Configurational forces and couples for crack propagation

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Abstract

Following the approach of Gurtin and Podio-Guidugli (1998), the problem of crack propagation based on the notion of configurational forces and couples in micropolar continua is considered.

1   Introduction

Very recently Gurtin and Podio-Guidugli (1998), developed a framework for dynamical fracture, concerning on the derivation of balance equations and constitutive equations that describe the motion of the crack tip in two-space dimensions. They worked within the nonlinear theory because the basic ideas are most easily explained within a framework that distinguishes between reference and deformed configurations; moreover, instead of laying down specific assumptions regarding the strength of the crack-tip singularities, they consider hypotheses motivated by the requirement that the underlying physical laws make sense. The theory is based on a configurational force balance and a mechanical version of the second law of thermodynamics.

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In classical continuum mechanics the response of a body to deformation is described by standard deformational forces consistent with balance laws for linear and angular momentum. But, as they stated, configurational forces are less intuitive: they are related to the intrinsic coherency of a body's material structure and perform work in the addition and removal of material and in the evolution of structural defects. Following Gurtin and Struthers (1990) and Gurtin (1995), they viewed configurational forces as basic primitive objects consistent with their own force balance. Configurational forces defined via the calculus of variations as derivatives of an energy have been introduced earlier, e.g. in the classic works of Eshelby (1951), (1975) on lattice defects.

The role of configurational forces, however, seems more pervasive and fundamental than problems susceptible to a variational formulation can indicate, a view they demonstrated within the context of fracture dynamics.

Particularly, we refer to the recent monograph by Maugin (1993), for a discussion of configurational forces within an Eshelbian framework, and for related references.

In this short communication we strictly follow the approach of Gurtin and Podio-Guidugli (1998), in a case of micropolar continua. Our final results are not different in form results well known (see Jarić (2000)). What is different is a derivation and a interpretation in terms of configurational forces, which we believe most accurately describe the underlying physics.

The scope of the paper is the following: at the beginning we review some basic formulae of the geometry and the kinematics of a cracked body given by Gurtin and Podio-Guidugli (1998), we need in our investigation; for the same reason we described the motion of the crack bodies; next we state the balances laws for deformational and configurational forces for micropolar continua; the last part belongs to the derivation of Eshelby tensor as a consequence of invariance under reparametrization (Gurtin, 1995).

2 Cracks; time-dependent control volumes

Let B denote a closed region of R^2 with boundary ∂B and, for each t in some open time interval, let $\mathcal{C}(t)$ be a smooth, connected, oriented curve in B with one end, Z_0, fixed at the boundary ∂B, with the remainder of $\mathcal{C}(t)$ - including the other end point $\mathbf{Z}(t)$ - contained in the interior of B, and with

$$\mathcal{C}(\tau) \subset \mathcal{C}(t) \quad \text{for all} \quad t \geq \tau \tag{2.1}$$

We view

$$\mathcal{B}(t) = B \backslash \mathcal{C}(t) \tag{2.2}$$

as a referential neighborhood of a growing crack $\mathcal{C}(t)$ with $\mathbf{Z}(t)$ the crack tip. (See Fig.1. Note that B contains the points of $\mathcal{C}(t)$ while $\mathcal{B}(t)$ does not; hence $\mathcal{B}(t)$ is cracked, while B is not. Note also that the assumed regularity of $\mathcal{C}(t)$ precludes singularities such as kinks and bifurcations.) We let $\mathbf{e}(t)$ denote the unit tangent to $\mathcal{C}(t)$ at $\mathbf{Z}(t)$ in the direction of (possible) propagation (Fig.1).
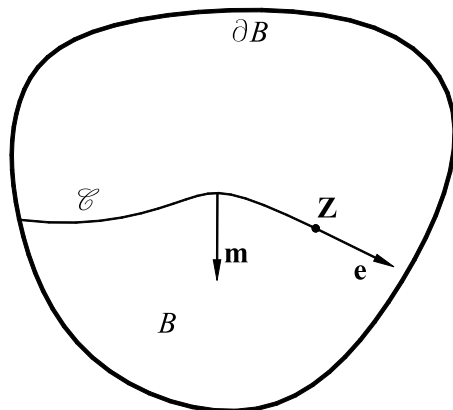

Figure 1:

Then the tip velocity

$$\mathbf{v}(t) = \mathrm{d}\mathbf{Z}(t)/\mathrm{d}t \tag{2.3}$$

may be written in the form

$$\mathbf{v}(t) = V(t)\mathbf{e}(t), \qquad V(t) \geq 0, \tag{2.4}$$

with V the speed. Finally, we choose a continuous unit normal field $\mathbf{m}(\mathbf{X}, t)$ for $\mathcal{C}(t)$.

By a control volume we mean a closed subregion $R(t)$ of B for which $\partial R(t)$ evolves smoothly with t, and for which

$$\mathcal{C}_R(t) = \mathcal{C}(t) \cap R(t), \tag{2.5}$$

the portion of the crack in $R(t)$, does not intersect $\partial R(t)$ at more than two points (Fig.2).
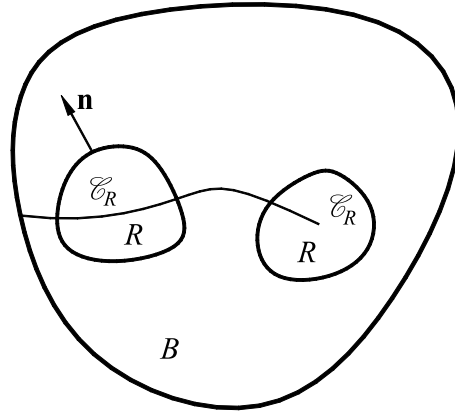

Figure 2:

For convenience, we limit our discussion to two classes of control volumes: those that do not intersect the tip and those that contain the tip in their interior. We view the dependence of $R(t)$ on t as resulting from the addition and removal of material points. So defined a control volume does not preclude control volumes R that are independent of time.

For $R(t)$ a control volume, $\mathbf{n}(\mathbf{X}, t)$ designates the outward unit normal to $\partial R(t)$, and $U_{\partial R}$ the (scalar) normal velocity of the boundary curve in the direction $\mathbf{n}$. A useful example of a time-dependent control volume is the tip disc

$$D_\delta(t) = \left\{ \mathbf{X} \in B : |\mathbf{X} - \mathbf{Z}(t)| \leq \delta \right\}, \tag{2.6}$$

a disc of radius δ centered at the tip $\mathbf{Z}(t)$; here the normal velocity is

$$U_{\partial D_\delta} = \mathbf{v} \cdot \mathbf{n}. \tag{2.7}$$

For convenience, we write

$$\mathcal{C}_\delta(t) = \mathcal{C}_{D_\delta}(t) = \mathcal{C}(t) \cap D_\delta(t). \tag{2.8}$$

2.1 Derivatives following the crack tip; tip integrals; transport theorems

According to Gurtin and Podio-Guidugli (1998), a field $\Phi(\mathbf{X}, t)$ is referred as smooth away from the tip if $\Phi(\mathbf{X}, t)$ is defined for all $\mathbf{X} \in \mathcal{B}(t)$ and all t, and if, away from the tip, $\Phi(\mathbf{X}, t)$ and its derivatives have limits up to the crack from either side. Then we write, for $\mathbf{X} \in \mathcal{C}(t)$, $\mathbf{X} \neq \mathbf{Z}(t)$

$$\Phi^\pm(\mathbf{X}, t) = \lim_{\varepsilon \to 0} \Phi(\mathbf{X} \pm \varepsilon \mathbf{m}(\mathbf{X}, t), t), \quad [\Phi] = \Phi^+ - \Phi^-. \tag{2.9}$$

Given such a field $\Phi(\mathbf{X}, t)$, consider the corresponding field $\hat{\Phi}(\mathbf{Y}, t)$ in which $\mathbf{Y}$ represents the position of the material point $\mathbf{X}$ relative to the tip $\mathbf{Z}(t)$

$$\hat{\Phi}(\mathbf{Y}, t) = \Phi(\mathbf{X}, t), \quad \mathbf{Y} = \mathbf{X} - \mathbf{Z}(t). \tag{2.10}$$

The partial derivative

$$\Phi^0(\mathbf{X}, t) = \frac{\partial \hat{\Phi}(\mathbf{Y}, t)}{\partial t}$$

with respect to t holding $\mathbf{Y}$ fixed represents the time derivative of $\Phi(\mathbf{X}, t)$ following the tip $\mathbf{Z}(t)$; by the chain rule,

$$\Phi^0 = \Phi^\bullet + \nabla\Phi \cdot \mathbf{v} \tag{2.11}$$

away from the tip, where

$$\Phi^\bullet(\mathbf{X}, t) = \frac{\partial \Phi(\mathbf{X}, t)}{\partial t}.$$

We will repeatedly take limits, as $\delta \to 0$, of integrals of fields over $\partial D_\delta(t)$; we refer to such limits, when meaningful, as tip integrals; examples, for φ a scalar field, $\mathbf{w}$ a vector field, and $\mathbf{T}$ a tensor field, are:

$$\oint_{tip} \varphi\mathbf{n} = \lim_{\delta \to 0} \int_{\partial D_\delta(t)} \varphi\mathbf{n}, \tag{2.12a}$$

$$\oint_{tip} (\mathbf{w} \otimes \mathbf{n}) = \lim_{\delta \to 0} \int_{\partial D_\delta(t)} \mathbf{w} \otimes \mathbf{n}, \qquad (2.12b)$$

$$\oint_{tip} \mathbf{Tn} = \lim_{\delta \to 0} \int_{\partial D_\delta(t)} \mathbf{Tn}. \qquad (2.12c)$$

Let $R(t)$ be a control volume that includes the tip and consider the region

$$R_\delta(t) = R(t) \backslash D_\delta(t), \qquad (2.13)$$

with $\delta > 0$ sufficiently small that $\partial R_\delta(t) = \partial R_\delta(t) \cup \partial D_\delta(t)$. Then, using the same letter $\mathbf{n}$ for the outward unit normal on both ∂R and ∂D_δ, and bearing in mind that the outward unit normal to ∂R_δ on ∂D_δ is $\mathbf{n}$, we may use the gradient theorem in the usual manner-with $\mathcal{C}_{R_\delta}$ considered as a "slit in R_δ" giving rise to an additional pair of boundary segments (Fig.3),
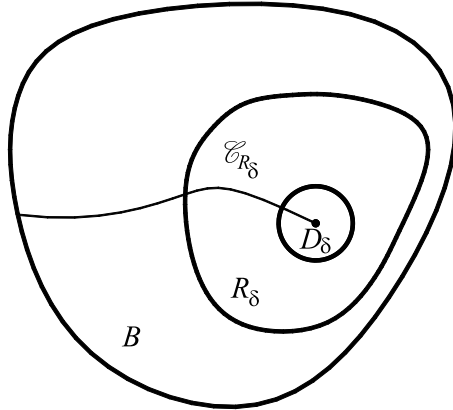

Figure 3:

and with $\int_{R_\delta} \nabla\Phi$ interpreted accordingly - to conclude that, for Φ smooth away from the tip,

$$\int_{R_\delta} \nabla\Phi = \int_{\partial R} \Phi\mathbf{n} - \int_{\mathcal{C}_{R_\delta}} [\Phi]\mathbf{m} - \int_{\partial D_\delta} \Phi\mathbf{n}. \qquad (2.14)$$

Here, for convenience, we have suppressed the argument t. Thus, if $\oint_{tip} \Phi \mathbf{n}$ exists, and if $[\Phi]$ is integrable on $\mathcal{C}$ then $\int_R \nabla \Phi$ exists as the limit

$$\lim_{\delta \to 0} \int_{R_\delta} \nabla \Phi$$

and we have the generalized gradient theorem

$$\int_R \nabla \Phi = \int_{\partial R} \Phi \mathbf{n} - \int_{\mathcal{C}_R} [\Phi]\mathbf{m} - \oint_{tip} \Phi \mathbf{n}. \tag{2.15}$$

The next definition allows us to state succinctly our hypotheses concerning momenta and energies. We will refer to Φ as regular if, in addition to being smooth away from the tip,

(R1) Φ is integrable on B; given any control volume $R(t)$, the mapping $t \to \int_{R(t)} \Phi$ is differentiable;

(R2) Φ^o is integrable on B and $[\Phi]\mathbf{m} \cdot \mathbf{v}$ is integrable on $\mathcal{C}(t)$, both uniformly in t;

(R3) $\oint_{tip} \Phi \mathbf{n}$ exists.

(The phrase "uniformly in t" signifies "uniformly for t in any compact interval".) By (R2) and (2.8),

$$\int_{\mathcal{C}_\delta(t)} [\Phi]\mathbf{m} \cdot \mathbf{v} \quad \text{approaches zero as } \delta \to 0. \tag{2.16}$$

The following well-known transport theorem is valid when $\Phi(\mathbf{X}, t)$ is smooth away from the tip and $R(t)$ does not contain the tip

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\{ \int_{R(t)} \Phi \right\} = \int_{R(t)} \Phi^\bullet + \int_{\partial R(t)} \Phi U_{\partial R}. \tag{2.17}$$

We now give two generalizations of (2.17) that account for the crack tip.

Transport theorem. For $R(t)$ a control volume that includes the tip, if $\Phi(\mathbf{X}, t)$ is regular, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{R(t)} \Phi\right\} = \int_{R(t)} \Phi^\bullet + \int_{\partial R(t)} \Phi U_{\partial R} - \oint_{tip} \Phi(\mathbf{v}\cdot\mathbf{n}), \qquad (2.18a)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{R(t)} \Phi\right\} = \int_{R(t)} \Phi^o + \int_{\partial R(t)} \Phi(U_{\partial R} - \mathbf{v}\cdot\mathbf{n}) + \int_{\mathcal{C}_{R(t)}} [\Phi]\mathbf{m}\cdot\mathbf{v} \quad (2.18b)$$

(with $\int_{R(t)} \Phi^\bullet$ defined as $\lim_{\delta\to 0} \int_{R_\delta} \Phi^\bullet$, which exists).

2.2   Motions of cracked bodies

The motion of micropolar body $\mathcal{B}(t)$ is described completely by $\mathbf{y}(\mathbf{X}, t)$ and $\chi(\mathbf{X}, t)$, where orthogonal χ is called microrotation (see Eringen (1976)). Let $\mathbf{y}(\mathbf{X}, t)$ be smooth away from the tip with $\mathbf{y}(\mathbf{X}, t)$ one-to-one in $\mathbf{X}$ on $\mathcal{B}(t)$ for each t. The deformation gradient

$$\mathbf{F} = \nabla\mathbf{y} \qquad (2.19)$$

and the material velocity $\mathbf{y}^\bullet$ is then smooth away from the tip.

Let $R(t)$ be a control volume. The boundary curve $\partial R(t)$ may be parameterized in a sufficiently small time interval and in a neighborhood of any of its points by a function of the form $\mathbf{X} = \hat{\mathbf{X}}(\sigma, t) = \hat{\mathbf{X}}(\sigma^1, \sigma^2, t)$; the field

$$\mathbf{u}(\mathbf{X}, t) = \frac{\partial\hat{\mathbf{X}}(\sigma, t)}{\partial t} \qquad (2.20)$$

then represents a velocity field for $\partial R(t)$ in that neighborhood. It is possible to use such parametrizations to construct a velocity field for $\partial R(t)$; that is, a smooth field $\mathbf{u}(\mathbf{X}, t)$ defined for all $\mathbf{X}$ on $\partial R(t)$ and all t in any (sufficiently small) time interval. A field $\mathbf{u}$ so constructed depends on the choice of local parametrizations, but its normal component is intrinsic (see Fig.4)

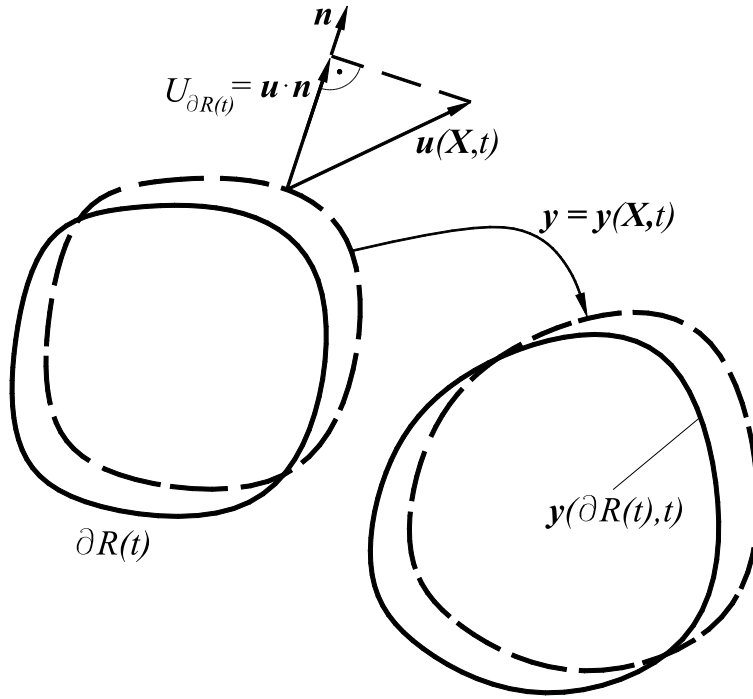$$\mathbf{u}\cdot\mathbf{n} = U_{\partial R}. \qquad (2.21)$$

Figure 4:

Each local parametrization $\mathbf{X} = \hat{\mathbf{X}}(\sigma, t)$ induces a corresponding local parametrization $\mathbf{x} = \hat{\mathbf{x}}(\sigma, t) = \mathbf{y}(\hat{\mathbf{X}}(\sigma, t), t)$ for the deformed boundary curve $\mathbf{y}(\partial R(t), t)$; the corresponding induced velocity field

$$\bar{\mathbf{u}}(\mathbf{X}, t) = \frac{\partial \hat{\mathbf{x}}(\sigma, t)}{\partial t} \tag{2.22}$$

for the deformed boundary $\mathbf{y}(\partial R(t), t)$ is related to $\mathbf{u}$ by the formula

$$\bar{\mathbf{u}} = \mathbf{y}^{\bullet} + \mathbf{F}\mathbf{u}. \tag{2.23}$$

The tip velocity $\mathbf{v}(t)$ may be considered as a velocity field for the boundary of the disc $D_\delta(t)$ using as a parametrization

$$\mathbf{X} = \hat{\mathbf{X}}_\delta(\sigma, t) = \mathbf{Z}(t) + \delta\mathbf{v}(\sigma) \tag{2.24}$$

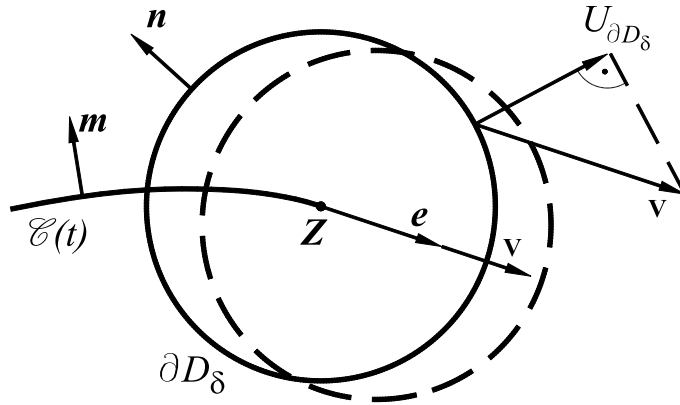with $\mathbf{v}(\sigma)$ a unit vector at an angle σ from a fixed axis (see Fig.5).

Figure 5:

Then

$$\mathbf{y}^o = \mathbf{y}^\bullet + \mathbf{F}\mathbf{v} \qquad (2.25)$$

the time derivative following $Z(t)$, represents the corresponding induced velocity field for $\mathbf{y}(\partial D_\delta(t), t)$. We assume that:

(Al) there is a function $\bar{\mathbf{v}}(t)$ such that

$$\mathbf{y}^o(\mathbf{X}, t) \to \bar{\mathbf{v}}(t) \quad \text{as } \mathbf{X} \to \mathbf{Z}(t) \text{ uniformly in } t. \qquad (2.26)$$

One might expect that $\bar{\mathbf{v}}(t)$ represents the velocity of the deformed crack tip. Granted sufficient regularity this is indeed the case. Assume for the moment that $\mathbf{y}(\mathbf{X}, t)$ has a limiting value $\mathbf{y}(\mathbf{Z}, t)$ as $\mathbf{X} \to \mathbf{Z}(t)$, so that the deformed crack tip is well defined. Then $\mathbf{y}(\mathbf{Z}, t)$ is differentiable in t and

$$\bar{\mathbf{v}}(t) = \frac{\mathrm{d}\mathbf{y}(\mathbf{Z}(t), t)}{\mathrm{d}t}. \qquad (2.27)$$

To verify (2.27) consider (2.24) with σ fixed, and let $\mathbf{y}_\delta = \mathbf{y}(\bar{\mathbf{X}}_\delta(\sigma, t), t)$. Then $\mathrm{d}\mathbf{y}_\delta/\mathrm{d}t = \mathbf{y}^o(\bar{\mathbf{X}}_\delta(\sigma, t), t)$, so that, by (2.26), $\mathrm{d}\mathbf{y}_\delta/\mathrm{d}t \to \bar{\mathbf{v}}(t)$ as $\delta \to 0$, uniformly in t. But, by hypothesis, $\mathbf{y}_\delta \to \mathbf{y}(\mathbf{Z}(t), t)$; thus $\mathbf{y}(\mathbf{Z}(t), t)$ is differentiable in t and (2.25) holds at $\mathbf{Z}(t)$.

We recall that for micropolar continua, the skew-symmetric gyration tensor $\boldsymbol{\nu}$ is defined by

$$\boldsymbol{\nu} = \boldsymbol{\chi}^\bullet \boldsymbol{\chi}^T, \qquad \boldsymbol{\nu} = -\boldsymbol{\nu}^T. \qquad (2.28)$$

The corresponding angular velocity ν is defined by

$$\nu = -\frac{1}{2}\boldsymbol{\varepsilon} \cdot \cdot \boldsymbol{\nu} \text{ or } \nu^k = -\frac{1}{2}\varepsilon^{klm}\nu_{lm}. \tag{2.29}$$

Then,

$$\boldsymbol{\chi}(\hat{\mathbf{X}}(\sigma,t),t) \equiv \mathbf{H}(\sigma,t) \tag{2.30}$$

and

$$\bar{\boldsymbol{\mu}} = \mathbf{H}^{\bullet}\mathbf{H}^T; \tag{2.31}$$

where

$$\mathbf{H}^{\bullet} = \frac{\partial \mathbf{H}}{\partial t} = \boldsymbol{\chi}^{\bullet} + \mathbf{u} \cdot \nabla \boldsymbol{\chi}.$$

Now, it is easy to show that

$$\bar{\boldsymbol{\mu}} = \boldsymbol{\nu} + (\mathbf{u} \cdot \nabla \boldsymbol{\chi})\boldsymbol{\chi}^T, \quad \text{or} \quad \bar{\mu}^{kl} = \nu^{kl} + u^L \chi^k_{K;L} \chi^{lK} \tag{2.32}$$

since

$$\hat{\mu}^k_{.l} = H^{\bullet k}{}_K H_l{}^K = \left(\frac{\partial \chi^k_K}{\partial t} + \chi^k_{KL} u^L\right)\chi_l{}^K.$$

Its corresponding angular velocity $\bar{\mu}$ is given by

$$\bar{\mu} = -\frac{1}{2}\boldsymbol{\varepsilon} \cdot \cdot \bar{\boldsymbol{\mu}} = -\frac{1}{2}\boldsymbol{\varepsilon} \cdot \cdot \left[\boldsymbol{\nu} + (\mathbf{u} \cdot \nabla \boldsymbol{\chi})\boldsymbol{\chi}^T\right] = \nu - \frac{1}{2}\boldsymbol{\varepsilon} \cdot \cdot (\mathbf{u} \cdot \nabla \boldsymbol{\chi})\boldsymbol{\chi}^T, \tag{2.33}$$

or in the componental form

$$\bar{\mu}_k = \nu_k - \frac{1}{2}\varepsilon_{klm}\chi^l_{K;L}\chi^{mk} u^L. \tag{2.33a}$$

Moreover, for the crack the boundary of the disc $D_\delta(t)$, using parameterization (2.24), we have

$$\hat{\boldsymbol{\mu}} = \boldsymbol{\nu} + (\mathbf{v} \cdot \nabla \boldsymbol{\chi})\boldsymbol{\chi}^T. \tag{2.34}$$

3   Basic laws

3.1   Balance laws for deformational and configurational forces

We let ρ denote the reference mass density, write

$$\mathbf{p} = \rho \mathbf{y}^{\bullet}, \qquad \rho\sigma = \Pi \tag{3.1}$$

for the momentum, and intrinsic angular momentum density due to rotation; σ is a spin density. Let $\mathbf{S}$ denote the Piola–Kirchhoff stress and $\mathbf{M}$ couple stress that arise in response to deformation. We neglect external body forces and body couples, and assume that the crack faces are traction-free, i.e

$$\mathbf{S}^{\pm}\mathbf{m} = 0, \qquad \mathbf{M}^{\pm}\mathbf{m} = 0, \quad \text{on } \mathcal{C}(t). \tag{3.2}$$

The balance laws for linear and angular momentum then take the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{R(t)} \mathbf{p}\right\} = \int_{\partial R(t)} (\mathbf{p}U_{\partial R} + \mathbf{S}\mathbf{n}) \tag{3.3a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{R(t)} (\mathbf{y} \times \mathbf{p} + \Pi)\right\} = \int_{\partial R(t)} \{(\mathbf{y} \times \mathbf{p} + \Pi)U_{\partial R} + \mathbf{y} \times \mathbf{S}\mathbf{n} + \mathbf{M}\mathbf{n}\}, \tag{3.3b}$$

for each control volume $R(t)$.

We consider, in addition, a configurational stress $\mathbf{C}$, a configurational force $\mathbf{f}$ distributed over $\mathcal{B}(t)$, and a configurational force $\mathbf{g}$ concentrated at the tip, Λ - a configurational couple stress, Δ - a configurational moment distributed over $\mathcal{B}(t)$, and $\mathfrak{M}(t)$ - a configurational moment concentrated at the tip; these are presumed consistent with the configurational force balance, if $R(t)$ does not contain the tip

$$\int_{\partial R(t)} \mathbf{C}\mathbf{n} + \int_{\mathcal{C}_{R(t)}} [\mathbf{C}]\mathbf{m} + \int_{R(t)} \mathbf{f} = 0, \tag{3.4a}$$

$$\int_{\partial R(t)} (\mathbf{X} \times \mathbf{C}\mathbf{n} + \Lambda\mathbf{n}) + \int_{\mathcal{C}_{R(t)}} (\mathbf{X} \times [\mathbf{C}]\mathbf{m} + [\Lambda]\mathbf{m}) + \int_{R(t)} (\mathbf{X} \times \mathbf{f} + \Delta) = 0, \tag{3.4b}$$

and, if $R(t)$ contain the tip

$$\int_{\partial R(t)} \mathbf{C}\mathbf{n} + \int_{\mathcal{C}_{R(t)}} [\mathbf{C}]\mathbf{m} + \int_{R(t)} \mathbf{f} + \mathbf{g}(t) = 0, \tag{3.5a}$$

$$\int_{\partial R(t)} (\mathbf{X} \times \mathbf{Cn} + \Lambda \mathbf{n}) + \int_{\mathcal{C}_{R(t)}} (\mathbf{X} \times [\mathbf{C}]\mathbf{m} + [\Lambda]\mathbf{m}) +$$

$$\int_{R(t)} (\mathbf{X} \times \mathbf{f} + \Delta) + \mathbf{Z}(t) \times \mathbf{g}(t) + \mathfrak{M}(t) = 0. \qquad (3.5b)$$

We assume that each of $\mathbf{f}$ and $\mathbf{g}$ consists of internal and inertial portions. While the decomposition of $\mathbf{f}$ is irrelevant to most of our discussion, determining the inertial portion of $\mathbf{g}$ will form a major part of our analysis. The same conclusion hold for Δ and $\mathfrak{M}$.

To ensure that the balances (3.4) are well defined and that their localization to the crack tip (in Section 4) is meaningful, we assume that:

(A2) ρ is continuous; $\mathbf{p}$ and $\mathbf{y} \times \mathbf{p} + \Pi$ are regular; $\mathbf{S}$, $\mathbf{M}$, Λ and $\mathbf{C}$ are smooth away from the tip; $\mathbf{f}$ and Δ are integrable over B; $\int_{\partial D_\delta} |\mathbf{Sn}|$ and $\int_{\partial D_\delta} |\mathbf{Mn}|$ remains bounded as $\delta \to 0$; $[\mathbf{C}]\mathbf{m}$ and $[\Lambda]\mathbf{m}$ are integrable on $\mathcal{C}(t)$.

Then the following local relations away from the crack hold:

$$\mathrm{div}\mathbf{S} = \mathbf{p}^\bullet, \qquad (3.6a)$$

$$\mathbf{\Pi}^\bullet = \mathrm{div}\mathbf{M} - \mathbf{s}^A, \qquad (3.6b)$$

where

$$s_i^A = \epsilon_{ijk} x_{;K}^j S^{kK}, \qquad (\mathbf{FS}^T)^A = \frac{1}{2}\epsilon \cdot \cdot \mathbf{s}^A.$$

From (3.4b) and (3.5b) it follows

$$\mathrm{div}\mathbf{C} + \mathbf{f} = 0, \qquad (3.7a)$$

$$-\mathbf{c}^A + \mathrm{div}\Lambda + \Delta = 0, \qquad (3.7b)$$

where

$$c_i^A = \epsilon_{ijk} x_{;K}^j C^{kK}, \qquad (\mathbf{FC}^T)^A = \frac{1}{2}\epsilon \cdot \cdot \mathbf{c}^A.$$

3.2   Mechanical version of the second law

In the absence of defects (such as cracks), of external body forces, and of thermal and compositional effects, micropolar continuum mechanics

may be based on a "second law" that utilizes stationary control volumes R and has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_R \psi\right\} + \mathcal{K}(R) \leq \mathcal{W}(R) \tag{3.8}$$

where ψ is the free energy density,

$$\mathcal{K}(R) = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_R (k + \kappa)\right\} \tag{3.9}$$

with
    - the kinetic energy density due to the translation motion,

$$k = \frac{1}{2}\rho|\mathbf{y}^\bullet|^2 \tag{3.10a}$$

    - the kinetic energy density due to the microrotation of the particle,

$$\kappa = \frac{1}{2}\rho\sigma \cdot \nu \tag{3.10b}$$

    - the total kinetic energy density,

$$k + \kappa, \tag{3.10c}$$

and where

$$\mathcal{W}(R) = \int_{\partial R} (\mathbf{Sn}\cdot\mathbf{y}^\bullet + \mathbf{Mn}\cdot\nu) \tag{3.11}$$

is the boundary working (Gurtin and Struthers, 1990; Gurtin, 1995).

For an evolving control volume $R(t)$ generalization of (3.8)-(3.11) is necessary, but by no means obvious. We consider the dependence of $R(t)$ on t as representing the addition of material to-or the removal of material from-the boundary $\partial R(t)$, and we write the second law in a manner reflecting this view. To begin with we take

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{R(t)} \psi\right\}$$

as the sole term involving free energy; we do not include the outflow term

$$\int\limits_{\partial R(t)} \psi U_{\partial R}$$

as we view noninertial interactions with the material exterior to R(t) in terms of working, rather than transport.

This leads to the main issue: generalization of the expression (3.11) to account for the work performed in the addition and removal of material at the boundary. We assume that $\mathbf{Cn}$ u represents the boundary working of the configurational stress $\mathbf{C}$, where $\mathbf{u}$ is the velocity field computed via a particular choice of local parametrizations $\mathbf{X} = \hat{\mathbf{X}}(\sigma, t)$ for $\partial R(t)$. The working of the deformational stress $\mathbf{S}$ must also be taken into account. When the control volume depends on time there is no intrinsic material description of its deformed boundary $\mathbf{y}(\partial R(t), t)$, as material is continually being added and removed, and it would seem appropriate to use, as a velocity for $\mathbf{y}(\partial R(t), t)$, the derivative $\bar{\mathbf{u}}(\mathbf{X}, t)$ of $\mathbf{y}(\hat{\mathbf{X}}(\sigma, t), t)$ with respect to t holding the surface parameter σ fixed; we therefore write the boundary working of $\mathbf{S}$ in the form $\mathbf{Sn} \cdot \bar{\mathbf{u}}$. The same reasoning holds for $\mathbf{\Lambda}$ and $\mathbf{M}$. Particulary, $\mathbf{Mn} \cdot \bar{\mu}$ is the boundary working of $\mathbf{M}$.

Finally, as we view the kinetic energy as "independent" of the internal structure of the material, we generalize $\mathcal{K}(R)$ in the standard manner, viz.,

$$\mathcal{K}(R(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\int\limits_R (k + \kappa)\right\} - \int\limits_{\partial R(t)} (k + \kappa)U_{\partial R}. \qquad (3.12)$$

In conclusion, we write the second law for an evolving control volume $R(t)$ - that may or may not contain the crack tip-in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int\limits_{R(t)} \psi\right\} + \mathcal{K}(R(t)) \leq \int\limits_{\partial R(t)} (\mathbf{Sn} \cdot \bar{\mathbf{u}} + \mathbf{Cn} \cdot \mathbf{u} + \mathbf{Mn} \cdot \bar{\mu} + \Lambda\mathbf{n} \cdot \mu,) \quad (3.13)$$

with $\mathbf{u}$ a velocity field for $\partial R(t)$; $\bar{\mathbf{u}}$ and $\bar{\mu}$ the corresponding induced velocity field for $\mathbf{y}(\partial R(t), t)$ and $\chi(\partial R(t), t)$. (The configurational forces

f and **g** perform no work, as their inertial components are accounted for by $\mathcal{K}(R(t))$, while their noninertial components are internal; moreover, there is no contribution from $\mathcal{C}(t)$ because of (3.2) and since only the tip of $\mathcal{C}(t)$ evolves.) Note that, by (2.23), the deformational working $\mathbf{Sn} \cdot \bar{\mathbf{u}}$ consists of a classical term $\mathbf{Sn} \cdot \mathbf{y}^{\bullet}$ plus a term $\mathbf{Sn} \cdot \mathbf{Fu}$; also $\mathbf{Mn} \cdot \bar{\mu}$ consist a clasical term $\mathbf{Mn} \cdot \nu$ and $-\frac{1}{2}\mathbf{Mn} \cdot (\epsilon \cdot \cdot (\mathbf{u} \cdot \nabla \chi) \chi^T)$. They account for the addition of strained material to ∂R. Note also that for R independent of time (3.13) reduces to the standard inequality (3.8), 50 there is no conflict with classical continuum mechanics.

To ensure that this version of the second law be meaningful, and to allow for its localization, we assume that:

(A3) ψ, k and κ are regular.

3.3 The Eshelby tensor as a consequence of invariance under reparameterization (Gurtin, 1995)

We require that our theory be independent of the choice of parametrization for $\partial R(t)$. This requirement of invariance under reparametrization has important consequences. In particular, the invariance of (3.13) is equivalent to invariance of boundary working, which, by (2.30), can be given the form

$$\mathcal{W}(R(t)) = \int_{\partial R(t)} (\mathbf{Sn} \cdot \bar{\mathbf{u}} + \mathbf{Cn} \cdot \mathbf{u} + \mathbf{Mn} \cdot \bar{\mu} + \Lambda \mathbf{n} \cdot \mu). \qquad (3.14)$$

Here, in order to simplify the investigation, we did not consider the rotation of $D_\delta(t)$. Then $\mu = 0$. We write

$$\mathcal{W}(R(t)) = \int_{\partial R(t)} \left\{ \mathbf{Sn} \cdot \mathbf{y}^{\bullet} + (\mathbf{F}^T\mathbf{Sn} + \mathbf{Cn}) \cdot \mathbf{u} + \mathbf{Mn} \cdot \nu - \right.$$

$$\left. \frac{1}{2}\mathbf{Mn} \cdot (\epsilon \cdot \cdot (\mathbf{u} \cdot \nabla \chi)\chi^T) \right\}, \qquad (3.14a)$$

where,

$$\frac{1}{2}\mathbf{Mn} \cdot (\epsilon \cdot \cdot (\mathbf{u} \cdot \nabla \chi)\chi^T) = \frac{1}{2}M_{kK}n^K \epsilon^{klm}\chi_{lL,M}\chi_m{}^L u^M,$$

$$\frac{1}{2}\epsilon^{klm}M_{lK}\chi_{mL} \equiv \mathcal{M}^k_{.LK}$$

$$-\frac{1}{2}\mathbf{Mn}\cdot(\epsilon\cdot\cdot[\mathbf{u}\cdot\nabla\chi)\chi^T] = u^M\chi_{lL,M}\mathcal{M}^{lL}_{K}n^K.$$

Also

$$\mathbf{F}^T\mathbf{S}\cdot\mathbf{u} = x^k_{;M}S_{kK}n^Ku^M$$

$$\mathbf{Cn}\cdot\mathbf{u} = C_{MK}n^Ku^M$$

$$(\mathbf{F}^T\mathbf{S} + \mathbf{Cn})\cdot\mathbf{u} - \frac{1}{2}\mathbf{Mn}\cdot(\epsilon\cdot\cdot[\mathbf{u}\cdot\nabla\chi)\chi^T] =$$
$$\left(C_{MK} + x^k_{;M}S_{kK} + \chi_{lL;M}\mathcal{M}^{lL}_{K}\right)n^Ku^M =$$
$$\mathrm{tr}(\mathbf{C} + \mathbf{F}^T\mathbf{S} + \nabla\chi^T\mathbf{M})\mathbf{n}\otimes\mathbf{u}.$$

Then

$$(\mathbf{C} + \mathbf{F}^T\mathbf{S} + \nabla\chi^T\mathcal{M})\cdot\mathbf{a}_\alpha.$$

Changes in parametrization affect the tangential component of $\mathbf{u}$, but leave the normal component unaltered. In fact, invariance of (3.14) under reparametrization is equivalent to the requirement that $(\mathbf{C} + \mathbf{F}^T\mathbf{S} + \nabla\chi\mathcal{M})\cdot\mathbf{a}_\alpha$ on $\partial R(t)$ for all tangential vector fields $\mathbf{a}_\alpha$ on $\partial R(t)$; thus, since $R(t)$ is arbitrary, $(\mathbf{C} + \mathbf{F}^T\mathbf{S})\cdot\mathbf{n}$ must be parallel to $\mathbf{n}$ for all $\mathbf{n}$, so that

$$\Rightarrow \ (\mathbf{C} + \mathbf{F}^T\mathbf{S} + \nabla\chi\mathcal{M})\cdot\mathbf{a}_\alpha = \pi\mathbf{I} \qquad (3.15a)$$

and, by (2.28), the working has the intrinsic form

$$\mathcal{W}(R(t),t) = \int_{R(t)} (\mathbf{Sn}\cdot\mathbf{y}^\bullet + \mathbf{Mn}\cdot\nu) + \int_{\partial R(t)} \pi U_{\partial R} \qquad (3.16a)$$

The scalar field π is a configurational tension that works to increase the volume of $R(t)$ through the addition of material at its boundary. Referring to the final term in (3.16) as the configurational working, (3.16) may be stated more suggestively as boundary working equals defrrmational working plus configurational working. Note that the configurational working $\pi U_{\partial R(t)}$ is not due solely to the action of the configurational stress $\mathbf{C}$; the deformational stress contributes also through the term $(\mathbf{Sn}\cdot\mathbf{Fn})U_{\partial R}$.

Next, assuming that $R(t)$ does not contain the crack, and using (2.17) and (3.16), the inequality (3.13) may be rewritten as

$$\int\limits_{R(t)} (\psi + k + \kappa)^\bullet \leq \int\limits_{\partial R(t)} \{\mathbf{Sn} \cdot \mathbf{y}^\bullet + \mathbf{Mn} \cdot \nu + (\pi - \psi)U_{\partial R}.\} \quad (3.17a)$$

Given a time τ, it is possible to find a second referential control volume $R'(t)$ with $R'(t) = R(t)$, but with $U_{\partial R'}(\mathbf{X}, \tau)$, the normal velocity of $\partial R'(\tau)$, an arbitrary scalar field on $\partial R'(\tau)$; satisfaction of (3.17) for all such $U_{\partial R'}$ implies

$$\pi = \psi. \quad (3.18)$$

Therefore, configurational tension coincides with free energy, a result analogous to the coincidence of surface tension and surface free-energy; what is more important, (3.16) and (3.18) yield the Eshelby relation

$$\mathbf{C} = \psi\mathbf{I} - \mathbf{F}^T\mathbf{S} - \nabla\chi^T\mathcal{M} \quad (3.19)$$

for the configurational stress $\mathbf{C}$.

4   Conclusion

The derivation of the Eshelby relation was accomplished without recourse to constitutive equations or to a variational principle; the derivation was based on a version of the second law appropriate to referential control volumes whose boundaries evolve with time. The result (3.18) is a consequence of the invariance of $\mathcal{W}(R)$ under reparametrization; it is independent of the particular form chosen for the second law and is hence more basic than (1.4).

Acknowledgements

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Konfiguracione sile i spregovi za prostiranje prsline

UDK 531.01, 539.421

Sledeći pristup Gurtin-a and Podio-Guidugli-a (1998), razmatra se problem prostiranja prsline zasnovan na pojmu konfiguracionih sila i spregova u mikropolarnom kontinuumu.