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On the Elasticity Tensor of Third Order

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Abstract

It was shown how the invariant third-order elastic coefficients can be derived by the procedure proposed by Srinivasan and Nigam. Particularly, the invariant elastic constants for triclinic, orthorhombic, cubic and isotropic crystals have been considered.

1 Introduction

We shall remind the reader on the notion of isotropic tensor.
Definition. *A tensor is called isotropic if its components retain the same values (are unchanged) by any proper orthogonal transformation of rectangular Cartesian coordinates.*

There are no isotropic tensors of the first order. The isotropic tensors of second and third and higher order can be constructed only by tensors δ_{ij}, Kronecker delta, and e_{ijk}, Ricci tensor of alternation. Obviously, tensors

$$\delta_{i_1 i_2}\delta_{i_3 i_4}\dots\delta_{i_{r-1} i_r} \tag{1}$$

and

$$e_{i_1 i_2 i_3}\delta_{i_4 i_5}\dots\delta_{i_{r-1} i_r} \tag{2}$$

are of even and odd order, respectively. They are isotropic, as well as any of their *isomers*, i.e. tensors which differ from original one by the arrangements of its indices. Therefore, any linear combination of such isomers is an isotropic tensor. It can also be proved that any isotropic tensor can be represented by linear combination of some isomers, [1], [2]. For example,

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}, \tag{3}$$

and

$$\begin{aligned} c_{ijklmn} = {} & \lambda_1\delta_{ij}\delta_{kl}\delta_{mn} + \lambda_2\delta_{ij}\delta_{km}\delta_{ln} + \lambda_3\delta_{ij}\delta_{kn}\delta_{lm} + \lambda_4\delta_{ik}\delta_{jl}\delta_{mn} + \\ & + \lambda_5\delta_{ik}\delta_{jm}\delta_{ln} + \lambda_6\delta_{ik}\delta_{jn}\delta_{lm} + \lambda_7\delta_{il}\delta_{jk}\delta_{mn} + \lambda_8\delta_{il}\delta_{jm}\delta_{kn} + \\ & + \lambda_9\delta_{il}\delta_{jn}\delta_{km} + \lambda_{10}\delta_{im}\delta_{jk}\delta_{ln} + \lambda_{11}\delta_{im}\delta_{jl}\delta_{kn} + \lambda_{12}\delta_{im}\delta_{jn}\delta_{kl} + \\ & + \lambda_{13}\delta_{in}\delta_{jk}\delta_{lm} + \lambda_{14}\delta_{in}\delta_{jl}\delta_{km} + \lambda_{15}\delta_{in}\delta_{jm}\delta_{kl} \end{aligned} \tag{4}$$

are general forms of isotropic tensors of fourth and sixth order, respectively. Isotropic tensors of eight and higher even order can be constructed in the same way. However, in these cases their isomers are not mutually independent. More precisely, the number of independent isomers, , is less then the number of all possible their isomers

$$N_r = \frac{r!}{2^n n!}$$

where $r = 2n$. The following table illustrate it for some r.

r	$=$	2	4	6	8	10
N_r	$=$	1	3	15	105	945
L_r	$=$	1	3	15	91	603

In order to calculate the method of theory of group representation is used, [3]. Because of huge number of L_r, for practical purposes in continuum mechanics, we usually confine attention for $r = 2, 4, 6$. Particularly, when dealing with elasticity tensors of second and third order we make use of their symmetric properties

$$c_{ijkl} = c_{jikl} = c_{klij}, \tag{5}$$

$$c_{ijklmn} = c_{jiklmn} = c_{klijmn} = c_{ijk\ln m} = c_{ijmnkl}, \tag{6}$$

so that their representation become quite simple:

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{7}$$

$$
\begin{aligned}
c_{ijklmn} = {} & a\delta_{ij}\delta_{kl}\delta_{mn}+ \\
& +b(\delta_{ij}\delta_{km}\delta_{ln}+\delta_{ij}\delta_{kn}\delta_{lm}+\delta_{im}\delta_{kl}\delta_{jn}+ \\
& +\delta_{in}\delta_{kl}\delta_{jm}+\delta_{ik}\delta_{jl}\delta_{mn}+\delta_{il}\delta_{jk}\delta_{mn})+ \\
& +c(\delta_{ik}\delta_{jm}\delta_{ln}+\delta_{ik}\delta_{jn}\delta_{lm}+\delta_{il}\delta_{jm}\delta_{kn}+ \\
& +\delta_{i\ln}\delta_{jn}\delta_{km}+\delta_{im}\delta_{jk}\delta_{ln}+\delta_{im}\delta_{jl}\delta_{kn}+ \\
& +\delta_{in}\delta_{jl}\delta_{km}+\delta_{in}\delta_{jm}\delta_{kl})
\end{aligned} \tag{8}
$$

It easy to see that

$$
\lambda = c_{1122}, \qquad \mu = \frac{1}{2}(c_{1111} - c_{1122}) \tag{9}
$$

and

$$
\begin{aligned}
a &= c_{112233}, \quad b = \frac{1}{2}(c_{112222} - c_{112233}), \\
c &= \frac{1}{8}(c_{111111} + 2c_{112233} - 3c_{112222}).
\end{aligned} \tag{10}
$$

The constants λ, μ and a, b, c are invariant with respect to the choice of the coordinate system. It is, therefore, appropriate to call them *the universal constants* for the isotropic materials. Contrary to them, elastic constants c_{ijkl} (or c_{ijklmn}) change their values under arbitrary orthogonal coordinate transformations; also the number of constants, required to specify the elastic property of a crystal changes from coordinate system to coordinate system. These two aspects are rather severe handicap in the treatment of various problems and may explain, in part at least, why the theory of cubic crystals in the elastic domain did not enjoy a development comparable to that of the classical or isotropic theory of elasticity. Because of that we have been for a long time in need to find universal constants for all crystal classes. It was Thomas [4] who obtained invariant constants λ, μ, α for cubic crystals similar to Lame's λ, μ for isotropic solids. He has proposed the following expression

$$
c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \alpha\nu_{ai}\nu_{aj}\nu_{ak}\nu_{al}, \tag{11}
$$

where ν_{ai} are the components of the unit vectors ν_a $(a = 1, 2, 3)$ which represent the crystallographic directions of cubic crystal in an arbitrary Cartesian system (The crystallographic axes represent in direction and magnitude the three non-parallel edges of the unit cell of a crystal.

The unit vectors along these axes are referred to as crystallographic directions [5]). Srinivasan and Nigam [6], proposed a procedure how to derive invariant constants of C_{ijkl} for all other crystal classes. The procedure is, as suggested by J.L. Synge, in some sense, based on the representation of tensors in anholonomic coordinate systems. For simplicity, the unit vectors $\mathbf{n}_a$ $(a = 1, 2, 3)$ along crystallographic directions are chosen as anholonomic basis. Generally, they are not orthogonal. In order to make this manuscript self-contained we proceed in explaining this procedure.

Given a vector ν. Then

$$\nu = \nu_i \mathbf{e}_i = w_a \mathbf{n}_a;$$

$\mathbf{e}_i$ and $\mathbf{n}_a$ $(i, a = 1, 2, 3)$ are two systems of basis vectors, respectively. Usually we take $\mathbf{e}_i$ orthonormal. Then

$$\nu_i = n_{ai} w_a, \qquad n_{ai} = \mathbf{n}_a \cdot \mathbf{e}_i, \tag{12}$$

where there is summation over a; a is not tensor index.

Let $\mathbf{m}_a$ be reciprocal basis to the basis vectors $\mathbf{n}_a$. Then

$$\mathbf{n}_a \cdot \mathbf{m}_b = \delta_{ab} \quad \Rightarrow \quad n_{ai} m_{bi} = \delta_{ab}. \tag{13}$$

Note that there is no distinction between *contravariant* and *covariant* indices since we are working in Cartesian frames of references. Then, from (12), we obtain

$$w_a = m_{ai} \nu_i. \tag{14}$$

But w_a do not depend on the choice of coordinate system with respect to the indices i. Therefore w_a are invariant and behave as a scalars with respect to any such coordinate transformation. The same approach can be applied to any tensor. T.P. Srinivasan & S.D. Nigam stated that this idea can be useful in finding the invariant dielectric constants, piezoelectric and photo-elastic coefficients. Because of that they confine they application of the procedure to the tensors of second, third and fourth order.

We shall consider the elasticity tensor C_{ijklmn} as a special case of a tensor of sixth order.

2 Invariant Elastic Constants of Triclinic Crystals

We write

$$C_{ijklmn} = n_{ai} n_{bj} n_{ck} n_{dl} n_{em} n_{fn} A_{abcdef}. \tag{15}$$

This relation can be inverted making use of (13) so that

$$A_{abcdef} = m_{ai} m_{bj} m_{ck} m_{dl} m_{em} m_{fn} C_{ijklmn}. \tag{16}$$

From (16) it can be seen that A_{abcdef} possesses the same symmetric properties as C_{ijklmn} defined by (6). In above form the scalars A_{abcdef} are the 56 invariant elastic constants for the *triclinic crystal* (no *axes* or *plane of symmetry*).

At this point, we wish to discuss the so called "matrix notation", which is widely used in the theory of elasticity. The matrix notation introduced by Voigt [6] can be defined by transformation which maps an index pair ij into a single index α, i.e. $ij \Leftrightarrow \alpha$, according to the rule: $11 \Leftrightarrow 1$, $22 \Leftrightarrow 2$, $33 \Leftrightarrow 3$, $23 = 32 \Leftrightarrow 4$, $13 = 31 \Leftrightarrow 5$, $12 = 21 \Leftrightarrow 6$. Using this transformation, we define symmetric "matrix", where α, β, γ are the images, respectively, of ij, kl, mn. The same mapping can be applied to $A_{abcdef} \Rightarrow A_{\alpha\beta\gamma}$ which enable us to write all 56 independent constants of $A_{\alpha\beta\gamma}$. It is convenient to write them in the following way:

$$\left.\begin{array}{cccccc} A_{111} & A_{112} & A_{113} & A_{114} & A_{115} & A_{116} \\ & A_{122} & A_{123} & A_{124} & A_{125} & A_{126} \\ & & A_{133} & A_{134} & A_{135} & A_{136} \\ & & & A_{144} & A_{145} & A_{146} \\ & & & & A_{155} & A_{156} \\ & & & & & A_{166} \end{array}\right\}, \quad N_1 = 21$$

$$\left.\begin{array}{ccccc} A_{222} & A_{223} & A_{224} & A_{225} & A_{226} \\ & A_{233} & A_{234} & A_{235} & A_{236} \\ & & A_{244} & A_{245} & A_{246} \\ & & & A_{255} & A_{256} \\ & & & & A_{266} \end{array}\right\}, \quad N_2 = 15$$

$$\left.\begin{array}{cccc} A_{333} & A_{334} & A_{335} & A_{336} \\ & A_{344} & A_{345} & A_{346} \\ & & A_{355} & A_{356} \\ & & & A_{366} \end{array}\right\}, \quad N_3 = 10$$

$$\begin{array}{ccc} A_{444} & A_{445} & A_{446} \\ & A_{455} & A_{456} \\ & & A_{466} \end{array} \quad N_4 = 6$$

$$\left.\begin{array}{cc} A_{555} & A_{556} \\ & A_{566} \end{array}\right\}, \quad N_5 = 3$$

$$\left.A_{666}\right\}, \quad N_6 = 1;$$

N_i, $(n = 1, 2, 3, 4, 5, 6)$ represents the number of independent components of $A_{\alpha\beta\gamma}$ for corresponding set in the above expressions.

Note the rule for writing each of the above sets. First set represents all $A_{1\alpha\beta} = A_{1\beta\alpha}$, $(\alpha, \beta = 1, 2, 3, 4, 5, 6)$; second set represents all $A_{2\alpha\beta} = A_{2\beta\alpha}$, $(\alpha, \beta = 2, 3, 4, 5, 6)$; third all $A_{3\alpha\beta} = A_{3\beta\alpha}$, $(\alpha, \beta = 3, 4, 5, 6)$ and so on.

From these expressions it is possible to obtain expressions in the case of crystals belonging to other systems by imposing the various point group symmetries on C_{ijklmn}. This is done by keeping the coordinates of C_{ijklmn} *unchanged* and *transforming only the vectors* $\mathbf{n}_a$ according to the symmetries present in the crystal whereas in dealing with elastic symmetry it has been customary to transform coordinates.

We shall illustrate it in case of orthorhombic and cubic crystals.

3 Invariant Elastic Constants of Orthorhombic Crystals

In orthorhombic crystals $\mathbf{n}_a$ $(a = 1, 2, 3)$ are orthonormal and each one of them is a two-fold axes of rotation. The atomic array is unchanged by inversion about the plane defined by $\mathbf{n}_a$; this will take $\mathbf{n}_a \Rightarrow -\mathbf{n}_a$.

i. One plane of symmetry defined by unit normal vector $\mathbf{n}_1$

The invariance of C_{ijklmn}, i.e. $C_{\alpha\beta\gamma}$, in (15) under the inversion $\mathbf{n}_1 \Rightarrow -\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$, unchanged, leads to following table of non-vanishing constants $A_{\alpha\beta\gamma}$:

$$
\begin{array}{cccccc}
A_{111} & A_{112} & A_{113} & A_{114} & \bullet & \bullet \\
 & A_{122} & A_{123} & A_{124} & \bullet & \bullet \\
 & & A_{133} & A_{134} & \bullet & \bullet \\
 & & & A_{144} & \bullet & \bullet \\
 & & & & A_{155} & A_{156} \\
 & & & & & A_{166}
\end{array}
$$

$$
\begin{array}{ccccc}
A_{222} & A_{223} & A_{224} & \bullet & \bullet \\
 & A_{233} & A_{234} & \bullet & \bullet \\
 & & A_{244} & \bullet & \bullet \\
 & & & A_{255} & A_{256} \\
 & & & & A_{266}
\end{array}
$$

$$
\begin{array}{cccc}
A_{333} & A_{334} & \bullet & \bullet \\
 & A_{344} & \bullet & \bullet \\
 & & A_{355} & A_{356} \\
 & & & A_{366}
\end{array}
$$

$$
\begin{array}{ccc}
A_{444} & \bullet & \bullet \\
 & A_{455} & A_{456} \\
 & & A_{466}
\end{array}
$$

$$
\begin{array}{cc}
\bullet & \bullet \\
 & \bullet
\end{array}
$$

$$
\bullet\ .
$$

Here the black points denote the corresponding vanishing constants, i.e. the constants which contain odd number of index 1.

ii. Two (three) orthogonal planes of elastic symmetry defined by unit normal vectors $\mathbf{n}_1$ *and* $\mathbf{n}_2$

Here we make use of the following very simple theorems and lemmas.

If an anisotropic elastic material possesses a material symmetry with the orthogonal matrix $\mathbf{Q}$*, it possesses the material symmetry with* $\mathbf{Q}^T = \mathbf{Q}^{-1}$*.*

If an anisotropic elastic material possesses a material symmetry with $\mathbf{Q}_1$ *and* $\mathbf{Q}_2$*, it possesses a material symmetry with respect to* $\mathbf{Q} = \mathbf{Q}_1\mathbf{Q}_2$.

The proves of these theorems can be find in [7].

Lemma 1.*A tensor of even order invariant under* $\mathbf{Q}$ *is also invariant under* $-\mathbf{Q}$.

Proof. Let $C_{i_1\ldots i_p}$ be invariant under $\mathbf{Q}$. Than

$$C_{i_1\ldots i_p} = Q_{i_1j_1}Q_{i_2j_2}\ldots Q_{i_pj_p}C_{j_1\ldots j_p}.$$

But,

$$\begin{aligned}(-Q_{i_1j_1})(-Q_{i_2j_2})\ldots(-Q_{i_pj_p})C_{j_1\ldots j_p} &= \\ &= (-1)^pQ_{i_1j_1}Q_{i_2j_2}\ldots Q_{i_pj_p}C_{j_1\ldots j_p} = \\ &= C_{i_1\ldots i_p}\end{aligned}$$

for $p = 2k$.

Lemma 2. *If a tensor of even order possesses two orthogonal planes of symmetry than the plane orthogonal to them is also the plane of symmetry.*

Proof. Let $\mathbf{n}_1, \mathbf{n}_2$ and $\mathbf{n}_3$ are orthonormal vectors. Let

$$\mathbf{Q}_i = \mathbf{I} - 2\mathbf{n}_i \otimes \mathbf{n}_i, \quad i = 1, 2, 3.$$

If $\mathbf{n}_1$ and $\mathbf{n}_2$ are the unite vectors of two planes of symmetry, than $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are corresponding orthogonal tensors of symmetry. Obviously

$$\mathbf{Q}_1\mathbf{Q}_2 = \mathbf{n}_3 \otimes \mathbf{n}_3 - \mathbf{I} = -\mathbf{Q}_3,$$

since

$$\mathbf{n}_1 \otimes \mathbf{n}_1 + \mathbf{n}_2 \otimes \mathbf{n}_2 + \mathbf{n}_3 \otimes \mathbf{n}_3 = \mathbf{I}.$$

Then by Lemma 1 and Theorem 2 $\mathbf{Q}_3$ is also tensor of symmetry, and thus $\mathbf{n}_3$ defines also plane of symmetry.

Then the value of constants $A_{\alpha\beta\gamma}$ are given in the following table:

$$\begin{array}{cccccc}
A_{111} & A_{112} & A_{113} & \bullet & \bullet & \bullet \\
 & A_{122} & A_{123} & \bullet & \bullet & \bullet \\
 & & A_{133} & \bullet & \bullet & \bullet \\
 & & & A_{144} & \bullet & \bullet \\
 & & & & A_{155} & \bullet \\
 & & & & & A_{166}
\end{array}$$

$$
\begin{array}{ccccc}
A_{222} & A_{223} & \bullet & \bullet & \bullet \\
 & A_{233} & \bullet & \bullet & \bullet \\
 & & A_{244} & \bullet & \bullet \\
 & & & A_{255} & \bullet \\
 & & & & A_{266}
\end{array}
$$

$$
\begin{array}{cccc}
A_{333} & \bullet & \bullet & \bullet \\
 & A_{344} & \bullet & \bullet \\
 & & A_{355} & \bullet \\
 & & & A_{366}
\end{array}
$$

$$
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
 & \bullet & A_{456} \\
 & & \bullet
\end{array}
$$

$$
\begin{array}{cc}
\bullet & \bullet \\
 & \bullet
\end{array}
$$

$$
\bullet
$$

Therefore, tensor of elasticity of third order for orthorhombic crystals has 20 invariant constants.

4 Invariant Elastic Constants of Cubic Crystals

The invariant constants of cubic crystals can be obtained from those corresponding to orthorhombic crystals by imposing the symmetric condition that C_{ijklmn} for orthorhombic crystals should be invariant under $120°$ rotation about the axis which is equally inclined to $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$. Such an operation will take $\mathbf{n}_1$ to $\mathbf{n}_2$, $\mathbf{n}_2$ to $\mathbf{n}_3$ and $\mathbf{n}_3$ to $\mathbf{n}_1$ and vice versa. Also each of $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ defines a two-fold axes of rotation. Hence, a rotation of $180°$ about $\mathbf{n}_3$ will take $\mathbf{n}_1$ to $\mathbf{n}_2$ and $\mathbf{n}_2$ to $-\mathbf{n}_1$, and so on. Again, from physical point of view it means that such operation will not alter the atomic array in cubic crystal. The requirement that C_{ijklmn} should be invariant under these symmetry operations leads to the conclusion that invariant coefficients of orthorhombic crystal with these changes of indices in the above table should be equal, i.e.

$$\begin{gathered}
A_{111} = A_{222} = A_{333}, \\
A_{112} = A_{113} = A_{122} = A_{133} = A_{223} = A_{233}, \\
A_{144} = A_{255} = A_{366}, \\
A_{155} = A_{166} = A_{244} = A_{266} = A_{344} = A_{355}, \\
A_{123}, \\
A_{456}.
\end{gathered} \tag{17}$$

It is of interest to write the table of invariant constants for cubic crystals:

$$\begin{array}{cccccc}
A_{111} & A_{112} & A_{112} & \bullet & \bullet & \bullet \\
 & A_{112} & A_{123} & \bullet & \bullet & \bullet \\
 & & A_{112} & \bullet & \bullet & \bullet \\
 & & & A_{144} & \bullet & \bullet \\
 & & & & A_{155} & \bullet \\
 & & & & & A_{155}
\end{array}$$

$$\begin{array}{ccccc}
A_{111} & A_{112} & \bullet & \bullet & \bullet \\
 & A_{112} & \bullet & \bullet & \bullet \\
 & & A_{155} & \bullet & \bullet \\
 & & & A_{144} & \bullet \\
 & & & & A_{155}
\end{array}$$

$$\begin{array}{cccc}
A_{111} & \bullet & \bullet & \bullet \\
 & A_{155} & \bullet & \bullet \\
 & & A_{155} & \bullet \\
 & & & A_{144}
\end{array}$$

$$\begin{array}{ccc}
\bullet & \bullet & \bullet \\
 & \bullet & A_{456} \\
 & & \bullet
\end{array}$$

$$\begin{array}{cc}
\bullet & \bullet \\
 & \bullet
\end{array}$$

$$\bullet$$

This is all we can obtain for the invariant coefficients of cubic crystal. Obviously, tensor of elasticity of third order for cubic crystal has 6 invariant constants.

5 Compact form of the tensor of elasticity of third order for cubic crystals

Once we know the invariant coefficients of elasticity tensor we can write compact form of the tensor of elasticity of third order. It can be done, formally as we do for the tensor of elasticity of second order, by collecting terms in (15) for corresponding invariant coefficients. Generally, so obtained expression for elasticity tensor of third order is very long and complicate because of a huge number of terms. This is the main difference between elasticity tensors of second and third order. In fact, only those terms which correspond to the invariant coefficients equal zero do not appear in the final form. All others have to be taken into account. But, in some cases, as it was seen for isotropic elastic tensor given by (8), these expressions can be quite simple. We shall proceed to show how such a form can be deduced for the tensor of elasticity of third order for cubic crystals.

First we introduce the following quantities:

$$M_{aijklmn} = n_{ai}n_{aj}n_{ak}n_{al}n_{am}n_{an} \quad \text{(no sum over } a\text{)},\ a = 1,2,3, \tag{18}$$

$$M_{ijklmn} = \sum_{a=1}^{3} n_{ai}n_{aj}n_{ak}n_{al}n_{am}n_{an}; \tag{19}$$

$$N_{aijkl} = n_{ai}n_{aj}n_{ak}n_{al} \quad \text{(no sum over } a\text{)}, \tag{20}$$

$$N_{ijklmn} = \sum_{a=1}^{3} n_{ai}n_{aj}n_{ak}n_{al}; \tag{21}$$

$$n_{aij} = n_{ai}n_{aj}, \quad \text{(no sum over } a\text{)}, \tag{22}$$

$$\delta_{ij} = \sum_{a=1}^{3} n_{ai}n_{aj}, \quad \delta_{ab} = \sum_{i=1}^{3} n_{ai}n_{bi}. \tag{23}$$

Note that $M_{aijklmn}$, M_{ijklmn}, N_{aijkl} and N_{ijkl} are tensors of even order symmetric with respect to all of their indices. Also, if we denote by $\mathfrak{g}$ the isotropy group of cubic crystals, then

$$n_{bi} = Q_{ij}n_{aj}, \quad a,b = 1,2,3 \tag{24}$$

for any orthogonal $\mathbf{Q}(Q_{ij}) \in \mathfrak{g}$. Because of that, $\mathbb{M}(M_{ijklmn})$ and $\mathbb{N}(N_{ijkl})$ are invariant under $\mathfrak{g}$, i.e. $\mathfrak{g}\mathbb{M} = \mathbb{M}$ and $\mathfrak{g}\mathbb{N} = \mathbb{N}$. We will see that these quantities form the basis of for the tensor of elasticity of third order for cubic crystals.

In order to show this we have to use the symmetry conditions given by (17). We start with conditions $(17)_1$. Obviously $A_{111} \Rightarrow A_{111111}$. Its coefficient is $n_{1i}n_{1j}n_{1k}n_{1l}n_{1m}n_{1n} = M_{1ijklmn}$. In the same way we conclude: $A_{222} \Rightarrow A_{222222}$ with corresponding coefficient $M_{2ijklmn}$; $A_{333} \Rightarrow A_{333333}$ and $M_{3ijklmn}$. But $A_{111} = A_{222} = A_{333}$ so that

$$M_{ijklmn} = \sum_{a=1}^{3} M_{aijklmn} \tag{25}$$

is their common coefficient.

Next we consider $(17)_2$. We shall write all terms for A_{112}. Then

$$A_{112} \Rightarrow N_{1ijkl}n_{2mn}, \quad A_{121} \Rightarrow N_{1ijmn}n_{2kl}, \quad A_{211} \Rightarrow N_{1klmn}n_{2ij}. \tag{26}$$

In the same way we can write the corresponding terms for each of A_{113}, A_{122}, A_{133}, A_{223}, A_{233}. In all we have 18 terms for $(17)_2$. After lengthy algebra, making use of (18)-(23), they can be written as

$$\delta_{ij}N_{klmn} + \delta_{kl}N_{ijmn} + \delta_{mn}N_{ijkl} - 3M_{ijklmn}. \tag{27}$$

Further we consider only term

$$\delta_{ij}N_{klmn} + \delta_{kl}N_{ijmn} + \delta_{mn}N_{ijkl} \tag{28}$$

since last term in (27) is already given by (25). The form of this term suggest us the term

$$\begin{aligned} &\delta_{ik}N_{jlmn} + \delta_{il}N_{jkmn} + \delta_{im}N_{jkln} + \delta_{in}N_{jklm}+ \\ &\delta_{jk}N_{ilmn} + \delta_{jl}N_{ikmn} + \delta_{jm}N_{ikln} + \delta_{jn}N_{iklm}+ \\ &\delta_{km}N_{ijln} + \delta_{kn}N_{ijlm} + \delta_{lm}N_{ijkn} + \delta_{ln}N_{ijkm} \end{aligned} \tag{29}$$

which is invariant under $\mathfrak{g}$ and has symmetric properties given by (6). These requirements are obviously satisfied by (8) which already has three invariant coefficients: a, b and c. Then (8), (25), (28) and (29) form the basis for the tensor of elasticity of third order for cubic crystals. Thus

$$\begin{aligned}
c_{ijklmn} = {} & \lambda\delta_{ij}\delta_{kl}\delta_{mn} + \\
& \mu\left(\delta_{ij}\delta_{km}\delta_{ln} + \delta_{ij}\delta_{kn}\delta_{lm} + \delta_{im}\delta_{kl}\delta_{jn} + \right. \\
& \left.\delta_{in}\delta_{kl}\delta_{jm} + \delta_{ik}\delta_{jl}\delta_{mn} + \delta_{il}\delta_{jk}\delta_{mn}\right) + \\
& \nu\left(\delta_{ik}\delta_{jm}\delta_{ln} + \delta_{ik}\delta_{jn}\delta_{lm} + \delta_{il}\delta_{jm}\delta_{kn} + \right. \\
& \delta_{il}\delta_{km}\delta_{jn} + \delta_{im}\delta_{jk}\delta_{ln} + \delta_{im}\delta_{jl}\delta_{kn} + \\
& \left.\delta_{in}\delta_{km}\delta_{jl} + \delta_{in}\delta_{jm}\delta_{kl}\right) + \\
& \alpha M_{ijklmn} + \\
& \beta\left(\delta_{ij}N_{klmn} + \delta_{kl}N_{ijmn} + \delta_{mn}N_{ijkl}\right) + \\
& \gamma\left(\delta_{ik}N_{jlmn} + \delta_{il}N_{jkmn} + \delta_{im}N_{jkln} + \delta_{in}N_{jklm} + + \right. \\
& \delta_{jk}N_{ilmn} + \delta_{jl}N_{ikmn} + \delta_{jm}N_{ikln} + \delta_{jn}N_{iklm} + \\
& \left.\delta_{km}N_{ijln} + \delta_{kn}N_{ijlm} + \delta_{lm}N_{ijkn} + \delta_{ln}N_{ijkm}\right)
\end{aligned} \tag{30}$$

is required form.

The reader will note that we did not consider terms of invariant coefficients $(17)_{3-6}$. We shorten our investigation defining the term given by (29) which completes the bases. However, if we do this investigation for $(17)_5$ we will obtain 6 terms:

$$\begin{aligned}
A_{123} &\Rightarrow n_{1ij}n_{2kl}n_{3mn} & A_{132} &\Rightarrow n_{1ij}n_{3kl}n_{2mn} \\
A_{213} &\Rightarrow n_{2ij}n_{1kl}n_{3mn} & A_{231} &\Rightarrow n_{2ij}n_{3kl}n_{1mn} \\
A_{312} &\Rightarrow n_{3ij}n_{1kl}n_{2mn} & A_{321} &\Rightarrow n_{3ij}n_{2kl}n_{1mn}
\end{aligned}$$

or, because of $(17)_5$, common term

$$\begin{aligned}
& n_{1ij}n_{2kl}n_{3mn} + n_{1ij}n_{3kl}n_{2mn} + n_{2ij}n_{1kl}n_{3mn} + \\
& + n_{2ij}n_{3kl}n_{1mn} + n_{3ij}n_{1kl}n_{2mn} + n_{3ij}n_{2kl}n_{1mn}.
\end{aligned} \tag{31}$$

Then we can write c_{ijklmn} in compact form with (31) instead of (29). The other terms basis are the same. Thus,

$$\begin{aligned}
c_{ijklmn} = {} & \lambda_1\delta_{ij}\delta_{kl}\delta_{mn} + \lambda_2\left(\delta_{ij}\delta_{km}\delta_{ln} + \delta_{ij}\delta_{kn}\delta_{lm} + \right. \\
& \left. + \delta_{im}\delta_{kl}\delta_{jn} + \delta_{in}\delta_{kl}\delta_{jm} + \delta_{ik}\delta_{jl}\delta_{mn} + \delta_{il}\delta_{jk}\delta_{mn}\right) + \\
& \lambda_3\left(\delta_{ik}\delta_{jm}\delta_{ln} + \delta_{ik}\delta_{jn}\delta_{lm} + \delta_{il}\delta_{jm}\delta_{kn} + \delta_{il}\delta_{km}\delta_{jn} + \right. \\
& \left.\delta_{im}\delta_{jk}\delta_{ln} + \delta_{im}\delta_{jl}\delta_{kn} + \delta_{in}\delta_{km}\delta_{jl} + \delta_{in}\delta_{jm}\delta_{kl}\right) + \\
& \lambda_4 M_{ijklmn} + \lambda_5\left(\delta_{ij}N_{klmn} + \delta_{kl}N_{ijmn} + \delta_{mn}N_{ijkl}\right) + \\
& \lambda_6\left(n_{1ij}n_{2kl}n_{3mn} + n_{1ij}n_{3kl}n_{2mn} + n_{2ij}n_{1kl}n_{3mn} + \right. \\
& \left. n_{2ij}n_{3kl}n_{1mn} + n_{3ij}n_{1kl}n_{2mn} + n_{3ij}n_{2kl}n_{1mn}\right)
\end{aligned} \tag{32}$$

(see V.A. Lubarda [11]). Of course, the values of constants in (30) and (32) are different.

6 Conclusion

The determination of third-order coefficients of crystals is of general interest, since they allow the evaluation of anharmonic properties such as thermal expansion and the interaction of thermal and acoustic phonons. For their evaluation, usually, the experimental technique involves measurements of the wave speeds of various small amplitude elastic modes in crystals under hydrostatic or uniaxial stresses, [8]. Here we consider another problem of a third-order coefficients of crystals: their invariance. The approach is simple an differs from so far used approaches. To see the differences it is advisable to consult the literature on this subject (see, for instance, [8], [9], [10], [11]). It is one of the reasons why we have confined our investigation to some classes of crystals, i.e. to triclinic, orthorhombic, cubic and isotropic materials.

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O tenzoru elastičnosti trećeg reda

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Pokazano je kako se invarijantni koeficijenti tenzora elastičnosti trećeg reda mogu odrediti na osnovu postupka Srinivasana i Nigama. Posebno su razmatrane invarijantne elastične konstante za triklinične, ortorombične, kubne i izotropne kristale.