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Two examples of pure radiation fields

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Abstract

We investigate two Riemannian metrics. Each metric is expressed through two functions, one of them depending on x^1 only and the other depending on x^4, the time, only. We compose the Ricci tensor. The functions composing the metric tensor are determined from the condition that the form of Ricci tensor corresponds to the energy-momentum tensor of the pure radiation field. This is the special case of the electromagnetic field in which the electric and magnetic three-vectors are equal and perpendicular.

1 Introduction

A) In this paper we shall to investigate a Riemannian metric in order to find out whether it could represent some cosmological model. We propose a metric

$$ds^2 = (dx^1)^2 + (dx^2)^2 + 2\,k\,f\,q\,dx^2\,dx^3 + (dx^3)^2 - (dx^4)^2 \qquad (1)$$

We suppose that f depends on x^1 only and q depends on x^4, the coordinate time, only, and k is a constant. We present the metric tensor in the matrix form

$$\{g_{\alpha\beta}\} = \left\{ \begin{array}{llll} 1 & 0 & 0 & 0 \\ 0 & 1 & kfq & 0 \\ 0 & kfq & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right\}$$

We shall calculate the Ricci tensor. It will be expressed by the functions appearing in the metric tensor. We shall determine these functions so that the form of the Ricci tensor correspond to the form of some known energy-momentum tensor. We present the components of the Ricci tensor different from zero. In the following expressions primes denote differentiation with respect to x^1, and the dots denote the differentiation with respect to x^4.

$$R_{11} = \frac{k^2 q^2}{2(k^2 f^2 q^2 - 1)^2} \left\{ 2(k^2 f^2 q^2 - 1) f f'' - (k^2 f^2 q^2 + 1)(f')^2 \right\}$$

$$R_{14} = \frac{k^2}{2(k^2 f^2 q^2 - 1)^2} \left\{ (k^2 f^2 q^2 - 3) f f' q \dot{q} \right\}$$

$$R_{22} = \frac{-k^2}{2(k^2 f^2 q^2 - 1)} \left[f^2 (\dot{q})^2 - q^2 (f')^2 \right] \tag{2}$$

$$R_{23} = -\frac{k}{2} \left[f \ddot{q} - q f'' \right]$$

$$R_{33} = \frac{-k^2}{2(k^2 f^2 q^2 - 1)} \left[f^2 (\dot{q})^2 - q^2 (f')^2 \right]$$

$$R_{44} = \frac{k^2 f^2}{2(k^2 f^2 q^2 - 1)^2} \left\{ 2(k^2 f^2 q^2 - 1)^2 q \ddot{q} - (k^2 f^2 q^2 + 1)(\dot{q})^2 \right\}$$

In order to have a better insight of Ricci tensor we shall present it in the form of matrix

$$\{R_{\alpha\beta}\} = \left\{ \begin{array}{llll} R_{11} & 0 & 0 & R_{14} \\ 0 & R_{22} & R_{23} & 0 \\ 0 & R_{32} & R_{33} & 0 \\ R_{41} & 0 & 0 & R_{44} \end{array} \right\}$$

To get the Ricci tensor which corresponds to a (part of) energy-momentum tensor of the perfect fluid or of the pure radiation field, the necessary condition is that R_{22}, R_{23} and R_{33} become zero. In our case $R_{22} = R_{33}$ and we get the system

$$\frac{(\dot{q})^2}{q^2} - \frac{(f')^2}{f^2} = 0$$

$$\frac{\ddot{q}}{q} - \frac{f''}{f} = 0$$

The solution of the system is

$$f = e^{\nu\, x^1}, \quad q = e^{\nu\, x^4}, \quad \nu = const$$

We obtained both unknown functions from the necessary condition. Putting this solution in the components of the Ricci tensor we obtain

$$R_{11} = R_{14} = R_{44} = \frac{k^2}{2} \frac{k^2\, f^2\, q^2 - 3}{(k^2\, f^2\, q^2 - 1)^2}\, \nu^2\ e^{2\nu\,(x^1 + x^4)}$$

Now we test whether these components of the Ricci tensor could correspond to (a part) of energy-momentum tensor of the perfect fluid. The density could be presented by

$$\rho = \frac{k^2\, \nu^2 (k^2\, f^2\, q^2 - 3)}{2\, (k^2\, f^2\, q^2 - 1)^2}, \quad k > \sqrt{3}$$

The condition $k > \sqrt{3}$ provides $\rho > 0$. We choose the fourvelocity to be expressed by

$$u_\alpha = e^{\nu\,(x^1 + x^4)} \{1, 0, 0, 1\}$$

The four-velocity is a time-like vector and the vector u_α, we have chosen to present the fourvelocity, is a null vector. But, when dealing with a null vector, energy-momentum tensor represents the pure radiation field. The pure radiation field is the special case of the electromagnetic field for which the electric and magnetic three-vectors are equal and perpendicular [1], [2]. Now we shall modify our interpretation of the

Ricci tensor components in order to obtain the energy-momentum tensor of a pure radiation field. The energy-momentum tensor of a pure radiation field is of the form

$$T_{\alpha\beta} = \Phi^2 \, v_\alpha v_\beta$$

where v_α is a null vector and $\Phi^2 = \frac{e^2+h^2}{2}$, $\vec{e}$ and $\vec{h}$ are three-vectors of electric and magnetic field. It is easy to see that, in the considered case, there is a null vector

$$v_\alpha = e^{\nu(x^1+x^4)}\{1,0,0,1\}$$

and the corresponding

$$\Phi^2 = \frac{k^2\,\nu^2(k^2\,f^2\,q^2-3)}{2\,(k^2\,f^2\,q^2-1)^2}, \quad k>\sqrt{3}$$

In such a way we got the metric representing the pure radiation field:

$$ds^2 = (dx^1)^2 + (dx^2)^2 + 2\,k\,e^{\nu\,(x^1+x^4)}\,dx^2\,dx^3 + (dx^3)^2 - (dx^4)^2$$

B) Let us consider one more "similar" metric. Namely, we propose the metric

$$ds^2 = (dx^1)^2 + (dx^2)^2 + 2\,a\,f\,q\,dx^2 dx^3 + n\,f^2\,q^2 (dx^3)^2 - (dx^4)^2$$

We suppose f and q as in previous case. The matrix of the metric tensor is of form

$$\{g_{\alpha\beta}\} = \left\{ \begin{array}{llll} 1 & 0 & 0 & 0 \\ 0 & 1 & afq & 0 \\ 0 & afq & nf^2q^2 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right\}$$

We calculate the Ricci tensor like in the previous example, and present the components different from zero

$$R_{11} = \frac{1}{2(a^2-n)}\left\{2(a^2-n)ff'' - a^2(f')^2\right\}$$

$$R_{14} = \frac{1}{2(a^2-n)fq}(a^2-n)f'\dot{q}$$

$$R_{22} = \frac{-a^2}{2(a^2-n)f^2q^2}\left[f^2(\dot{q})^2 - q^2(f')^2\right]$$

$$R_{23} = \frac{-a}{2(a^2-n)fq}\left[(n-a^2)q^2ff'' - f^2q^2(f')^2 + (a^2-n)f^2q\ddot{q} + nf^2(\dot{q})^2\right] \tag{3}$$

$$R_{33} = \frac{-n}{2(a^2-n)}\left[2(n-a^2)q^2ff'' - a^2q^2(f')^2 + 2(a^2-n)f^2q\ddot{q} + a^2f^2(\dot{q})^2\right]$$

$$R_{44} = \frac{1}{2(a^2-n)}\left\{2(a^2-n)q\ddot{q} - a^2(\dot{q})^2\right\}$$

We obtained the same structure of the Ricci tensor as in the previous example. Setting the condition

$$R_{22} = R_{23} = R_{33} = 0$$

we obtain the following system of equations:

$$\frac{(f')^2}{f^2} - \frac{(\dot{q})^2}{q^2} = 0$$

$$(n-a^2)q\frac{f''}{f} - n\frac{(f')^2}{f^2} - \left\{(n-a^2)\frac{\ddot{q}}{q} - n\frac{(\dot{q})^2}{q^2}\right\} = 0$$

$$2(n-a^2)\frac{f''}{f} - a^2\frac{(f')^2}{f^2} - \left\{2(n-a^2)\frac{\ddot{q}}{q} - a^2\frac{(\dot{q})^2}{q^2}\right\} = 0$$

This is the system of three equations with only two unknown functions. But, the system is consistent. Namely, subtracting the third equation from the second one which is multiplied by two, one obtains the equation equivalent to the first one. The solution of this system is in the form

$$f = e^{\mu x^1}, \quad q = e^{\mu x^4}, \quad \mu = const \tag{4}$$

Putting this solution in the components of the Ricci tensor we get

$$R_{11} = R_{14} = R_{44} = \frac{\mu^2}{2}\,\frac{a^2 - 2n}{a^2 - n} \qquad (= const)$$

The Ricci tensor we got corresponds to the energy-momentum tensor of the pure radiation field presented by a constant null vector

$$v_\alpha = \frac{\mu}{\sqrt{2}}\{1,0,0,1\}$$

and corresponding

$$\Phi^2 = \frac{a^2 - 2n}{a^2 - n}, \quad a^2 > 2n \quad \text{or} \quad a^2 < n$$

Now, we can present the proposed metric, expressed by the functions f and q which we determined from the condition that the form of the Ricci tensor corresponds to the form of the energy-momentum tensor of the pure radiation field.

$$ds^2 = (dx^1)^2 + (dx^2)^2 + 2\,a\,e^{\mu\,(x^1+x^4)}\,dx^2\,dx^3 + n\,e^{2\,\mu\,(x^1+x^4)}(dx^3)^2 - (dx^4)^2$$

So, we obtained one more metric representing the pure radiation field.

References

[1] A. Lichnerowicz: Théories relativites de la gravitation et de l'électromagnétisme, *Masson (1955)*

[2] D. Kramer, H. Stephani et al: Exact solutions of Einstein's field equations, *Cambridge UP (1980)*

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Razmatramo dve Rimanske metrike. Obe su izražene pomoću dve funkcije. Jedna od njih zavisi od x^1, a druga samo od vremena x^4. Komponujemo Ričijev tenzor. Funkcije koje čine metrički tenzor se odredjuju iz uslova da oblik Ričijevog tenzora odgovara energetsko-momentnom tenzoru čisto radijacionog polja. Ovo je specijalni slučaj elektromagnetskog polja u kojem su električni i magnetni tri-vektori jednaki i normalni jedan na drugog.