2\boldsymbol{F},$$

and we denote by ${}_i\boldsymbol{S}$ the partial derivative of $\widehat{\sigma}$ with respect to ${}_i\boldsymbol{F}$:

$${}_i\boldsymbol{S} = \partial_{{}_i\boldsymbol{F}}\,\widehat{\sigma} \quad (i = 1, 2, 3), \tag{3.5}$$

that is to say, the stress-like construct which is work-conjugate to the deformation measure ${}_i\boldsymbol{F}$. For these materials, a rotation of gradient $\boldsymbol{R}$ is a material symmetry transformation if

$$\begin{aligned} \widehat{\sigma}({}_1\boldsymbol{F}^{+}, {}_2\boldsymbol{F}^{+}, {}_3\boldsymbol{F}^{+}) &= \widehat{\sigma}({}_1\boldsymbol{F}, {}_2\boldsymbol{F}, {}_3\boldsymbol{F}), \\ ({}_2\boldsymbol{F}^{+}\boldsymbol{a}^{+})\boldsymbol{b}^{+} &= \boldsymbol{R}({}_2\boldsymbol{F}\boldsymbol{a})\boldsymbol{b}, \\ (({}_3\boldsymbol{F}^{+}\boldsymbol{a}^{+})\boldsymbol{b}^{+})\boldsymbol{c}^{+} &= \boldsymbol{R}(({}_3\boldsymbol{F}\boldsymbol{a})\boldsymbol{b})\boldsymbol{c}, \\ \boldsymbol{a}^{+} &= \boldsymbol{R}\boldsymbol{a}, \text{ etc.,} \end{aligned} \tag{3.6}$$

[3] The stored-energy mapping is frame indifferent if it satisfies

$$\widehat{\sigma}(\boldsymbol{Q}\boldsymbol{F}) = \widehat{\sigma}(\boldsymbol{F}) \quad \text{for all orthogonal } \boldsymbol{Q} \text{ and invertible } \boldsymbol{F}$$

(cf. footnote 1).

for all vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, all invertible second-order tensors ${}_1\boldsymbol{F}$, and all third- and fourth-order tensors ${}_2\boldsymbol{F}$ and ${}_3\boldsymbol{F}$ having the symmetries specified in Remark 1 below.

We anticipate that it is the dependence of the stored energy on the second gradient that allows for *edge* contact interactions, interpreted as forces per unit length, while the dependence on the third makes *vertex* contact interactions possible, under form of concentrated forces. Moreover, a dependence on deformation gradients higher than the first induces surface interactions additional to the forces per unit area which are typical of simple materials: first-order *couples* per unit area are induced by a dependence on the second gradient, (first- and) second-order couples by a dependence on the third.

Remarks

1. In view of their definitions, the stress constructs ${}_2\boldsymbol{S}$ and ${}_3\boldsymbol{S}$ have the same symmetries as, respectively, ${}_2\boldsymbol{F}$ and ${}_3\boldsymbol{F}$:
(i) $({}_2\boldsymbol{S}\boldsymbol{a})\boldsymbol{b} = ({}_2\boldsymbol{S}\boldsymbol{b})\boldsymbol{a}$ for all vectors $\boldsymbol{a}, \boldsymbol{b}$;
(ii) the vector $(({}_3\boldsymbol{S}\boldsymbol{a})\boldsymbol{b})\boldsymbol{c}$ is the same whatever the permutation, possibly with repetitions, of the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$.
2. As to physical dimensions, we have that

$$\dim({}_i\boldsymbol{S}) = \text{force} \times \text{length}^{i-3};$$

edge interactions and first-order couples have the dimensions of ${}_2\boldsymbol{S}$, vertex interactions and second-order couples have those of ${}_3\boldsymbol{S}$.

4 Equilibrium of a hyperelastic cube

To see what types of contact interactions between body parts or with the environment are to be expected in the case of gradient hyperelasticity, we let Ω be a cubic body part and insist that

$$\int_\Omega \widehat{\sigma}(\nabla f, \nabla^{(2)} f, \nabla^{(3)} f) = \text{extr}\,. \tag{4.1}$$

Then, by repeated use of the standard divergence theorem, of more or less well-known decompositions of the first and second gradient tensors into tangential and normal parts, and of the surface divergence theorem and its consequences (see the Appendix), the extremum condition (4.1) yields the stationarity condition

$$0 = \int_{\Omega} \boldsymbol{d} \cdot \boldsymbol{v} + \int_{\partial\Omega^*} ({}_0\boldsymbol{s} \cdot \boldsymbol{v} + {}_1\boldsymbol{s} \cdot \partial_{\boldsymbol{n}} \boldsymbol{v} + {}_2\boldsymbol{s} \cdot \partial_{\boldsymbol{n}}^{(2)} \boldsymbol{v}) \qquad (4.2)$$
$$+ \sum_{i=1}^{12} \int_{\mathcal{E}_i} (\boldsymbol{f}_i^E \cdot \boldsymbol{v} + \boldsymbol{T}_i^E \cdot {}^{\boldsymbol{t}^\perp}\nabla \boldsymbol{v}) + \sum_{i=1}^{8} \boldsymbol{f}_i^V \cdot \boldsymbol{v}(V_i),$$

where $\boldsymbol{d}$, the internal *distance interaction*, has the following representation in terms of the stress-like constructs ${}_i\boldsymbol{S}$:

$$\boldsymbol{d} = -\operatorname{Div} {}_1\widetilde{\boldsymbol{S}}, \qquad (4.3)$$

with

$${}_1\widetilde{\boldsymbol{S}} := {}_1\boldsymbol{S} - \operatorname{Div} {}_2\widetilde{\boldsymbol{S}}, \quad {}_2\widetilde{\boldsymbol{S}} := {}_2\boldsymbol{S} - \operatorname{Div} {}_3\boldsymbol{S}; \qquad (4.4)$$

and where a variety of internal *contact interactions* occurs, on the smooth part of $\partial\Omega$ (denoted by $\partial\Omega^*$), the edges, and the vertices (here $\mathcal{E}_i$ is the $i-$th edge, and V_i the $i-$th vertex). We see that the contact interaction of the cubic part Ω with the rest of the body has manifold manifestations, which change according to the type of boundary point: on a face, we have a *force* ${}_0\boldsymbol{s}$, a *first-order couple* ${}_1\boldsymbol{s}$, and a *second-order couple* ${}_2\boldsymbol{s}$, all *per unit area*; on an edge, a *force* $\boldsymbol{f}^E$ and a *couple* $\boldsymbol{T}^E$,[4] both *per unit length*; on a vertex, a *force* $\boldsymbol{f}^V$.

We now specify the representations these contact interactions have in terms of the stress-like constructs ${}_i\boldsymbol{S}$, building on the preparatory developments collected into the Appendix. As to the generality of our results, we emphasize that, while the representation (4.3)-(4.4) of distance interaction is shape invariant, the representations of contact interactions are for general body shapes more complicated than those given below for a cube, whose faces are flat and edges straight, and whose vertices top right trihedra.

[4]At the variance with the other manifestations of contact interactions, all vectorial in nature, $\boldsymbol{T}^E$ is tensorial, and is work-conjugate to the variation gradient taken *perpendicularly to the edge*, namely,

$${}^{\boldsymbol{t}^\perp}\nabla \boldsymbol{v} := (\nabla \boldsymbol{v})(\boldsymbol{1} - \boldsymbol{t} \otimes \boldsymbol{t}),$$

where $\boldsymbol{t}$ is the tangent to the edge.

We let $\partial\Omega_i$ denote the face of the cube Ω having normal $\boldsymbol{n}_i$, and choose: as typical edge of Ω, the segment $\mathcal{E}$ that the contours of faces $\partial\Omega_1$ and $\partial\Omega_2$ have in common; as typical vertex, the triple-junction point V where the contours of faces $\partial\Omega_1$, $\partial\Omega_2$ and $\partial\Omega_3$ meet. We find that
- at a point of $\partial\Omega^*$, the contact interaction consists in three vector fields, having the representations

$$\begin{aligned} {}_0\boldsymbol{s} &= {}_1\widetilde{\boldsymbol{S}}\boldsymbol{n} - {}^s\mathrm{Div}\,({}_2\widetilde{\boldsymbol{S}}\boldsymbol{n}) + {}^s\mathrm{Div}\,({}^s\mathrm{Div}\,{}_3\boldsymbol{S}\boldsymbol{n}), \\ {}_1\boldsymbol{s} &= ({}_2\widetilde{\boldsymbol{S}}\boldsymbol{n})\boldsymbol{n} - 2\,{}^s\mathrm{Div}\,(({}_3\boldsymbol{S}\boldsymbol{n})\boldsymbol{n}), \\ {}_2\boldsymbol{s} &= (({}_3\boldsymbol{S}\boldsymbol{n})\boldsymbol{n})\boldsymbol{n}; \end{aligned} \tag{4.5}$$

- at a point of the edge $\mathcal{E}$, the contact interaction consists in a vector field and a tensor field, having the representations

$$\begin{aligned} \boldsymbol{f}^E &= [\![({}_2\widetilde{\boldsymbol{S}}\boldsymbol{n}_1)\boldsymbol{n}_2 - ({}^s\mathrm{Div}\,{}_3\boldsymbol{S}\boldsymbol{n}_1)\boldsymbol{n}_2 + ((({}_3\boldsymbol{S}\boldsymbol{n}_1)\boldsymbol{n}_2)\boldsymbol{n}_3)_{,s}]\!]\,, \\ \boldsymbol{T}^E &= 3\,({}_3\boldsymbol{S}\boldsymbol{n}_1)\boldsymbol{n}_2 \end{aligned} \tag{4.6}$$

(here $[\![\Psi]\!]$ equals twice the edge average of Ψ and $(\cdot)_{,s}$ denotes differentiation with respect to the edge parametrization).
- at the vertex V, the contact interaction consists in the force

$$\boldsymbol{f}^V = 6\,(({}_3\boldsymbol{S}\boldsymbol{n}_1)\boldsymbol{n}_2)\boldsymbol{n}_3\,. \tag{4.7}$$

In the absence of external actions, all these six fields must be null everywhere they are defined; otherwise, they are to balance the relative applied forces and couples.

5 Final remarks

Our present work indicates what diverse types of external actions should enter a general power form for third-gradient material bodies with fairly arbitrary boundary shape. Our picture of internal interactions is consistent with those of Forte & Vianello [9], Noll & Virga [10], and Degiovanni, Marzocchi, & Musesti [8], as far as a comparison is possible (all these works consider only second-gradient materials, for which edge couples and vertex forces do not arise); see also the paper by DiCarlo &

Tatone [6]. As to the work of Mindlin [11] and Wu [12], it is interesting to consider as an example their *cohesive stress*

$$ {}_3\boldsymbol{S}^C = \frac{1}{3}\beta_0(\boldsymbol{1}\otimes\boldsymbol{1}+\boldsymbol{I}+\boldsymbol{T}), \tag{5.1} $$

where $\boldsymbol{I}$ and $\boldsymbol{T}$ are, respectively, the fourth-order identity and transposer.[5] In this case,

$$ ({}_3\boldsymbol{S}^C\boldsymbol{a})\boldsymbol{b} = \frac{1}{3}\beta_0((\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{1}+\boldsymbol{a}\otimes\boldsymbol{b}+\boldsymbol{b}\otimes\boldsymbol{a}) \quad \text{for all vectors } \boldsymbol{a},\ \boldsymbol{b}, \tag{5.2} $$

so that, e.g.,

$$ {}_2\boldsymbol{s} = \beta_0\boldsymbol{n}, \quad \boldsymbol{T}_1^E = \frac{1}{3}\beta_0(\boldsymbol{n}_1\otimes\boldsymbol{n}_2+\boldsymbol{n}_2\otimes\boldsymbol{n}_1), \quad \boldsymbol{f}_1^V = \boldsymbol{0}\,. \tag{5.3} $$

We regard our present work as preliminary to deeper research in various directions. For one, it remains to be seen whether a Cauchy-like, constructive Stress Theorem would be possible for continua being neither simple nor hyperelastic. As recalled in the beginning of Section 2, Cauchy considered only one type of contact interactions, that he assumed to depend only on a first-order geometrical character of the contact surface, the normal; interpreted as forces per unit area; and part-wise balanced together with the distance force. For general nonsimple materials, even if we restrict ourselves to body parts with everywhere smooth boundaries, we have to deal with a number of diverse contact interactions, whose dependence on the geometrical characters – up to the third order – of the surface is quite complex, whose balance principles are to be appropriately spelled out, and whose related stresses must be constructed. In this connection, a research line that might be generalized and extended is the variational approach developed in [18] as an alternative to Cauchy's construction.

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[5] For $\boldsymbol{A}$ a second-order tensor, $\boldsymbol{IA}=\boldsymbol{A}$, $\boldsymbol{TA}=\boldsymbol{A}^T$.

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A Appendix

With the definitions in (4.4), the variation of the functional in (4.1) yields:

$$0 = \int_{\Omega} (-\operatorname{Div}{}_1\widetilde{\boldsymbol{S}}) \cdot \boldsymbol{v} + \int_{\partial\Omega} ({}_1\widetilde{\boldsymbol{S}}\boldsymbol{n} \cdot \boldsymbol{v} + {}_2\widetilde{\boldsymbol{S}}\boldsymbol{n} \cdot \nabla\boldsymbol{v} + {}_3\boldsymbol{S}\boldsymbol{n} \cdot \nabla^{(2)}\boldsymbol{v})\,. \quad \text{(A.1)}$$

To arrive at (4.2), we find it convenient to begin by supposing that $\partial\Omega$ is the complete boundary of a region Ω of arbitrary shape.

A.1 Gradient decompositions

At a regular point of $\partial\Omega$, we decompose the variation gradients in (A.1) as follows:

$$\begin{aligned} \nabla\boldsymbol{v} &= {}^s\nabla\boldsymbol{v} + \partial_{\boldsymbol{n}}\boldsymbol{v} \otimes \boldsymbol{n}, \\ \nabla^{(2)}\boldsymbol{v} &= {}^s\nabla^{(2)}\boldsymbol{v} + \partial_{\boldsymbol{n}}\boldsymbol{v} \otimes {}^s\nabla\boldsymbol{n} + \partial_{\boldsymbol{n}}^{(2)}\boldsymbol{v} \otimes \boldsymbol{n} \otimes \boldsymbol{n} \\ &\quad + (\partial_{\boldsymbol{n}}\boldsymbol{v})_{,\alpha} \otimes (\boldsymbol{n} \otimes \boldsymbol{e}^\alpha + \boldsymbol{e}^\alpha \otimes \boldsymbol{n}) \\ &\quad - \boldsymbol{v}_{,\alpha} \otimes (({}^s\nabla\boldsymbol{n})^T\boldsymbol{e}^\alpha) \otimes \boldsymbol{n} \end{aligned} \quad \text{(A.2)}$$

(here $\boldsymbol{e}^\alpha$ is a contravariant base vector; see [5] for more details). With these relations, we find that

$$\begin{aligned}
{}_2\widetilde{\boldsymbol{S}}\boldsymbol{n}\cdot\nabla\boldsymbol{v} &= {}_2\widetilde{\boldsymbol{S}}\boldsymbol{n}\cdot{}^s\nabla\boldsymbol{v} + ({}_2\widetilde{\boldsymbol{S}}\boldsymbol{n})\boldsymbol{n}\cdot\partial_{\boldsymbol{n}}\boldsymbol{v},\\
{}_3\boldsymbol{S}\boldsymbol{n}\cdot\nabla^{(2)}\boldsymbol{v} &= {}_3\boldsymbol{S}\boldsymbol{n}\cdot{}^s\nabla^{(2)}\boldsymbol{v} + ({}_3\boldsymbol{S}\boldsymbol{n})[{}^s\nabla\boldsymbol{n}]\cdot\partial_{\boldsymbol{n}}\boldsymbol{v}\\
&\quad + (({}_3\boldsymbol{S}\boldsymbol{n})\boldsymbol{n})\boldsymbol{n}\cdot\partial^{(2)}_{\boldsymbol{n}}\boldsymbol{v} + 2\,({}_3\boldsymbol{S}\boldsymbol{n})\boldsymbol{n}\cdot{}^s\nabla(\partial_{\boldsymbol{n}}\boldsymbol{v})\\
&\quad - (({}_3\boldsymbol{S}\boldsymbol{n})\boldsymbol{n})({}^s\nabla\boldsymbol{n})^T\cdot{}^s\nabla\boldsymbol{v}.
\end{aligned}\tag{A.3}$$

On integration over $\partial\Omega$, the terms in the need of further treatment are those containing surface gradients, that is, the first in $(\text{A.3})_1$ and the first, fourth and fifth in $(\text{A.3})_2$. If, as is the case for a cube, $\partial\Omega$ is a finite union of smooth surfaces, each of which has a simple, closed, piece-wise smooth boundary curve, then the following divergence identity becomes useful.

A.2 A surface divergence identity and its consequences

Let the surface $\mathcal{S}$ be oriented by its normal $\boldsymbol{n}$; let Γ, the boundary curve of $\mathcal{S}$, be so oriented as to leave it on the left; and, for $\boldsymbol{t}$ the unit tangent to Γ, let $\boldsymbol{m} := \boldsymbol{t}\times\boldsymbol{n}$. Then, for $\boldsymbol{a}$ a vector field over $\mathcal{S}$, we have that

$$\int_{\mathcal{S}} {}^s\mathrm{Div}\,\boldsymbol{a} = -\int_{\mathcal{S}} 2H(\boldsymbol{a}\cdot\boldsymbol{n}) + \int_\Gamma \boldsymbol{a}\cdot\boldsymbol{m} \tag{A.4}$$

(here H denotes the mean curvature of $\mathcal{S}$). Two applications of this identity are:

- for $\boldsymbol{a} = \boldsymbol{A}^T\boldsymbol{v}$,

$$\int_{\mathcal{S}} \boldsymbol{A}\cdot{}^s\nabla\boldsymbol{v} = -\int_{\mathcal{S}} \boldsymbol{v}\cdot({}^s\mathrm{Div}\,\boldsymbol{A} + 2H\boldsymbol{A}\boldsymbol{n}) + \int_\Gamma \boldsymbol{v}\cdot\boldsymbol{A}\boldsymbol{m}\,; \tag{A.5}$$

- for $\boldsymbol{a} = \boldsymbol{\Lambda}^T[{}^s\nabla\boldsymbol{v}]$, with $\boldsymbol{\Lambda}$ a third-order tensor,

$$\int_{\mathcal{S}} \boldsymbol{\Lambda}\cdot{}^s\nabla^{(2)}\boldsymbol{v} = -\int_{\mathcal{S}} {}^s\nabla\boldsymbol{v}\cdot({}^s\mathrm{Div}\,\boldsymbol{\Lambda} + 2H\boldsymbol{\Lambda}\boldsymbol{n}) + \int_\Gamma {}^s\nabla\boldsymbol{v}\cdot\boldsymbol{\Lambda}\boldsymbol{m}\,.^6 \tag{A.6}$$

[6] Note that

$$\boldsymbol{\Lambda}^T\cdot\boldsymbol{d}\otimes\boldsymbol{b}\otimes\boldsymbol{c} = \boldsymbol{\Lambda}\cdot\boldsymbol{b}\otimes\boldsymbol{c}\otimes\boldsymbol{d},\quad (\boldsymbol{\Lambda}^T[\boldsymbol{A}])\boldsymbol{a} = (\boldsymbol{\Lambda}\boldsymbol{a})\cdot\boldsymbol{A}.$$

Generally speaking, edge forces per unit length are induced by line integrals of the type $\int_\Gamma \boldsymbol{v}\cdot\boldsymbol{\Lambda m}$, while line integrals of the type $\int_\Gamma {}^s\nabla\boldsymbol{v}\cdot\boldsymbol{\Lambda m}$ induce both edge forces per unit length and forces concentrated at edge junctions. To show that the former part of this statement holds true, a direct use of (A.5) suffices; for the latter part, further manipulations are needed to make use of (A.6). To begin with, combination of (A.5) and (A.6) yields:

$$\begin{aligned}\int_{\mathcal{S}} \boldsymbol{\Lambda}\cdot{}^s\nabla^{(2)}\boldsymbol{v} &= \int_{\mathcal{S}} \boldsymbol{v}\cdot({}^s\mathrm{Div}\,({}^s\mathrm{Div}\,\boldsymbol{\Lambda}+2H\boldsymbol{\Lambda n})+2H({}^s\mathrm{Div}\,\boldsymbol{\Lambda}+2H\boldsymbol{\Lambda n})\boldsymbol{n})\\ &\quad-\int_\Gamma \boldsymbol{v}\cdot({}^s\mathrm{Div}\,\boldsymbol{\Lambda}+2H\boldsymbol{\Lambda n})\boldsymbol{m}+\int_\Gamma {}^s\nabla\boldsymbol{v}\cdot\boldsymbol{\Lambda m}\,.\end{aligned}\tag{A.7}$$

Next, by representing the surface gradient on Γ in the intrinsic form

$${}^s\nabla\boldsymbol{v}=\partial_{\boldsymbol{t}}\boldsymbol{v}\otimes\boldsymbol{t}+\partial_{\boldsymbol{m}}\boldsymbol{v}\otimes\boldsymbol{m},$$

the last integral in (A.7) can be written as

$$\begin{aligned}\textstyle\int_\Gamma {}^s\nabla\boldsymbol{v}\cdot\boldsymbol{\Lambda m} &= \textstyle\int_\Gamma (\,\boldsymbol{v}\cdot((\boldsymbol{\Lambda m})\boldsymbol{t})_{,s}+\partial_{\boldsymbol{m}}\boldsymbol{v}\cdot(\boldsymbol{\Lambda m})\boldsymbol{m}\,)\\ &+ \textstyle\int_\Gamma(\boldsymbol{v}\cdot(\boldsymbol{\Lambda m})\boldsymbol{t})_{,s}\,.\end{aligned}\tag{A.8}$$

Finally,

$$\begin{aligned}&\int_{\mathcal{S}} \boldsymbol{\Lambda}\cdot{}^s\nabla^{(2)}\boldsymbol{v}=\int_{\mathcal{S}} \boldsymbol{v}\cdot({}^s\mathrm{Div}\,({}^s\mathrm{Div}\,\boldsymbol{\Lambda}+2H\boldsymbol{\Lambda n})+2H({}^s\mathrm{Div}\,\boldsymbol{\Lambda}+2H\boldsymbol{\Lambda n})\boldsymbol{n})\\ &-\int_\Gamma(\boldsymbol{v}\cdot(({}^s\mathrm{Div}\,\boldsymbol{\Lambda}+2H\boldsymbol{\Lambda n})\boldsymbol{m}-((\boldsymbol{\Lambda m})\boldsymbol{t})_{,s})-\partial_{\boldsymbol{m}}\boldsymbol{v}\cdot(\boldsymbol{\Lambda m})\boldsymbol{m})+\int_\Gamma(\boldsymbol{v}\cdot(\boldsymbol{\Lambda m})\boldsymbol{t})_{,s}\,.\end{aligned}\tag{A.9}$$

It is because of the last integral that concentrated forces may occur at edge junctions.

A.3 Application to a cube

Let now Ω be a cube, and let $\partial\Omega_i$ denote the face having normal $\boldsymbol{n}_i$ and contour Γ_i. Moreover, let the typical edge $\mathcal{E}$ and the typical vertex V be the intersections of the closures, respectively, of faces $\partial\Omega_1$ and $\partial\Omega_2$ and of faces $\partial\Omega_1$, $\partial\Omega_2$, and $\partial\Omega_3$.

Firstly, we apply identity (A.5) for $\mathcal{S}=\partial\Omega_\alpha$, $\boldsymbol{A}={}_2\widetilde{\boldsymbol{S}}\boldsymbol{n}_\alpha$ $(\alpha=1,2)$, and obtain that

$$\int_{\partial\Omega_\alpha} {}_2\widetilde{\boldsymbol{S}}\boldsymbol{n}_\alpha\cdot{}^s\nabla\boldsymbol{v}=-\int_{\partial\Omega_\alpha}\boldsymbol{v}\cdot{}^s\mathrm{Div}\,({}_2\widetilde{\boldsymbol{S}}\boldsymbol{n}_\alpha)+\int_{\Gamma_\alpha}\boldsymbol{v}\cdot({}_2\widetilde{\boldsymbol{S}}\boldsymbol{n}_\alpha)\boldsymbol{m}\,,$$

whence

$$\begin{aligned}\int_{\partial\Omega_1\cup\partial\Omega_2}{}_2\widetilde{\boldsymbol{S}}\boldsymbol{n}\cdot{}^s\nabla\boldsymbol{v}=&-\int_{\partial\Omega_1\cup\partial\Omega_2}\boldsymbol{v}\cdot({}^s\mathrm{Div}\,({}_2\widetilde{\boldsymbol{S}}\boldsymbol{n}))\\&+\sum_{\alpha=1}^{2}\int_{\Gamma_\alpha\setminus\mathcal{E}}\boldsymbol{v}\cdot({}_2\widetilde{\boldsymbol{S}}\boldsymbol{n}_\alpha)\boldsymbol{m}+\int_{\mathcal{E}}\boldsymbol{v}\cdot[\![({}_2\widetilde{\boldsymbol{S}}\boldsymbol{n}_1)\boldsymbol{n}_2]\!]\end{aligned}\tag{A.10}$$

Thus, among other things, at a point of edge $\mathcal{E}$ we expect the term

$$[\![({}_2\widetilde{\boldsymbol{S}}\boldsymbol{n}_1)\boldsymbol{n}_2]\!]$$

to appear into a specific edge balance of forces per unit length (cf. $(4.6)_1$).

Secondly, we apply identity (A.5) for $\mathcal{S}=\partial\Omega_\alpha$, just as before, but for $\boldsymbol{A}=2\,({}_3\boldsymbol{S}\boldsymbol{n}_\alpha)\boldsymbol{n}_\alpha$ $(\alpha=1,2)$ and with $\boldsymbol{v}$ replaced by $\partial_{\boldsymbol{n}_\alpha}\boldsymbol{v}$. We now obtain that

$$\begin{aligned}\int_{\partial\Omega_\alpha}(2\,{}_3\boldsymbol{S}\boldsymbol{n}_\alpha)\boldsymbol{n}_\alpha\cdot{}^s\nabla(\partial_{\boldsymbol{n}_\alpha}\boldsymbol{v})\;=&\;-\int_{\partial\Omega_\alpha}\partial_{\boldsymbol{n}_\alpha}\boldsymbol{v}\cdot{}^s\mathrm{Div}\,(2\,({}_3\boldsymbol{S}\boldsymbol{n}_\alpha)\boldsymbol{n}_\alpha)\\&+\;\int_{\Gamma_\alpha}\partial_{\boldsymbol{n}_\alpha}\boldsymbol{v}\cdot((2\,{}_3\boldsymbol{S}\boldsymbol{n}_\alpha)\boldsymbol{n}_\alpha)\boldsymbol{m}\,,\end{aligned}$$

whence the term

$$2\,{}^s\mathrm{Div}\,(({}_3\boldsymbol{S}\boldsymbol{n})\boldsymbol{n})$$

in a surface balance of first-order couples per unit area (cf. (4.5)) and the term

$$2\,({}_3\boldsymbol{S}\boldsymbol{n}_1)\boldsymbol{n}_2$$

in an edge balance of second-order couples per unit length (cf. $(4.6)_2$).[7]

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[7]To see this, consider the following integrals

$$\sum_{\alpha=1}^{2}\int_{\Gamma_\alpha\setminus\mathcal{E}}\partial_{\boldsymbol{n}_\alpha}\boldsymbol{v}\cdot(({}_3\boldsymbol{S}\boldsymbol{n}_\alpha)\boldsymbol{n}_\alpha)\boldsymbol{m}+\int_{\mathcal{E}}\big(\partial_{\boldsymbol{n}_1}\boldsymbol{v}\cdot(({}_3\boldsymbol{S}\boldsymbol{n}_1)\boldsymbol{n}_1)\boldsymbol{n}_2+\partial_{\boldsymbol{n}_2}\boldsymbol{v}\cdot(({}_3\boldsymbol{S}\boldsymbol{n}_2)\boldsymbol{n}_2)\boldsymbol{n}_1\big),$$

and note that the second integrand can be written as

$${}^{\boldsymbol{t}^\perp}\nabla\boldsymbol{v}\cdot({}_3\boldsymbol{S}\boldsymbol{n}_1)\boldsymbol{n}_2\,.$$

Thirdly, we apply identity (A.6) for $\mathcal{S} = \partial\Omega_i$ and $\boldsymbol{\Lambda} = {}_3\boldsymbol{S}\boldsymbol{n}_i$ $(i = 1,2,3)$, and find that

$$\int_{\partial\Omega_i} {}_3\boldsymbol{S}\boldsymbol{n}_i \cdot {}^s\nabla^{(2)}\boldsymbol{v} = \int_{\partial\Omega_i} \boldsymbol{v} \cdot {}^s\mathrm{Div}\,({}^s\mathrm{Div}\,{}_3\boldsymbol{S}\boldsymbol{n}_i)$$
$$-\int_{\Gamma_i}\Big(\boldsymbol{v}\cdot(({}^s\mathrm{Div}\,{}_3\boldsymbol{S}\boldsymbol{n}_i)\boldsymbol{m}_i - ((({}_3\boldsymbol{S}\boldsymbol{n}_i)\boldsymbol{m}_i)\boldsymbol{t}_i)_{,s}\,) - \partial_{\boldsymbol{m}_i}\boldsymbol{v}\cdot(({}_3\boldsymbol{S}\boldsymbol{n}_i)\boldsymbol{m}_i)\boldsymbol{m}_i\Big)$$
$$+\int_{\Gamma_i}(\boldsymbol{v}\cdot(({}_3\boldsymbol{S}\boldsymbol{n}_i)\boldsymbol{m}_i)\boldsymbol{t}_i)_{,s}\;. \qquad \text{(A.11)}$$

Repeated use of this relation allows us to conclude that
- at a point of the edge $\mathcal{E}$, we expect the term

$$[\![\,({}^s\mathrm{Div}\,{}_3\boldsymbol{S}\boldsymbol{n}_1)\boldsymbol{n}_2 - ((({}_3\boldsymbol{S}\boldsymbol{n}_1)\boldsymbol{n}_2)\boldsymbol{n}_3)_{,s}\,]\!]$$

to contribute to the balance of forces per unit length, and the term

$$({}_3\boldsymbol{S}\boldsymbol{n}_1)\boldsymbol{n}_2$$

to contribute to an edge balance of couples per unit length;
- at the vertex point V, we expect the term

$$6\,(({}_3\boldsymbol{S}\boldsymbol{n}_1)\boldsymbol{n}_2)\boldsymbol{n}_3$$

to contribute to a specific vertex balance of concentrated forces (cf. (4.7))[8].

[8] To see this, it is sufficient to focus on the last integral in (A.11) and note that

$$\int_{\Gamma_1\cup\Gamma_2}(\boldsymbol{v}\cdot(({}_3\boldsymbol{S}\boldsymbol{n})\boldsymbol{m})\boldsymbol{t})_{,s} = \sum_{\alpha=1}^{2}\int_{\Gamma_\alpha\setminus\mathcal{E}}\boldsymbol{v}\cdot((({}_3\boldsymbol{S}\boldsymbol{n})\boldsymbol{m})\boldsymbol{t})_{,s} + 2\left[\boldsymbol{v}\cdot(({}_3\boldsymbol{S}\boldsymbol{n}_1)\boldsymbol{n}_2)\boldsymbol{n}_3)\right]_{V_*}^{V},$$

where V_* denotes the vertex (1,1,-1) of the cube Ω.

Kontaktna medjudejstva, napon i materijalna simetrija

UDK 531.01, 539.32

Za neproste materijale je argumentovana potreba da se koncepti interakcije i materijalne simetrije kao i povezujući koncept napona pazljivo revidiraju i generalizuju.