THEORETICAL AND APPLIED MECHANICS
vol. 26, pp. 169-177, 2001

The Hamiltonian principle in the theory of elasticity

Aleksandar Bakša

Submitted 17 February, 1999

Abstract

This paper attempts to contribute to the canonical formalism in the non-linear theory off elasticity. It has been shown that the basic balance equations, in the Hamiltonian form, represent the extremal of the functionals of action. We have used the canonical formalism upon an infinitesimal-dimensional phase manifold in the material description.

1 Introduction

Let (B, G) and (R^3, g) be the smooth Riemannian manifolds, which represent the reference configuration and the ambient space of body $\mathcal{B}$ respectively. We define the configuration space as a set of all orientation-preserving embeddings of B in R^3 which do not change the orientation:

$$S := Emb(B, R^3) \tag{1}$$

This set can always be completed in the Banach space, and therefore can be considered as an infinitesimal-dimensional phase manifold.

We are constructing the tangent bundle upon S in the following way: Let us consider the smooth curve χ_ε upon $S : \varepsilon \in R \rightarrowtail \chi_\varepsilon \in S$.

The tangent upon S in point χ_0 is

$$v := \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \chi_\varepsilon \tag{2}$$

Consequently, the tangent space upon S in point χ is

$$T_\chi S := \{v_\chi : B \to TR^3 \big| \, v_\chi(X) \in T_x R^3, \forall X \in B\}, \; x = \chi(X) \tag{3}$$

while the tangent bundle is

$$TS = \bigcup_{\chi \in S} T_\chi S. \tag{4}$$

The cotangent space $T^*_\chi S$ in point χ is

$$T^*_\chi S := \{v^*_\chi : B \to T^* R^3 \big| \, v^*_\chi(X) \in T^*_x R^3, \forall X \in B\}. \tag{5}$$

The natural pairing of the tangent and cotangent space in the same point is performed by using the weak nondegenerate bilinear form

$$\langle \cdot, \cdot \rangle : T^*_\chi S \times T_\chi R \to R \tag{6}$$

expressed by the formula

$$\langle w^*_\chi, v_\chi \rangle = \int_B w^*_\chi(v_\chi) \mu(G), \tag{7}$$

where $w^*_\chi(v_\chi) = g(w_\chi, v_\chi)$ - the value of the l-form w^* upon vector v_χ (in the rest of the paper we shall omit index χ) and $\mu(G) = (\det G)^{1/2} dX$ are the metric element of the volume.

If we denote the charts of the local coordinates with $\Phi(X) = (X^1, X^2, X^3)$ and $\varphi(x) = (x^1, x^2, x^3)$, we shall have

$$\begin{aligned} &v = v^i \frac{\partial}{\partial x^i}, \; w^* = w_i dx^i, \; dX = dX^1 \wedge dX^2 \wedge dX^3, \\ &g(w, v) = g_{ij} w^i w^j = w_j v^j. \end{aligned} \tag{8}$$

2 Lagrangian and Hamiltonian

The motion of a body $\mathcal{B}$ is a uniparametric mapping of $\chi_t : B \times R \to R^3$. The motion of point $X \in B$ is, consequently, curve $\chi_t(X) \equiv \chi(X, t)$ upon R^3. The velocity of point B in the material description is, according to definition

$$v_t(X) = \frac{\partial}{\partial t}\chi(X.t) = \frac{\partial \chi^k}{\partial t}\frac{\partial}{\partial x^k} \in T_x S; \; x = \chi(X, t). \tag{9}$$

Let us denote the deformation gradient in the following way

$$F(X) := T_\chi(X, t) = \frac{\partial \chi^k}{\partial X^K}\frac{\partial}{\partial x^k} \otimes dX^K. \tag{10}$$

Let us also introduce into the consideration the metric tensor spaces $\mathcal{M}_G$ (in the referential configuration) and $\mathcal{M}_g$ (in the ambient space). We call Descartes' product

$$Q = \mathcal{M}_g \times TS \times \mathcal{M}_G \tag{11}$$

the material configuration space.

We shall consider the motion of an elastic body determined by the Lagrangian $\mathcal{L}$, by the boundary and the initial conditions. A characteristic of elastic bodies is that they possess the function of the deformation energy $W : Q \to R$ such that the first Piola-Kirchhoff's tensor P is equal to the partial derivative of function W according to F. We shall assume that the volume forces acting upon a body have the potential $\pi : Q \to R$. We assume that the boundary conditions are given in the mixed form:

$$\begin{aligned} &\chi(X) = \varphi(X), \;\; X \in \partial_1 B, \;\; (\varphi - \text{given function}) \\ &P(X)n(X) = t(X), \;\; X \in \partial_2 B, \;\; (t - \text{given function}) \end{aligned} \tag{12}$$

$Cl(\partial_1 B \cup \partial_2 B) = Cl\, \partial B, \; \partial_1 B \cap \partial_2 B = \emptyset$, n 1-form upon ∂B which is annulled at each vector from the tangential bundle upon ∂B, whereby we take that ∂B represents a submanifold of manifold B. The Lagrangian is a functional upon Q, which, under the conditions mentioned, can be taken in the following form

$$\mathcal{L}(g;\chi,v,G) := \int\limits_B \left\{\rho(X)\left[\frac{1}{2}\left|v(X)\right|_g^2 - W(g(\chi(X)),F(X),G(X))\right] - \right.$$

$$\left. -\pi(g(\chi(X)),F(X),G(X))\right\}\mu(G) + \int\limits_{\partial_2 B}(\chi(X),t(X))\Big|_{\partial_2 B}\gamma(G) \tag{13}$$

where $\gamma(G)$ is the restriction of the volume element $\mu(G)$ upon $\partial_2 B$ whereby $n \wedge \gamma(G) = \mu(G)$. In the remaining part of the paper we shall assume that the functional (13) is differentiable according to Fréchet.

We shall now pass on to the construction of the Hamiltonian. With this aim, let us define the momentum in the material description. One-form of $M_\chi \in T^*_\chi S$, satisfying equation

$$\langle M_\chi, U\rangle = \mathcal{L}_v[U], \quad \forall U \in T_\chi S, \tag{14}$$

where $\mathcal{L}_v[U]$ is the value, partial with respect to U, of Fréchet's derivative of the Lagrangian on vector U, we call the material momentum that corresponds to velocity v. It can easily be seen that

$$M_\chi = \rho v^\flat_\chi = \rho(g_{ij}\circ\chi)v^j dx^i \tag{15}$$

($\flat$ the operator of the index lowering). The Hamiltonian $\mathcal{H}$ is a functional upon the function set $\mathcal{F}Q^*$defined upon the material phase space $Q^* = M_g \times T^*S \times M_G$, which represents Legendre's transformation of the Lagrangian. If we denote Legendre's transformation induced by the Lagrangian $\mathcal{L}$ with ℓ,

$$\ell : \{\chi, v_\chi\} \in TS \to \{\chi, M_\chi\} \in T^*S \tag{16}$$

the Hamiltonian is, by definition,

$$\mathcal{H} := \left\langle M, \ell^{-1}(M)\right\rangle - \mathcal{L}\circ\ell^{-1}(M). \tag{17}$$

Accordingly, the Hamiltonian

$$\mathcal{H}(g;M,\chi;G) := \int\limits_B \left\{ \frac{1}{2\rho(X)} \left|M(X)\right|_g^2 + \rho(X)W(g(\chi(X)),F(X),G(X)) + \right.$$

$$\left. +\pi(g(\chi(X)),F(X),G(X)) \right\} \mu(G) + \int\limits_{\partial_2 B} (\chi(X),t(X))\Big|_{\partial_2 B} \gamma(G) \tag{18}$$

corresponds to the Lagrangian (13).

For the sake of easier writing, we shall express the former in the form

$$\mathcal{H}(g;M,\chi;G) := \int\limits_B H(g,M,\chi,G)\mu(G) - \int\limits_{\partial_2 B} (\chi,t)\gamma(G). \tag{19}$$

3 The Hamiltonian principle

Let us define the 1-form on an extended phase space $Q^* \times R$

$$\omega := \int\limits_B (M_i(X)d\chi^i(X))\mu(G) - \mathcal{H}(g;M,\chi,G)dt. \tag{20}$$

Let $c : [0.1] \to T^*S$ be a smooth curve. Let us consider the functional

$$\mathcal{J}(c) = \int\limits_c \omega \tag{21}$$

(which is, traditionally, called the functional of action). We shall say that curve $\hat{c}$ is the extremal of functional (21) if $\mathcal{J}(\hat{c}) \leq \mathcal{J}(c)$ or $\mathcal{J}(c) \leq \mathcal{J}(\hat{c})$ for each c from an environment of $\hat{c}$.

Theorem 1 *(The Hamiltonian Principle) The extremal $\hat{c}$ of functional (21) upon the set Ω of all smooth curves with joint ends in fixed points $\hat{c}(0)$ and $\hat{c}(1)$ is the motion that corresponds to Hamiltonian (18).*

Proof. Let us first define the environment of point $\hat{c}$ in T^*S. In order to avoid possible complications, we shall assume that curve $\hat{c}$ has no self-crossings or any other similar irregularities. Let us use $\mathcal{A}$ to denote the group of single-parametric diffeomorphisms α_s upon Ω. The open set

$$S := \{c_s = \alpha_s \circ \hat{c} \mid \alpha_s \in \mathcal{A},\, s \in (-\varepsilon, \varepsilon), c_o = \hat{c}\} \subset \Omega \tag{22}$$

represents the environment of point $\hat{c}$.

Since the extremal of the functionals is at the same time its critical point too, we have

$$\left.\frac{d}{ds}\right|_{s=0} \mathcal{J}(c_s) = 0 \tag{23}$$

The derivative on the left-hand side of this equation, according to definition, is

$$\left.\frac{d}{ds}\right|_{s=0} \mathcal{J}(c_s) = \int_{\hat{c}} \lim_{s\to 0} \frac{1}{s}(\alpha_s^*\omega - \omega) = \int_{\hat{c}} L_v\omega, \tag{24}$$

where $L_v\omega$ is Lie's derivative of 1-form ω in the direction of the varied vector field v, induced by diffeomorphism $s \to \alpha_s$. By utilizing Cartan's formula of homotopy, Lie's derivative can be expressed in the following way

$$L_v\omega = i_v d\omega + d(i_v\omega). \tag{25}$$

Keeping in mind the fact that the varied vector field is annulled at the fixed ends of curve $\hat{c}$, from (23), with regard to (24) and (25), we obtain

$$\int_{\hat{c}} i_v d\omega = 0. \tag{26}$$

Let us write the varied vector field v in the form

$$v = \xi_i \frac{\partial}{\partial M_i} + \eta^i \frac{\partial}{\partial \chi^i} \equiv \xi + \eta \tag{27}$$

and substitute it in (26), we obtain

$$\int\limits_{[0,1]} \left\{ \int\limits_{B} \left(\dot{\chi} \cdot \xi - \dot{M} \cdot \eta \right) \mu(G) - \mathcal{H}_M[\xi] - \mathcal{H}_\chi[\eta] \right\} dt = 0 \tag{28}$$

where $\mathcal{H}_M[\xi]$ and $\mathcal{H}_\chi[\eta]$ are partial Fréchet's derivatives (according to M and χ, respectively) of the Hamiltonian upon ξ and η,respectively. Regarding the calculation of these derivatives, we are giving a short remark. The partial derivative with respect to M is calculated in accordance with the standard definition of the Fréchet's derivatives (the same as we had in determining the partial derivative of the Lagrangian with respect to v). Namely, the mapping of $M \to \mathcal{H}(g, \cdot, \chi, G)$ is differentiated, according to Fréchet, at fixed χ. In that way, we obtain

$$\mathcal{H}_M[\xi] = \int\limits_{B} \frac{1}{\rho}(M^{\#} \cdot \xi)\mu(G) = \int\limits_{B} v \cdot \xi \mu(G) \tag{29}$$

($\#$ is the symbol denoting the operation of raising the index). This definition cannot be applied in the calculation of the partial derivative with respect to χ, since M cannot be fixed when χ changes. We shall therefore use formula ([1])

$$\mathcal{H}_\chi[\eta] = \lim_{\lambda \to 0} \frac{1}{\lambda}[\mathcal{H}(g; M_{\chi+\lambda\eta}, \chi + \lambda\eta; G) - \mathcal{H}(g; M_\chi, \chi; G)] \tag{30}$$

by the application of which, we obtain

$$\mathcal{H}_\chi[\eta] = \int\limits_{B} \left(\rho \frac{\partial W}{\partial F} : F_\chi[\eta] + \frac{\partial \pi}{\partial \chi} \cdot \eta \right) \mu(G) - \int\limits_{\partial_2 B} \chi \cdot \eta \Big|_{\partial_2 B} \gamma(G). \tag{31}$$

Taking into account that

$$\rho \frac{\partial W}{\partial F} : F_\chi[\eta] = P : grad \eta \tag{32}$$

as well as the theorem of divergence (cf. [2]), (31) can get the form

$$\mathcal{H}_\chi[\eta] = - \int\limits_{B} \left(div P - \frac{\partial \pi}{\partial \chi} \right) \cdot \eta \mu(G) + \int\limits_{\partial_2 B} Pn \cdot \eta \Big|_{\partial_2 B} \gamma(G). \tag{33}$$

When (29) and (33) are substituted in (28), we obtain

$$\int_{[0,1]} \left\{ \int_B \left[(\dot{\chi} - v) \cdot \xi - \left(\dot{M} - divP + \frac{\partial \pi}{\partial \chi} \right) \cdot \eta \right] \mu(G) + \right.$$

$$\left. + \int_{\partial_2 B} (Pn - t) \cdot \eta \Big|_{\partial_2 B} \gamma(G) \right\} dt = 0. \tag{34}$$

The variational vector field, as well as its restriction on $\partial_2 B$ can in $intB$ be selected arbitrarily. Hence, assuming that the Hamiltonian is continuously differentiable, it follows from (34) that

$$\frac{\partial \chi(X,t)}{\partial t} = v_t(X), \ \frac{\partial M(X,t)}{\partial t} = divP(X) - \frac{\partial \pi(g, \chi(X), G)}{\partial \chi} \text{ for } X \in intB, \tag{35}$$

$$\chi(X,t) = \varphi(X) \text{ for } X \in \partial_1 B; \ P(X)n(X) = t(X), \text{ for } X \in \partial_2 B. \tag{36}$$

The equations obtained represent the familiar canonical equations of motion (balance) in material description ([1]). Thereby the theorem is proved.

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Aleksandar Bakša
Faculty of Mathematics
University of Belgrad
11000 Beograd
Studenski trg 16
Yugoslavia

Hamiltonov princip u teoriji elastičnosti
UDK 531.01; 539.31

U radu je pokušana formulacija kanonskog formalizma u nelinearnoj teoriji elastičnosti. Pokazano je da osnovne jednačine balansa, u Hamiltonovom obliku, predstavljaju ekstremale funkcionala dejstva. Korišćen je kanonski formalizam na infinitezimalno-dimenzionalnoj faznoj mnogostrukosti u materijalnom opisu.