THEORETICAL AND APPLIED MECHANICS
vol. 27, pp. 79-86, 2002

# Some geometrical aspects of Bianchi type-I space time

**G.Mohanty, R.C.Sahu, P.K.Sahoo**

Submitted 1 March, 2000

**Abstract**

A problem of spatially homogeneous Bianchi type-I space-time is investigated in Einstein theory without source of gravitation. Some geometrical natures of the space-time are discussed.

## 1 Introduction

Using the fundamental principle of Einstein space, Radojevic [1] derived solution for Bertotti-Robinson like space-time [2] and studied some geometrical aspects. Subsequently Mohanty and Pattanaik [3] studied some more geometrical aspects. In this paper we discuss mainly the geometrical natures of the Bianchi type-I space-time. We have evaluated Riemannian curvature tensor, Ricci tensor, Ricci scalar and derived solution to the Einstein's vacuum field equations. When the curvature of the Riemannian manifold $K \neq 0$, it is shown that the space-time does not represent the space of constant curvature [4] in general curved space. When $K = 0$, the space-time being a flat space represents trivially a space of constant curvature. The solution for the space-time Bianchi type-I to be a symmetric space is derived. Moreover it is found that the Bianchi type-I space-time does not represent a recurrent space in general [4].

## 2 Einstein's vacuum solution

We consider Bianchi type-I [5] metric in the form

$$ds^2 = -dt^2 + A^2dx^2 + B^2dy^2 + C^2dz^2, \tag{1}$$

where $A, B$ and $C$ are functions of cosmic time '$t$' which insures that the space-time is spatially homogeneous.

The nonzero components of Ricci tensor for metric (1) are

$$R_{11} = -A\left(A_{44} + \frac{A_4B_4}{B} + \frac{A_4C_4}{C}\right), \tag{2}$$

$$R_{22} = -B\left(B_{44} + \frac{B_4C_4}{C} + \frac{B_4A_4}{A}\right), \tag{3}$$

$$R_{33} = -C\left(C_{44} + \frac{C_4A_4}{A} + \frac{C_4B_4}{B}\right), \tag{4}$$

$$R_{44} = \frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{C_{44}}{C}, \tag{5}$$

where the suffix *4* denotes the exact derivative with respect to time.

Using (2)-(5), Ricci scalar $R$ can be evaluated as

$$R = -2\left[\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{A_4B_4}{AB} + \frac{A_4C_4}{AC} + \frac{B_4C_4}{BC}\right]. \tag{6}$$

In the empty space, the law of gravitation chosen by Einstein is $R_{ij} = 0$ and hence the Einstein vacuum field equations can be written in the following explicit form:

$$A_{44} + \frac{A_4B_4}{B} + \frac{A_4C_4}{C} = 0, \tag{7}$$

$$B_{44} + \frac{B_4C_4}{C} + \frac{B_4A_4}{A} = 0, \tag{8}$$

$$C_{44} + \frac{C_4 A_4}{A} + \frac{C_4 B_4}{B} = 0, \tag{9}$$

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{C_{44}}{C} = 0. \tag{10}$$

The solution to the above field equations (7)-(10) can be easily obtained as

$$A = (\alpha_1 t + \alpha_2)^{n1}\,, \;\; B = (\alpha_1 t + \alpha_2)^{n2}\,, \;\; C = (\alpha_1 t + \alpha_2)^{n3}\,. \tag{11-13}$$

where $\alpha_1(\neq 0)$ and $\alpha_2$ are nonzero constants of integration, $\sum_{i=1}^{3} n_i = 1$ and

$$\sum_{i,j=1}^{3} n_i n_j = 0, \; i \neq j.$$

The vacuum model for the metric (1) can be written as

$$ds^2 = -dt^2 + (\alpha_1 t + \alpha_2)^{2n_1} dx^2 + + (\alpha_1 t + \alpha_2)^{2n_2} dy^2 + (\alpha_1 t + \alpha_2)^{2n_3} dz^2. \tag{14}$$

With suitable coordinate transformations the above model can be reduced to the form

$$ds^2 = -dT^2 + T^{2n_1} dx^2 + T^{2n_2} dy^2 + T^{2n_3} dz^2. \tag{15}$$

For this model, we observe that

(i) the model is not asymptotically flat at infinite future as in the case of other symmetrical space times and

(ii) the model collapses at initial epoch.

# 3 Space of constant curvature

A space satisfying the condition

$$R_{hijk} = k(g_{hj} g_{ik} - g_{hk} g_{ij}) \tag{16}$$

is called a space of constant curvature. Here the constant $k$ denotes the curvature of Riemannian manifold.

For metric (1), eqn. (16) yields the following non-vanishing components.

$$R_{1212} = kA^2B^2, R_{1313} = kA^2B^2, R_{2323} = kB^2C^2 \qquad \text{(17a-17c)}$$

$$R_{1414} = -kA^2,\ R_{2424} = -kB^2,\ R_{3434} = -kC^2. \qquad \text{(17d-17f)}$$

The Riemannian curvature tensor

$$R_{hijk} = \frac{1}{2}\left[\frac{\partial^2 g_{hk}}{\partial x^i \partial x^j} + \frac{\partial^2 g_{ij}}{\partial x^h \partial x^k} - \frac{\partial^4 g_{ik}}{\partial x^h \partial x^j} - \frac{\partial^2 g_{hj}}{\partial x^i \partial x^k}\right] + g_{ab}\Gamma^a_{ij}\Gamma^b_{hk} - g_{ab}\Gamma^a_{ik}\Gamma^b_{hj} \qquad (18)$$

for the metric (1) yields the following non-vanishing components:

$$R_{1212} = AA_4BB_4,\ R_{1313} = AA_4CC_4,\ R_{2323} = BB_4BB_4, \qquad \text{(19a-19c)}$$

$$R_{1414} = -AA_{44},\ R_{2424} = -BB_{44},\ R_{3434} = -CC_{44}. \qquad \text{(19d-19f)}$$

Matching the corresponding components of (17) and (19) the value of $k$ can be obtained as

$$k = \frac{A_4B_4}{AB} + \frac{A_4C_4}{AC} + \frac{B_4C_4}{BC} = \frac{A_{44}}{A} = \frac{B_{44}}{B} = \frac{C_{44}}{C}. \qquad (20)$$

From (20), it is inferred that the space-time (1) is not a space of constant curvature in general curved space. Otherwise it leads to a flat space-time. Moreover if $A = B = C$, $K = k(t)$ which shows that the curvature varies with $t$. However it may be verified from the definition the space-time is not Einstein's space, which confirms the above result as regards to this geometrical nature.

# 4 Symmetric space

The symmetric spaces are defined by

$$R_{hijk;l} = 0 \qquad (21)$$

The semicolon ';' denotes the covariant differentiation with respect to $g_{ij}$. Hence one can find six distinct non-vanishing components of Riemannian curvature tensor for the metric (1). They are

$$R_{1212} = AA_4BB_4, \; R_{2323} = AA_4CC_4, \; R_{2323} = BB_4CC_4 \qquad (22\text{-}24)$$

$$R_{1414} = -AA_{44}, \; R_{2424} = -BB_{44}, \; R_{3434} = -CC_{44}. \qquad (25\text{-}27)$$

For the above six surviving components of $R_{hijk}$ for the metric (1), equation (21) yields

$$R_{1212;4} = \frac{\partial}{\partial x^4}(AA_4BB_4) - 2\left(\frac{A_4}{A} + \frac{B_4}{B}\right)AA_4BB_4 = 0, \qquad (28)$$

$$R_{1313;4} = \frac{\partial}{\partial x^4}(AA_4CC_4) - 2\left(\frac{A_4}{A} + \frac{C_4}{C}\right)AA_4CC_4 = 0, \qquad (29)$$

$$R_{2323;4} = \frac{\partial}{\partial x^4}(BB_4CC_4) - 2\left(\frac{B_4}{B} + \frac{C_4}{C}\right)BB_4CC_4 = 0, \qquad (30)$$

$$R_{1414;4} = \frac{\partial}{\partial x^4}(-AA_{44}) + 2A_4A_{44} = 0, \qquad (31)$$

$$R_{2424;4} = \frac{\partial}{\partial x^4}(-BB_{44}) + 2B_4B_{44} = 0, \qquad (32)$$

$$R_{3434;4} = \frac{\partial}{\partial x^4}(-CC_{44}) + 2C_4C_{44} = 0. \qquad (33)$$

Now we inferred that except above six equations i.e., (28)-(33) of $R_{hijk;l} = 0$, the space-time described by metric (1), is a symmetric space. We can design the metric (1) as a symmetric space by solving equations (28)-(33).

The first integral of eqn. (31) yields

$$AA_{44} = (A_4)^2 + \alpha_1, \qquad (34)$$

where $\alpha_1$ is a constant of integration.

In order to avoid the mathematical complexity for the integration of highly non-linear differential equation, we may take suitable constants of integration according to our choice. So choosing $\alpha_1 = 0$ and then integrating twice, we get

$$A = e^t. \tag{35}$$

Similarly from (32) & (33) we can get

$$B = e^t, \tag{36}$$

$$C = e^t. \tag{37}$$

With the above choice of metric potentials, we can confirm that the space-time (1) is a symmetric space.

## 5 Recurrent space

The recurrent spaces are defined by

$$R_{hijk;l} = K_l R_{hijk}, \tag{38}$$

where $(K_l)$ is the recurrent vector.

For $l = 1, 2, 3$ the covariant differentiations of the surviving components of $R_{hijk}$ are zero. Therefore $k_1 = k_2 = k_3 = 0$

For $l = 4$, using equations (22)-(27) in (38), the different values $k_4$ are obtained as:

$$k_4 = (AA_4BB_4)_4 - 2\left(\frac{A_4}{A} + \frac{B_4}{B}\right) AA_4BB_{44},$$

$$k_4 = (AA_4CC_4)_4 - 2\left(\frac{A_4}{A} + \frac{C_4}{C}\right) AA_4CC_{44},$$

$$k_4 = (BB_4CC_4)_4 - 2\left(\frac{B_4}{B} + \frac{C_4}{C}\right) BB_4CC_{44},$$

$$k_4 = (-AA_{44})_4 + 2A_4A_{44},$$

$$k_4 = (-BB_{44})_4 + 2B_4B_{44},$$

$$k_4 = (-CC_{44})_4 + 2C_4CC_{44}.$$

Here $k_4$ is unique only when $A = B = C = e^t$, then the space time (1) represents a recurrent space trivially as $k_4 = 0$. Hence the recurrent vector takes the form

$$K_l = (0, 0, 0, k_4) = (0, 0, 0, 0)$$

Thus Bianchi type-I space-time in general is not a recurrent space.

# 6 Conclusion

Here it is observed that the space-time described by the metric (1) is not a space of constant curvature and hence it is not an Einstein space. Moreover the space-time (1) becomes a symmetric space only when $A = B = C = e^t$. In this case the space-time is flat at initial epoch and admits a singularity at infinite future, which is similar to the behaviour of the flat space given by (15). However the flat space-time given by (15) is neither symmetric space nor recurrent space.

**Acknowledgment** The authors thank Prof. U.K Panigrahi, U.C.P., Engineering School, Berhampur, India, for his constant encouragement and interest in this work.

# References

[1] Radojevic, D.: One example of Einstein space, Teor. Prim. Meh. 13, (1988), 77-81.

[2] Godel, K: An example of a new type of cosmological solutions to Einstein's field equation of gravity, Rev. Mod. Phys. 21, (1949), 447-450.

[3] Mohanty, G and Pattanaik, R.R: Some geometrical aspects of Bertotti-Robinson like space-time, Teor. Prim. Meh. 16, 75-78. (1990).

[4] Willmore, T.J: An Introduction to Differential Geometry, Oxford University Press, Oxford, 236-38, 1978.

[5] Venkateswarlu, R and Reddy, D.R.K: Bianchi type-I models in self-creation theory of gravitation, Astrophys.& Space Sc. 168, (1990),193-199.

**G.Mohanty, P.K.Sahoo**
School of Mathematical Sciences
Sambalpur University, Jyotivihar
Burla,Sambalpur-768019
Orissa, INDIA

**R.C.Sahu,**
Department of Mathematics,
K.S.U.B. College, Bhanjanagar,
Ganjam, Orissa, INDIA

## Neki geometrijski aspekti prostor-vremena prvog Biankijevog tipa

UDK 530.12; 514.82

U radu je proučen problem prostorno homogenog prostor-vremena prvog Biankijevog tipa u Ajnštajnovoj teoriji bez gravitationog izvora. Neke geometrijske prirode ovog prostor-vremena se diskutuju.