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Bianchi type I inhomogeneous anisotropic cosmological models with perfect fluid

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Abstract

The problem of inhomogeneous anisotropic Bianchi type I space time with perfect fluid is considered and exact solutions to the field equations are derived when the metric potentials are functions of cosmic time only. Some physical and geometrical properties of the solutions are also discussed.

1 Introduction

The study of Einstein's field equations in the presence of perfect fluid has attracted the attention of many workers. Banerjee and Santos (1981) have solved cosmological equations for a Bianchi type I metric in general scalar tensor theory proposed by Nordtvedt (1970) and Barker (1978) and shown that the universe expands from the initial singularity of zero volume and then contracts back. Venkateswarlu and Reddy (1990) have obtained spatially homogeneous and anisotropic Bianchi type I models in self creation theory of gravitation proposed by Barber (1982), Mansouri and Mahazzab (1993) have considered homogeneous anisotropic Bianchi type I models and have calculated tunreling rate using Euclidean approach. Mohanty and Daud (1995) have shown that there exists spatially homogeneous and isotropic Bianchi type I cosmological models in Wesson's (1981) scale invariant theory of gravitation when the source of gravitational field is a perfect fluid characterized by equation of state $p = \rho C^2$.

None of the above authors has attempted the problem of LRS Bianchi type I space time (MacCallum, 1979) with perfect fluid in Einstein's theory. However without solving the field equations, Hajj-Boutros and Sfeila (1987) and ShriRam (1989) have obtained solutions of the above problem using solution generation techniques. Recently Mazumder (1994) has shown that the field equations of the above problem are solvable for any arbitrary cosmic scale functions where he has taken particular solutions analogous to those of Hajj-Boutros (1987) and Shri Ram (1989). Here we have taken an attempt to solve the field equations to obtain general solutions of this problem, when the space time is inhomogeneous and anisotropic. In order to avoid the mathematical complexity due to inhomogeneity, we have derived solutions for homogeneous case and studies some physical and geometrical properties of the solutions.

In Sec.2, we have derived Einstein's field equations for LRS Bianchi type I metric with perfect fluid.

In Sec.3, we have obtained consequences of Einstein's field equations considering the cosmic scale functions to be the functions of time only. In Sec.4, we have solved the field equations for three different cases and obtained explicit exact solutions. We have also studied some physical and geometrical properties of the solutions in Sec.5 and have given concluding remarks in Sec.6.

2 Einstein's field equations

The metric of inhomogeneous and anisotropic space time is given by

$$ds^2 = dt^2 - A^2dx^2 - B^2(dy^2 + dz^2) \tag{1}$$

where A and B are functions of x and t only. The energy momentum tensor for a perfect fluid is given by

$$T_{ij} = (\rho + p)u_i u_j - p g_{ij} \tag{2}$$

together with

$$g_{ij}u^i u^j = 1 \tag{3}$$

where u^i is the four velocity vector of the fluid, p and ρ are proper pressure and density of the distribution respectively.

The Einstein field equations can be written as

$$R_{ij} - \frac{1}{2}g_{ij}R = -8\pi T_{ij} \tag{4}$$

where the units are chosen such that $G = 1 = C$. With the help of equations (2) and (3), equation (4) for the metric (1) in commoving coordinate system i.e. $u_i = (0\ 0\ 0\ 1)$ take the following explicit forms:

$$-8\pi p = \frac{2B_{44}}{B} - \frac{B_1^2}{A^2B^2} + \frac{B_4^2}{B^2}, \tag{5}$$

$$-8\pi p = \frac{B_{44}}{B} - \frac{B_{11}}{A^2B} + \frac{A_1B_1}{A^3B} + \frac{A_4B_4}{AB} + \frac{A_{44}}{A}, \tag{6}$$

$$-8\pi p = \frac{2B_{11}}{A^2B} - \frac{2A_1B_1}{A^3B} - \frac{2A_4B_4}{AB} + \frac{B_1^2}{A^2B^2} - \frac{B_4^2}{B^2}, \tag{7}$$

$$0 = B_{41} - \frac{B_1A_4}{A} \tag{8}$$

Here after wards the suffixed 1 and 4 after a after a field variable represent partial differentiation with respect to x and t respectively.

The conservation equations $T^{ij}_{;j} = 0$ for $i = 1$ leads to

$$p_1 = 0 \tag{9}$$

which on integration yields

$$p = p(t). \tag{10}$$

The equations of motion for $i = 2, 3$ are identically zero and the other surviving equation of continuity for $i = 4$ yields

$$\frac{A_4}{A} + \frac{2B_4}{B} = -\frac{\rho_4}{p+\rho}. \tag{11}$$

As the field equations are highly nonlinear in nature, in general it is difficult to obtain the explicit solution of the field equations. We therefore consider the case where the cosmic scale functions are functions of time only.

3 Consequence of field equations

Here the cosmic scale functions are assumed in the forms:

$$A = f(t) \text{ and } B = g(t). \tag{12}$$

In this case (8) is identically satisfied. Substitution of (12) in (5)-(6) yield

$$\frac{g_{44}}{g} + \frac{g_4^2}{g^2} = \frac{f_4 g_4}{fg} + \frac{f_{44}}{f}. \tag{13}$$

In view of (13), one can take

$$g = F(f(t)). \tag{14}$$

With the help of (14), equation.(13) can be expressed in the form

$$\left(\frac{F'}{F} - \frac{1}{f}\right) f_{44} + \left(\frac{F''}{F} + \frac{F'^2}{F^2} - \frac{F'}{Ff}\right) f_4^2 = 0 \tag{15}$$

Here a superscript prime indicates the differentiation with respect to the argument. This equation is satisfied for the following cases:

$$\textbf{Case I}: \frac{F'}{F} - \frac{1}{f} = 0 \text{ and } \frac{F''}{F} + \frac{F'^2}{F^2} - \frac{F'}{Ff} = 0. \tag{16ab}$$

$$\textbf{Case II}: f_{44} = 0 \text{ and } \frac{F''}{F} + \frac{F'^2}{F^2} - \frac{F'}{Ff} = 0. \tag{17ab}$$

$$\textbf{Case III}: f_4 = 0. \tag{18}$$

$$\textbf{Case IV}: f_4 = 0 \text{ and } \frac{F'}{F} - \frac{1}{f} = 0. \tag{19ab}$$

4 Solutions

In this section we intend to derive explicit exact solutions of the field equations for each of the above cases.

4.1 Case I

Using (16a) in equation (16b) and integrating, we obtain

$$F = K_1 f + K_2. \tag{20}$$

where $K_1(\neq 0)$ and K_2 are constants of integration. Now from (16a) and (20), one gets

$$K_2 = 0. \tag{21}$$

In view of (21), equation.(14) now yields

$$g = K_1 f. \tag{22}$$

which leads to isotropic model. Substitution of (22) in equations (5) and (6) yields same value for the pressure of the distribution i.e.

$$-8\pi p = \frac{2f_{44}}{f} + \frac{f_4^2}{f^2}. \tag{23}$$

Further substitution of (22) in equation.(7), the density of the distribution is found to be

$$\rho = \frac{3}{8\pi}\left(\frac{f_4}{f}\right)^2. \tag{24}$$

In order to get the explicit forms of the physical and geometrical parameters, we consider the fluid distribution obeying the barotropic equation of state i.e.

$$p = (\gamma - 1)\,\rho,\ 0 \leq \gamma \leq 2. \tag{25}$$

In view of (25), (23) and (24) yield

$$f = C_2\left(C_1 + \frac{3}{2}\gamma t\right)^{2/3\gamma}. \tag{26}$$

Where C_1 and $C_2(\neq 0)$ are constants of integration. Now substitution of (26) in (23) and (24) yields

$$p = \frac{3(\gamma - 1)}{8\pi}\ \left(C_1 + \frac{3}{2}\gamma t\right)^{-2} \tag{27}$$

and

$$\rho = \frac{3}{8\pi}\left(C_1 + \frac{3}{2}\gamma t\right)^{-2} \tag{28}$$

Thus we get the following homogeneous barotropic fluid model in the form

$$ds^2 = dt^2 - \left(C_1 + \frac{3}{2}\gamma t\right)^{4/3\gamma}(dx^2 + dy^2 + dz^2) \tag{29}$$

With proper choice of coordinate system the metric can be written in the form

$$ds^2 = dT_1^2 - T_1^{4/3\gamma}(dX^2 + dY^2 + dZ^2). \tag{30}$$

4.2 Case II

In this case (17a)-(17b), immediately integrated, yield

$$f = K_1 t + K_2. \tag{31}$$

and

$$F = \left(K_3 f^2 + K_4\right)^{1/2}, \tag{32}$$

where $K_1(\neq 0), K_2, K_3(> 0), K_4(< 0)$ are constants of integration.

In view of (32), equation.(14) now yields

$$g = \left(K_3 f^2 + K_4\right)^{1/2} \tag{33}$$

With the help of (31) and (33), equations.(5)-(7) yield the pressure and density of the distribution as

$$p = \frac{-K_3K_1^2}{8\pi}\left[\frac{2K_4 + K_3(K_1t + K_2)^2}{(K_3(K_1t + K_2)^2 + K_4)^2}\right] \tag{34}$$

and

$$\rho = \frac{K_3K_1^2}{8\pi}\left[\frac{3\left\{K_3(K_1t + K_2)^2 + K_4\right\} - K_4}{\left\{K_3(K_1t + K_2)^2 + K_4\right\}^2}\right]. \tag{35}$$

Consequently the corresponding metric of our solution can be written as

$$ds^2 = dt^2 - (K_1 t + K_2)^2 dx^2 - \left[K_3(K_1 t + K_2)^2 + K_4\right](dy^2 + dz^2) \tag{36}$$

which can be transformed through a proper choice of coordinates to the form

$$ds^2 = dT_2^2 - T_2^2 dX^2 - (T_2^2 + K)(dY^2 + dZ^2) \tag{37}$$

where $K = \dfrac{K_4}{K_3 K_1^2}(< 0)$ is a constant.

4.3 Case III

In this case (18) integrated to yield

$$f = const \tag{38}$$

In view of (38), equations .(5)-(6) yield

$$g = (K_1 t + K_2)^{1/2} \tag{39}$$

where $K_1(\neq 0)$ and K_2 are constants of integration.

Substitution of (38) and (39) into equations (5)-(7) yields the same value for the pressure and density of the distribution i.e.

$$p = \rho = \frac{K_1^2}{32\pi (K_1 t + K_2)^2}. \tag{40}$$

In this case the metric can be written as

$$ds^2 = dt^2 - dx^2 - (K_1 t + K_2)(dy^2 + dz^2) \tag{41}$$

which can be transformed through a proper choice of coordinates to the form

$$ds^2 = dT_2^2 - dX^2 - T_2(dY^2 + dZ^2) \tag{42}$$

4.4 Case IV

In this case integrating (19a), one gets

$$f = const \tag{43}$$

In view of (43), (19b) yield

$$F = const \tag{44}$$

Consequently, we obtain

$$A = const \tag{45a}$$

and

$$B = const \tag{45b}$$

Substitution of (45a)-(45b) into equations (5)-(7) yields the pressure and density of the distribution as

$$p = \rho = 0. \tag{46}$$

which concludes that the perfect fluid does not survive in this case and the space time becomes Minkowskian.

5 Some physical and geometrical properties

In this section we study the following physical and geometrical properties of the models obtained in the preceding sections. The pressure and density of the homogeneous barotropic fluid model represented by (30) are given by

$$p = \frac{3(\gamma - 1)}{8\pi T_1^2} \quad \text{and} \quad \rho = \frac{3}{8\pi T_1^2} \tag{47ab}$$

It is interesting to note that one can get "False Vacuum" model for $\gamma = 0$ and "Stiff Fluid" model for $\gamma = 2$. These cases have already been studied by Mohanty and Pradhan (1990, 1991) in case of viscous fluid distribution. The reality conditions in this case i.e. $p > 0$ and $\rho > 0$ are satisfied for $1 < \gamma \leq 2$, the strong energy condition i.e. $\rho + 3p > 0$ is satisfied for $\frac{2}{3} < \gamma \leq 2$ and weak energy condition i.e. is satisfied for $0 < \gamma \leq 2$. The spatial volume is

$$V = AB^2 = K_5 T_1^{2/\gamma} \tag{48}$$

where $K_5 = \left(\frac{3}{2}\gamma\right)^{2/\gamma} K_1^2 C_2^3$ is a constant. For $\gamma \neq 0$ it clearly shows the isotropic expansion of the universe with time.

The magnitude of scalar expansion θ and shear σ^2 for the model (30) are given by

$$\theta = \frac{2}{\gamma T_1} \quad \text{and} \quad \sigma^2 = 0 \tag{49ab}$$

As θ blows up and the space time becomes Minkowskian for $\gamma = 0$, this case does not admit "False Vacuum" model. Moreover this model preserves the shape and size of the universe during evolution. The scalar expansion becomes indefinitely large or indefinitely small according as $T_1 \to 0$ or $T_1 \to \infty$. For $T_1 > 0$, the expansion scalar $\theta > 0$ indicates that the model is expanding in nature but the expansion is decelerating. Thus model (30) is spatially homogeneous and isotropic.

The pressure and density for the model (37) are given by

$$p = -\frac{1}{8\pi} \cdot \frac{T_2^2 + 2K}{\left(T_2^2 + K\right)^2} \quad \text{and} \quad \rho = \frac{1}{8\pi} \cdot \frac{3T_2^2 + 2K}{\left(T_2^2 + K\right)^2} \tag{50ab}$$

In this case the strong energy condition is satisfied for $K < 0$ and the weak energy condition is identically satisfied. The spatial volume is

$$V = K_6 \cdot T_2 \left(T_2^2 + K\right) \tag{51}$$

($K_6 = K_3 K_1^2$ is a constant) which shows the anisotropic expansion of the universe with time for $K_3 > 0$.

The scalar expansion θ and shear σ^2 for the model (37) are given by

$$\theta = \frac{3T_2^2 + K}{T_2\left(T_2^2 + K\right)} \quad \text{and} \quad \sigma^2 = \frac{1}{3}\frac{K^2}{T_2^2\left(T_2^2 + K\right)} \tag{52ab}$$

Here $\left(\sigma^2/\theta^2\right) \to 0$ as $T_2 \to \infty$ which implies that the model approaches isotropy for large value of T_2. For $K = 0$ this case violates the strong energy condition.

Further the pressure, density, scalar expansion and shear σ^2 for the model (42) are given by

$$p = \rho = \frac{1}{32\pi T_2^2} \tag{53ab}$$

$$\theta = \frac{1}{T_2} \quad \text{and} \quad \sigma^2 = \frac{1}{12T_2^2} \tag{53cd}$$

Thus this case degenerates only stiff fluid model (Mohanty and Panigrahi, 1990). In this case the reality and energy conditions are identically satisfied. $\sigma^2/\theta^2 = 1/12$ being independent of cosmic time implies that model does not approach isotropy. As in the preceding cases, this model is also expanding in nature, but expansion is decelerating. The spatial volume

$$V = T_2 \tag{54}$$

which implies anisotropic expansion of the universe with time.

In view of (47ab), (50ab) and (53ab), it is seen that $T_1, T_2 = 0$ represent Big bang singularity. It is also observed that the acceleration a_i and rotation w_{ij} are identically zero in all the cases. It is interesting to note that $T_1 = \infty$ for the first case and $T_2 = \infty$ for the remaining two cases represent geometrical singularities.

In the fourth case it is shown that the fluid distribution does not exist and space time becomes Minkowskian.

6 Conclusion

In this paper we have solved field equations for LRS Bianchi type I homogeneous cosmological models with perfect fluid and obtained solutions for homogeneous barotropic fluid model in first case, perfect fluid model in second case and stiff fluid model in third case. The fourth case does not admit any solution for perfect fluid distribution. Thus the present work completes the work of Mazumder (1994), wherein he has taken the particular solutions of the field equations. It is observed that the models are non rotating in nature with null acceleration and expanding during the course of evolution. Except first case the model of first case tends to isotropy and that of third case does not lead to isotropy for large cosmic time. In all the three cases the spatial volume increases with time which supports the analysis done earlier in support of expansion of the models.

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Nehomogeni anizotropni kosmološki modeli prvog Bjankijevog tipa sa idealnim fluidom

UDK 530.12; 532.51

Posmatra se problem nehomogenog anizotropnog prostor-vremena prvog Bjankijevog tipa sa idealnim fluidom. Izvedena su tačna rešenja jednačina polja kada su metrički potencijali funkcije samo kosmičkog vremena. Neke fizičke i geometrijske osobine tih rešenja su takodje diskutovane.