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Condition for instability of conservative gyroscopic systems

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Abstract

The note investigates the stability of linear conservative gyroscopic systems. A new theorem which provide sufficient condition for instability is established. This condition is stated in terms of the coefficients of system matrices.

1 Introduction

Systems of interest here are linear conservative gyroscopic systems described by the equation

$$M\ddot{q} + B\dot{q} + Cq = 0, \tag{1}$$

where q is the $n-$dimensional position vector and M, B and C are $n \times n$ constant real matrices; C is symmetric ($C^T = C$), M is symmetric and positive definite ($M > 0$), and B is skew ($B^T = -B$). Stability properties of the systems have been studied for more than one hundred years. Applications vary from the classical problem of the spinning top to more complicated rotating bodies (such as spaceship, elastic shafts and CD players), and to the motion of fluids in flexible pipes.

It is well known that the positive definiteness of C is a sufficient condition for stability of system (1). While gyroscopic forces $B\dot{q}$ can never destabilize a stable conservative system ($C > 0$), they can possibly stabilize an unstable conservative system ($C \not> 0$). Many attempts have been made to establish related stability and/or instability criteria when C is not positive definite, and the references on the subject can be found in [1] (see, also, subsequent papers [2], [3], [4], [5]).

2 Preliminaries

It is convenient, although not necessary, to rewrite equation (1) in the form

$$\ddot{x} + G\dot{x} + Kx = 0, \tag{2}$$

using the congruent transformation $x = M^{1/2}q$, $G = M^{-1/2}BM^{-1/2}$, $K = M^{-1/2}CM^{-1/2}$.

The system is said to be stable if all solutions $x(t)$ of (2) are bounded for all non-negative t. All solutions of (2) can be characterized algebraically using properties of the quadratic matrix polynomial

$$L(\lambda) := \lambda^2 I + \lambda G + K, \tag{3}$$

where I is identity matrix. The eigenvalues of the system (2) are zeros of the characteristic polynomial

$$\Delta(\lambda) := \det(L(\lambda)) \tag{4}$$

and the multiplicity of an eigenvalue is the order of the corresponding zero in $\Delta(\lambda)$. If λ is an eigenvalue, the nonzero vectors in the nullspace of $L(\lambda)$ are the eigenvectors associated with λ. In general, eigenvalues and eigenvectors may be real or complex. Since $G^T = -G$ and $K^T = K$, then $L(\lambda)^T = L(-\lambda)$ and, consequently, $\Delta(\lambda) = \Delta(-\lambda)$. Thus, the system (2) is stable only when every eigenvalue is on the imaginary axis (purely imaginary or zero) and semi-simple, i.e., if the eigenvalue has multiplicity k, there are k linearly independent associated eigenvectors.

3 Nonspectral Conditions for Instability

The most simple nonspectral criteria for instability of the system (2) are as follows:

(A) The system is unstable if $detK < 0$ (Kelvin-Tait-Chetayev theorem);

(B) The system is unstable if $4K - G^2 < 0$ [6] (see also [7]);

(C) The system is unstable if $2Tr(K) + \|G^2\| \leq 0$, where $Tr(K)$ is the trace of K and $\|G\|$ is Euclidean matrix norm of G [4].

The criteria given above are supplemented by the following theorem.

Theorem 1 *If the condition*

$$\left[2Tr(K) + \|G^2\|\right]^2 > 2n\left[2\|K\|^2 + \|G^2\|^2 + 4Tr(G^TKG)\right] \qquad (5)$$

holds, then the dynamical system (2) is unstable.

Proof. The characteristic polynomial (4) contains only even powers of λ , i. e.,

$$\Delta(\lambda) = \lambda^{2n} + a_1\lambda^{2n-2} + ... + a_{n-1}\lambda^2 + a_n, \qquad (6)$$

since, as mentioned in Section 2, $\Delta(\lambda) = \Delta(-\lambda)$. Denoting $\beta = \lambda^2$ we get the polynomial of n-th order in β. If the gyroscopic system (2) is stable then all roots β of (6) are real and nonpositive.

Consider the quadratic form

$$S = y^THy, \quad y \in \Re^n \qquad (7)$$

with

$$H = \begin{bmatrix} s_0 & s_1 & s_2 & . & . & . & s_{n-1} \\ s_1 & s_2 & s_3 & . & . & . & s_n \\ s_2 & s_3 & s_4 & . & . & . & s_{n+1} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ s_{n-1} & s_n & s_{n+1} & . & . & . & s_{2n-2} \end{bmatrix},$$

where

$$s_k = \sum_{j=1}^{n} \beta_j^k, \quad k = 0, 1, 2, ... \qquad (8)$$

A theorem (see [8], Chapter XVI, Theorem 6) states that rank and sign of the quadratic form (7) are equal to the number of unequal roots of (6) and the number of unequal real roots β, respectively. It follows that all roots β are real if and only if S is positive semi-definite.

Suppose that

$$s_0 s_2 - s_1^2 < 0 \tag{9}$$

Then S$\ngeq$0 and, consequently, the polynomial $\Delta(\beta)$ has at least one non-real root. Therefore, condition (9) implies instability. Now it will be proved that (9) is equivalent to (5).

The power sums (8) can be expressed by the coefficients of the polynomial $\Delta(\beta)$ through Newton recurrence formulae [9]:

$$\begin{aligned} &s_0 = n, \quad s_k = -s_{k-1}a_1 - \ldots - s_1 a_{k-1} - k a_k, \quad k \leq n \\ &s_k = -s_{k-1}a_1 - \ldots - s_{k-n}a_n, \quad k > n+1 \end{aligned} \tag{10}$$

From (10) we have

$$s_1 = -a_1 \tag{11}$$

and

$$s_2 = a_1^2 - 2a_2 \tag{12}$$

On the other hand, it is well-known that

$$\Delta(\lambda) = \det(A - \lambda I) \tag{13}$$

where

$$A = \begin{bmatrix} -G & -K \\ I & 0 \end{bmatrix} \tag{14}$$

According to the method of Fadaev [8], we have

$$a_1 = -\frac{1}{2}Tr\left(A^2\right) = \frac{1}{2}\left[2Tr(K) - Tr\left(G^2\right)\right] \tag{15}$$

and

$$a_2 = -\frac{1}{4}Tr\left(A^4 + a_1 A^2\right) =$$

$$= \frac{a_1^2}{2} - \frac{1}{4}\left[2Tr(K^2) + Tr\left(G^4\right) + 4Tr(G^T KG)\right] \tag{16}$$

Since $G = -G^T$ there exists an orthogonal matrix T such that (see [10])

$$T^T GT = \tilde{G} = diag\left(g_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, ..., g_k \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 0, ..., 0\right), \tag{17}$$

and

$$T^T G^2 T = \tilde{G}^2 = -diag\left(g_1^2, g_1^2, ..., g_k^2, g_k^2, 0, ..., 0\right), \tag{18}$$

where $k \leq [n/2]$. From (17) and (18), we have

$$Tr\left(G^2\right) = -\left\|G\right\|^2, \tag{19}$$

because orthogonal transformation preserve Euclidean norm and trace of a matrix. Similarly,

$$Tr\left(K^2\right) = \left\|K\right\|^2, \tag{20}$$

and

$$Tr\left(G^4\right) = \left\|G^2\right\|^2. \tag{21}$$

Finally, from (9), according to (15), (16), (19), (20) and (21), we obtain condition (5). This proves the result.

Remark 1 *Condition* (C) *can be easily established by the following consideration. If the system (2) is stable then* $\sum_1^{2n} \lambda_i^2 = Tr\left(A^2\right) < 0$. *Consequently,* $Tr\left(A^2\right) = -2Tr(K) - \left\|G\right\|^2 \geq 0$ *implies instability.*

Theorem 1 is a new criterion for instability and the following examples show that neither condition (C) nor condition (5) implies the other one.

Example 1 *Let*

$$G = 4\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } K = -\begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}.$$

We have

$$\|G\|^2 + 2Tr(K) = 6 > 0$$

so that criterion (C) does not apply. However,

$$2\|K\|^2 + \left\|G^2\right\|^2 + 4Tr(G^T K G) = -126$$

and instability follows from Theorem 1.

Example 2 *Let*

$$G = 4\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } K = -\begin{pmatrix} 1 & 0 \\ 0 & 26 \end{pmatrix}.$$

We have

$$2Tr(K) + \|G\|^2 = -22$$

and instability follows from criterion (C). On the other hand,

$$\left[2Tr(K) + \|G\|^2\right]^2 - 4\left[2\|K\|^2 + \left\|G^2\right\|^2 + 4Tr(G^T K G)\right] = -68$$

and Theorem 1 tells us nothing.

References

[1] R.M.Bulatovic, The stability of linear potential gyroscopic systems when the potential energy has a maximum, Prikl. Mat. Mekh. (PMM),61, (1997),385-389, (in Russian).

[2] R.M.Bulatovic, A nonspectral stability criterion for linear conservative gyroscopic systems, ASME J. Appl. Mech., 65, (1998), 539-541.

[3] R.M.Bulatovic, A stability theorem for gyroscopic systems, Acta Mechanica, 136, (1999).

[4] P.Lancaster and P.Zizler, On the stability of gyroscopic systems, ASME J. Appl. Mech., 65, (1998), 519-522.

[5] R.Hryniv, W.Kliem, P.Lancaster and C.Pommer, A precise bound for gyroscopic stabilization, DTU, Report No. 18, (1998), p.17.

[6] G.K.Pozharicki, On the instability of motions of conservative systems, Prikl. Math. Mekh. (PMM), 20, (1956), 429-433, (in Russian).

[7] P.Hagedorn, Uber die Instabilitat konservativer Systeme mit gyroscopishen Kraften, Arch. Rat. Mech. Anal., 58, (1975), 1-9.

[8] F.R.Gantmacher, The Theory of Matrices (in Russian), Nauka, Moscow 1988.

[9] G.A.Korn and T.M.Korn, Mathematical Handbook for Scientists and Engineers, McGraw-Hill, New York 1961.

[10] R.Bellman, Introduction to Matrix Analysis, Mc Graw-Hill, New York 1970.

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Uslov nestabilnosti konzervativnih žiroskopskih sistema

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Proučava se stabilnost linearnih konzervativnih žiroskopskih sistema. Uspostavljena je nova teorema koja obezbedjuje dovoljan uslov nestabilnosti. Ovaj uslov je izražen pomoću koeficijenata matrica sistema.