THEORETICAL AND APPLIED MECHANICS
vol. 26, pp. 107-126, 2001

On viscoplasticity of ferromagnetics[1]

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Submitted 30 October 1999

Abstract

The paper deals with viscoplasticity of ferromagnetic materials. Evolution equations are derived either from inelastic materials of differential type or from loading function generalized normality. In both cases tensor representation is applied to such a set of evolution equations. Restrictions to the set of field equations are established by means of the extended irreversible thermodynamics. Small magnetoelastic strains of isotropic insulators are considered in detail in two special cases of finite as well as small plastic strain. As one example low-cycle fatigue of ferromagnetics is considered with special account to time delay between stress and magnetic field histories. To describe such an experimental evidence an integro-differential equation is proposed whose equivalent plastic strain dependent kernel covers the observed delay.

1 Introduction

The principal objective of this work is to present a rational thermodynamic approach to inelasticity of ferromagnetic materials in a simplified version which should serve primarily to subsequent nondestructive electromagnetic examination of inelastic behavior of reactor steels (cf.

[1] *Dedicated to Professor Henryk Zorski with gratitude.*

[1, 2, 3]). Due to limited space however some second order effects have to be dropped from the consideration.

In this paper like in [4, 5, 6]) associativity of flow rule (the normality of the plastic strain rate tensor onto a yield surface has not been taken as granted even if such an approach is accepted in the majority of the papers dealing with the subject (cf. [7, 8] and references mentioned in these papers). Such normality is seriously questioned not only by the theoretical but by experimental results as well (see [9]) for a comparison between tension and torsion and [10] for experiments dealing with cruciform specimens). For these reasons the normality is abandoned and instead of such an assumption evolution equations (exposed in the second section of this paper) are based on the appropriate geometry of deformation and the extended irreversible thermodynamics. This geometry is founded on the continuum theory of dislocations (compare with [11, 12, 13]) and is shortly reviewed in the next section. A very attractive approach to the extended thermodynamics has been proposed in [14] with a rational analysis of thermodynamic processes leading to the desired thermodynamic restrictions of general constitutive equations. This approach with the Liu's theorem [15] was applied to viscoplastic materials in [4] and to inelastic composite materials in [5]. However despite of its beauty an inherent coldness function (which is not quite clear from the experimental point of view) is inevitable. Herein an alternative approach [16, 17, 18] (already applied to thermoplasticity of irradiated materials in [6]) is chosen instead.

As a prerequisite, a correct geometric description of an inelastic deformation process analyzed is necessary. Consider a crystalline body in a real configuration (k) with dislocations and an inhomogeneous temperature field $T(X,t)$ (where t stands for time and X for the considered particle of the body)subject to surface tractions. Corresponding to (k) there exists, usually, an initial reference configuration (K) with (differently distributed) dislocations at a homogeneous temperature T_0 without surface tractions. Due to these defects such a configuration is not stressfree but contains an equilibrated residual stress (often named as "back-stress").

It is generally accepted that linear mapping function $\mathbf{F}(.,t) : (K) \rightarrow (k)$ is compatible second rank *total deformation gradient tensor*. Here time t as scalar parameter allows for family of deformed configura-

tions (k). In the papers dealing with continuum representations of dislocation distributions configuration (k) is imagined to be cut into small elements denoted by (n) , these being subsequently brought to the temperature of (K) free of neighbors. The deformation tensor $\mathbf{F}_E(.,t) : (n) \rightarrow (k)$ obtained in such a way is incompatible and should be called the *thermoelastic distortion tensor* whereas (n)-elements are commonly named as *natural state local reference configurations* (cf. for instance [11, 12, 13]). Of course, the corresponding *plastic distortion tensor*

$$\mathbf{F}_P(.,t) := \mathbf{F}_E(.,t)^{-1} \cdot \mathbf{F}(.,t), \tag{1}$$

is also incompatible. Here $\mathbf{F}$ is found by comparison of material fibres in (K) and (k) while $\mathbf{F}_E$ is determined by crystallographic vectors in (n) and (k). Multiplying above formula from the left hand side by $\mathbf{F}(.,t)$ we reach at Kröner's decomposition rule which is often wrongly named as Lee's decomposition formula. It is worthy of note that $\mathrm{curl}\mathbf{F}(.,t)^{-1} \neq \mathbf{O}$ and this *incompatibility* is commonly connected to an asymmetric second order *tensor of dislocation density*.

In the paper [13] the authors connected to the natural state elements magnetization vectors in such a way that they are isoclinic in (n) and inhomogeneous in (k) the inhomogeneity being responsible for magnetostrictive strains. Such an assumption is very much in accord with the above geometrical argument and is accepted in the sequel.

2 Evolution and constitutive equations

Ignoring ferroelectric effects the next set of objective and Galilei-invariant state variables [1, 20] (with acceptance of internal variables approach) should be introduced in general

$$\Gamma := \{\mathbf{E}, \mathbf{E}_P, A, T, GRADT, Q, \vec{\mathcal{Q}}, \vec{\mathcal{J}}, \vec{\mathcal{P}}, \vec{\mathcal{M}}, \vec{\mathcal{M}}_R\}, \quad \Gamma \in \mathcal{G}, \tag{2}$$

where $2\,\mathbf{E} = \mathbf{F}^T \cdot \mathbf{F} - \mathbf{1} \equiv \mathbf{C} - \mathbf{1}$ is the Lagrangean total strain tensor,
$2\mathbf{E}_P = \mathbf{F}_P^T \cdot \mathbf{F}_P - \mathbf{1} \equiv \mathbf{C}_P - \mathbf{1}$ - is the Lagrangean plastic strain tensor,
$GRADT \equiv \mathbf{F}^{-1} \cdot gradT$ - temperature gradient,
$Q \equiv J^{-1} q_f$ - the electric charge, $(J^2 \equiv \det \mathbf{C})$
$\vec{\mathcal{Q}} \equiv J\mathbf{F}^{-1} \cdot \vec{q}$ - the heat flux vector ,
$\vec{\mathcal{J}} \equiv J\mathbf{F}^{-1} \cdot (\vec{j} - \vec{v} q_f\ )$ - the electric current vector,
$\vec{\mathcal{P}} \equiv J\mathbf{F}^{-1} \cdot \vec{p}$ - the polarization vector,
$\vec{\mathcal{M}} \equiv J\mathbf{F}^{-1} \cdot (\vec{m} + \vec{v} \times \vec{p}/c)$ - the magnetization vector,
$\vec{\mathcal{M}}_R$ - the (K)-defined irreversible (residual) magnetization vector,
A - the volume defined dislocation density,

with $GRADT, Q, \vec{\mathcal{Q}}, \vec{\mathcal{J}}, \vec{\mathcal{P}}, \vec{\mathcal{M}}, \vec{\mathcal{M}}_R$ and A being (K)-defined, whereas by $grad \equiv \nabla\otimes$ and $GRAD \equiv \nabla_0 \otimes$ are denoted, respectively, vectorial differential operators in (k) and (K). The above set may be otherwise understood as a point belonging to the extended configuration (deformation-temperature-electromagnetic) space $\mathcal{G}$. To this configuration point there corresponds a reaction point represented by the set

$$\Delta_1 := \{\mathbf{T}, u, s, \vec{S}, \vec{\mathcal{E}}, \vec{\mathcal{B}}\}, \quad \Delta_1 \in \mathcal{D}, \tag{3}$$

where $\mathbf{T} = J\mathbf{F}^{-1}\mathbf{T}_k \cdot \mathbf{F}^{-T}$ - the symmetric Piola-Kirchhoff stress tensor of the second kind related to (K) wherein $\mathbf{T}_k$ is the Cauchy stress,
u and s - the internal energy and the entropy densities,
$\vec{S} \equiv J\mathbf{F}^{-1} \cdot \vec{s}$ - the entropy flux vector,
$\vec{\mathcal{E}} = (\vec{e} + \vec{v} \times \vec{b}/c) \cdot \mathbf{F} \equiv \vec{E} + \mathbf{F} \cdot \vec{V} \times \vec{B}/c$ - the electromotive intensity and
$\vec{\mathcal{B}} = (\vec{b} - \vec{v} \times \vec{e}/c) \cdot \mathbf{F} \equiv \vec{B} - \mathbf{F} \cdot \vec{V} \times \vec{E}/c$ - the magnetic induction vector,

and $\mathcal{D}$ is the extended stress space, whose (K)-defined objective and Galilei-invariant elements are listed above. At this place the constitutive equations are simply stated by the bijective mapping:

$$\Delta_1 = \mathcal{R}(\Gamma) \equiv \Delta_1(\Gamma) \quad or \quad \mathcal{R} : \mathcal{G} \to \mathcal{D}, \tag{4}$$

which is too general so that the thermodynamic analysis presented henceforth is aimed to supply restrictions concordant with the second

law of thermodynamics.

The evolution functions are proposed here in such a way to be compatible with (2 - 4) and are collected into the set

$$\Delta_2 := \{\vec{\mathcal{Q}}^*, \vec{\mathcal{J}}^*, \mathbf{E}^*, \vec{M}^*, A^*\}, \quad \Delta_2 \in \mathcal{D}, \tag{5}$$

such that objective evolution equations simply read:

$$D\vec{\mathcal{Q}} = \vec{Q}^*(\Gamma), \tag{6}$$

$$D\vec{\mathcal{J}} = \vec{J}^*(\Gamma), \tag{7}$$

$$D\mathbf{E}_P = \mathbf{E}^*(\Gamma), \tag{8}$$

$$D\vec{\mathcal{M}}_R = \vec{M}^*(\Gamma), \tag{9}$$

$$D\,A = A^*(\Gamma), \tag{10}$$

where the material time derivative is designated by D. The simplicity of left hand sides of (6 - 10) owing to the absence of corotational time derivatives has the origin in the accepted Lagrangian description of constitutive functions and variables listed in (2 - 3) .

A thermodynamic process occurring in the considered body is described by the evolution equations and by the following balance laws (ρ_0 and ρ are mass densities in (K) and (k) while $\vec{v}$ is the velocity of the particle) which are equivalent but slightly modified with regard to those of [1]:

$$\begin{aligned} \rho Du - \{\mathbf{T} + (\vec{\mathcal{P}} \otimes \vec{\mathcal{E}} - \vec{B} \otimes \vec{\mathcal{M}} + \mathbf{1}\vec{B} \cdot \vec{\mathcal{M}}) \cdot \mathbf{C}^{-1}\} \quad &: \quad D\mathbf{E} - \\ \vec{\mathcal{J}} \cdot \vec{\mathcal{E}} - \vec{\mathcal{E}} \cdot D\vec{\mathcal{P}} - \vec{B} \cdot D\vec{\mathcal{M}} + DIV\vec{\mathcal{Q}} - \rho_0 h \quad &= \quad 0, \end{aligned} \tag{11}$$

$$\rho_0 - \rho J = 0, \tag{12}$$

$$\rho D\vec{v} - \vec{f} - \vec{f}^{em} - DIV(\mathbf{T}\cdot\mathbf{F}^T)/J = \vec{0} \tag{13}$$

with conventional notation

$$\begin{aligned} J\vec{f}^{em} \;\equiv\; & Q\vec{\mathcal{E}} + (\vec{\mathcal{J}} + D\vec{\mathcal{P}}) \times \frac{1}{c}\vec{B}\cdot\mathbf{F}^{-1} + \vec{\mathcal{P}}\cdot\nabla_0 \otimes (\vec{\mathcal{E}}\cdot\mathbf{F}^{-1}) + \\ & J\mathbf{F}^{-1}\cdot\nabla_0 \otimes (\frac{1}{J}\vec{B}\cdot\mathbf{F})\cdot\mathbf{F}^{-1}\cdot\vec{\mathcal{M}}, \end{aligned}$$

$$\mathbf{T} - \mathbf{T}^T + (\vec{\mathcal{P}}\otimes\vec{\mathcal{E}} - (\vec{B}\otimes\vec{\mathcal{M}})\cdot\mathbf{C}^{-1} - \mathbf{C}^{-1}\cdot(\vec{\mathcal{E}}\otimes\vec{\mathcal{P}} - (\vec{\mathcal{M}}\otimes\vec{B}) = \vec{0}, \tag{14}$$

$$DQ + JDIV\vec{\mathcal{J}} = 0, \tag{15}$$

$$Q - JDIV(\vec{P} + J\mathbf{C}^{-1}\cdot\vec{E}) = 0, \tag{16}$$

$$CURL\vec{\mathcal{E}} + \frac{1}{c}D\vec{B} = 0, \tag{17}$$

$$DIV\vec{B} = 0, \tag{18}$$

$$CURL(J^{-1}\mathbf{C}\cdot\vec{B} - \vec{\mathcal{M}}) - \frac{1}{c}(D\vec{E} + D\vec{\mathcal{P}} + \vec{\mathcal{J}}) = 0, \tag{19}$$

which are, respectively, the equation of energy balance , the mass conservation law, the equation of balance of momentum , the equation of balance of angular momentum, the balance of electric charge and Maxwell equations.

Let us specify more precisely the scope of this work by the assumptions:

(A1) ferroelectric and ferrimagnetic effects, intrinsic spin, exchange forces

and gyromagnetic effects are ignored (with negligible precessional velocity of magnetization - cf. [19]

(A2) all time rates are small i.e. the considered process is slow.

The consequence of *(A1)* is simplification of the set of internal variables losing from it gradient of the magnetization vector assuming in such a way that balance law for magnetization [21] (i.e. balance of angular momentum of spin continuum in wording of [1]) is identically satisfied. One consequence of (A2) is that all the terms of the form $(\vec{v}/c) \times (\cdot)$ in electromagnetic vector fields listed in (1 - 2) may be disregarded.

The above listed balance laws imply constraints on the elements of the set $\{\Gamma\} U \{D\Gamma\}$ causing breaking of their independence which is the essence of the Liu's theorem [15]. There is still another constraint on these elements in the case of inelastic deformation process: the essential notion of yield surface which divides sharply two regions of material behavior. Let us define dynamic and static scalar yield functions in the following way:

$$f = f^*(\mathbf{T}, T, \mathbf{E}_P, \vec{\mathcal{M}}_R) \equiv h(\Gamma), \tag{20}$$

$$f_0 = f^*(\mathbf{T}^{\#}, T, \mathbf{E}_P, \vec{\mathcal{M}}_R) \equiv h_0(\Gamma), \tag{21}$$

where $\mathbf{T}^{\#}$ is static stress corresponding to the dynamic viscoplastic stress $\mathbf{T}$. Their difference is usually named the *overstress tensor* and may be represented by a linear function of $D\mathbf{E}_P$ (linearity appears as consequence of *(A2)*):

$$\Delta\mathbf{T} := \mathbf{T} - \mathbf{T}^{\#} = \mathcal{T}(\Gamma) : D\mathbf{E}_P, \tag{22}$$

with $\mathcal{T}(\Gamma)$ being fourth rank *tensor of plastic viscosity coefficients.* Introducing the *plastic strain rate intensity* by

$$Dp := (D\mathbf{E}_P^T \, : \, D\mathbf{E}_P)^{1/2} \equiv \|D\mathbf{E}_P\| \geq 0, \tag{23}$$

the classification:

$f > 0, f_0 = 0, Dp > 0$ - viscoplastic behavior;

$f = f_0 = 0, Dp = 0$ - elastoplastic frontier;

$f = f < 0, Dp = 0$ - elastic behavior;

and the kinematic constraint:

$$< f > Df_0 = 0, \tag{24}$$

may be formulated in a straightforward way, where $< x >= 1$ for x positive and $< x >= 0$ otherwise.

All thermodynamic processes must obey the master law of nature i.e. the second law of thermodynamics which in our case reads:

$$\rho Ds + DIV\vec{S} - \rho\frac{r}{T} = 0, \tag{25}$$

where r/T is the entropy source. Precisely defined a thermodynamic process is a solution of evolution and balance equations which obeys (25). The analysis of the above entropy inequality (25) by the Liu's theorem may be described as follows. Replacing $s^*(\Gamma)$ and $\vec{S}^*(\Gamma)$ into (25) this becomes a differential inequality linear with respect to the elements of the set $\{D\Gamma\} \cup \{GRAD\Gamma\}$ namely:

$$\begin{aligned}
&\rho_0 Ds + DIV\vec{S} - \rho\frac{r}{T} - \Lambda^u \Big\{ -[\mathbf{T} + (\vec{\mathcal{P}} \otimes \vec{\mathcal{E}} - \vec{B} \otimes \vec{\mathcal{M}} + \\
&\mathbf{1}\vec{B} \cdot \vec{\mathcal{M}}) \cdot \mathbf{C}^{-1}] : D\mathbf{E} + \rho_0 Du - \vec{\mathcal{J}} \cdot \vec{\mathcal{E}} - \vec{\mathcal{E}} \cdot D\vec{\mathcal{P}} - \vec{B} \cdot D\vec{\mathcal{M}} + \\
&DIV\vec{\mathcal{Q}} - \rho_0 h \Big\} - \vec{\Lambda}^v \cdot [\rho D\vec{v} - \vec{f} - \vec{f}^{em} - J^{-1} DIV(\mathbf{T} \cdot \mathbf{F}^T)] - \\
&\Lambda^\ell : [\mathbf{T} - \mathbf{T}^T + (\vec{\mathcal{P}} \otimes \vec{\mathcal{E}} - \vec{B} \otimes \vec{\mathcal{M}}) \cdot \mathbf{C}^{-1} - \mathbf{C}^{-1} \cdot (\vec{\mathcal{E}} \otimes \vec{\mathcal{P}} - \\
&\vec{\mathcal{M}} \otimes \vec{B})] - \Lambda^f < f > Df_0 - -\vec{\Lambda}^Q \cdot [D\vec{\mathcal{Q}} - \vec{Q}^*(\Gamma) - \vec{\Lambda}^J \cdot [D\vec{\mathcal{J}} - \\
&\vec{J}^*(\Gamma)] - \vec{\Lambda}^M \cdot [D\vec{\mathcal{M}}_R - \vec{M}^*(\Gamma)] - \Lambda^E : [D\mathbf{E}_P - \mathbf{E}^*(\Gamma] - \\
&\Lambda^A [DA - A^*(\Gamma)] - \Lambda^q [DQ + JDIV\vec{\mathcal{J}}] - \Lambda^1 [Q - JDIV(\vec{P} + \\
&J\mathbf{C}^{-1} \cdot \vec{E})] - \vec{\Lambda}^2 \cdot [CURL\vec{\mathcal{E}} + \frac{1}{c} D\vec{B}] - \Lambda^3 DIV\vec{B} - \\
&\vec{\Lambda}^4 \cdot [CURL(J^{-1}\mathbf{C} \cdot \vec{B} - \vec{M}) - \frac{1}{c}(DE + D\vec{\mathcal{P}} + \vec{\mathcal{J}}] \geq 0.
\end{aligned} \tag{26}$$

By introducing Lagrange multipliers all the elements of the set $\{D\Gamma\} \cup \{GRAD\Gamma\}$ except $GRADT$ (which is already included into

Γ) become mutually independent. Hence, in thus extended inequality all the coefficients with the elements of the set $\{D\Gamma\} \cup \{GRAD\Gamma\}$ must vanish. This gives rise to the following constitutive restrictions (cf. [6]):

$$\vec{S} = \Lambda^u(T)\vec{Q} + J\Lambda^q(T)\vec{J} \equiv (\vec{Q} + J\lambda^q\vec{J})/T, \tag{27}$$

$$\mathbf{T} = \rho\partial_{\mathbf{E}}F + < f > T\Lambda^f\partial_{\mathbf{E}}f_0 + \mathbf{C}^{-1}\cdot(\vec{\mathcal{E}}\otimes\vec{\mathcal{P}} - \vec{\mathcal{M}}\otimes\vec{B} + \mathbf{1}\vec{B}\cdot\vec{\mathcal{M}}), \tag{28}$$

$$s = \partial_T F + < f > T\rho^{-1}\Lambda^f\partial_T f_0, \tag{29}$$

$$\vec{\mathcal{E}} = -\rho_0\partial_{\vec{\mathcal{P}}}F - < f > T\Lambda^f\partial_{\vec{\mathcal{P}}}f_0, \tag{30}$$

$$\vec{B} = -\rho\partial_{\vec{\mathcal{M}}}F - < f > T\Lambda^f\partial_{\vec{\mathcal{M}}}f_0, \tag{31}$$

$$\vec{0} = \partial_{GRADT}F + < f > T\rho^{-1}\Lambda^f\partial_{GRADT}f_0, \tag{32}$$

and the residual dissipation inequality

$$\begin{aligned}\vec{\Lambda}^Q\cdot\vec{Q}^*(\Gamma) \quad + \quad & \vec{\Lambda}^J\cdot\vec{J}^*(\Gamma) + \vec{\Lambda}^M\cdot\vec{M}^*(\Gamma) + \Lambda^E : \mathbf{E}^*(\Gamma) + \Lambda^A A^*(\Gamma) + \\ & \partial_T\vec{S}\cdot GRAD\,T + T^{-1}\vec{\mathcal{J}}\cdot\vec{\mathcal{E}} \geq 0,\end{aligned} \tag{33}$$

where $F := u - s(\Lambda^u)^{-1} \equiv u - Ts$ is the free energy density. If the thermodynamic process is very near to equilibrium (cf. [6]) then the above residual inequality permits the direct application of Onsager-Casimir reciprocity relations. The above Lagrange multipliers are explicitly given by:

$$\vec{\Lambda}^Q = -\rho_0 T^{-1}\partial_{\vec{Q}}F - < f > \Lambda^f\partial_{\vec{Q}}f_0,$$

$$\vec{\Lambda}^J = -\rho_0 T^{-1}\partial_{\vec{J}}F - < f > \Lambda^f \partial_{\vec{J}} f_0,$$

$$\vec{\Lambda}^M = -\rho_0 T^{-1}\partial_{\vec{M}}F - < f > \Lambda^f \partial_{\vec{M}} f_0,$$

$$\Lambda^E = -\rho_0 T^{-1}\partial_{\mathbf{E}}F - < f > \Lambda^f \partial_{\mathbf{E}} f_0,$$

$$\Lambda^A = -\rho_0 T^{-1}\partial_{A}F - < f > \Lambda^f \partial_{A} f_0,$$

$$\vec{\Lambda}^v = 0,\ \Lambda^1 = 0,\ \vec{\Lambda}^2 = 0,\ \Lambda^3 = 0,\ \vec{\Lambda}^4 = 0,\ \Lambda^\ell = 0.$$

The details of the above procedure are presented elsewhere ([6]).

3 Small magnetoelastic strains of isotropic plastically deformed insulators

In order to illustrate the above derived constitutive and evolution equations we accept in this section the following very simplifying assumptions:

(A3) elastic strain, reversible and irreversible magnetization are small of the same order but plastic strain itself is finite (cf. also [6]);

(A4) thermal and electric effects are neglected,

holding for an isotropic body.

Such assumptions correspond to the so called piezomagnetism processes when magnetization is generated by straining processes (cf. for instance [27] and [28]).

Let us take into account that by its very nature the mechanical stress disappears when pure elastic strain vanishes and, similarly, the local magnetic field equals to zero if the reversible magnetization vanishes. Then, according to [13] it is reasonable to introduce magnetostrictive strain by means of

$$\mathbf{E}_E := \mathbf{E} - \mathbf{E}_P = \mathbf{F}_P^T \cdot (\mathbf{F}_E^T \cdot \mathbf{F}_E - \mathbf{1}) \cdot \mathbf{F}_P, \tag{34}$$

or, in other words:

$$\mathbf{E}_E \equiv \mathbf{E}_e + \mathcal{L} : (\overrightarrow{\mathcal{M}}_0 \otimes \overrightarrow{\mathcal{M}}_0) \equiv \mathbf{E}_e + \mathbf{E}_{mag}. \tag{35}$$

Here $\mathcal{L}$ is the fourth rank tensor of magnetostriction constants symmetric only in indices of the first as well as the second pair whereas the notation $\vec{\mathcal{M}}_0$ stands for the unit vector of the magnetization vector $\vec{\mathcal{M}}$. The constituents of the *Lagrangian elastic strain tensor* $\mathbf{E}_E$, namely, $\mathbf{E}_e$ as well as $\mathbf{E}_{mag}$ are both incompatible and are referred to as *pure elastic strain* and *magnetostrictive strain* .

With these facts taken into account and the above assumptions *(A3-A4)* the constitutive equations for mechanical part of the stress tensor and the local magnetic field specialize into:

$$\begin{aligned} \mathbf{T} &= (c_1\mathbf{1} + c_2\mathbf{E}_P + c_3\mathbf{E}_P^2)\ tr\mathbf{E}_e + 2c_4\ \mathbf{E}_e + \\ &\quad c_5\ (\mathbf{E}_P\cdot\mathbf{E}_e + \mathbf{E}_e\cdot\mathbf{E}_P) + c_6\ (\mathbf{E}_P^2\cdot\mathbf{E}_e + \mathbf{E}_e\cdot\mathbf{E}_P^2), \end{aligned} \tag{36}$$

$$\mathbf{H} = c_7\mathbf{M}_r + c_8(\mathbf{E}_P\cdot\mathbf{M}_r + \mathbf{M}_r\cdot\mathbf{E}_P) + c_9(\mathbf{E}_P^2\cdot\mathbf{M}_r + \mathbf{M}_r\cdot\mathbf{E}_P^2), \tag{37}$$

where instead of magnetic induction field the internal magnetic field vector $\vec{H}$ (opposing the local magnetic field vector under assumption *(A1)*) has been derived from (31) by making use of tensorial representations for the proper orthogonal group [22]. In the above "magnetic" constitutive equation

$$\vec{\mathcal{M}}_r := \vec{\mathcal{M}} - \vec{\mathcal{M}}_R, \tag{38}$$

is the reversible magnetization vector while the above antisymmetric second rank tensors $\mathbf{H}, \mathbf{M}_r$ and $\mathbf{M}_R$ are made from the corresponding vectors by means of the Ricci third rank permutation tensor defined in (K)-configuration in the following way:

$$\mathbf{H} \equiv \mathcal{E}\cdot\vec{H} = -\mathbf{H}^T, \mathbf{M}_r \equiv \mathcal{E}\cdot\vec{\mathcal{M}}_r = -\mathbf{M}_r^T, \mathbf{M}_R \equiv \mathcal{E}\cdot\vec{\mathcal{M}}_R = -\mathbf{M}_R^T. \tag{39}$$

They are favored instead of the corresponding vectors for convenience and more compact representation. Of course, an equivalent formulation using cross products of vectors $\vec{\mathcal{M}}_r$ and $\vec{\mathcal{M}}_R$ with symmetric

second rank tensor $\mathbf{E}_P$ is also possible. Equation (36) is the generalized Hooke's law accounting for plastic strain induced mechanical anisotropy while the constitutive equation for internal magnetic field predicts magnetic anisotropy induced by the same cause.

The free energy function generating (36) and (37) i.e. :

$$\begin{aligned} F &= \frac{1}{2}c_1\ i_1^2 + \frac{1}{2}c_2\ i_2^2 + \frac{1}{2}c_3\ i_3^2 + c_4\ i_4 + c_5\ i_5 + c_6\ i_6 + \\ &\quad \frac{1}{2}c_7\ i_7 + \frac{1}{2}c_8\ i_8 + \frac{1}{2}c_9\ i_9, \end{aligned} \tag{40}$$

where the following proper and mixed invariants appearing in the above scalar function must be introduced (cf. [22]):

$$\begin{aligned} i_1 &= tr\mathbf{E}_e,\ i_2 = tr\{\mathbf{E}_P \cdot \mathbf{E}_e\},\ i_3 = tr\{\mathbf{E}_P^2 \cdot \mathbf{E}_e\}, \\ i_4 &= tr\{\mathbf{E}_e^2\},\ i_5 = tr\{\mathbf{E}_P \cdot \mathbf{E}_e^2\},\ i_6 = tr\{\mathbf{E}_P^2 \cdot \mathbf{E}_e^2\}, \\ i_7 &= tr\{\mathbf{M}_r^2\},\ i_8 = tr\{\mathbf{E}_P \cdot \mathbf{M}_r^2\},\ i_9 = tr\{\mathbf{E}_P^2 \cdot \mathbf{M}_r^2\}. \end{aligned} \tag{41}$$

In the sequel inverse forms of (36) and (37) will be useful. They can be written in the following way:

$$\begin{aligned} \mathbf{E}_e &= (\gamma_1\mathbf{1} + \gamma_2\mathbf{E}_P + \gamma_3\mathbf{E}_P^2)\ tr\mathbf{T} + 2\gamma_4\ \mathbf{T} + \\ &\quad \gamma_5\ (\mathbf{E}_P \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{E}_P) + \gamma_6\ (\mathbf{E}_P^2 \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{E}_P^2), \end{aligned} \tag{42}$$

and

$$\mathbf{M}_r = \gamma_7\mathbf{H} + \gamma_8(\mathbf{E}_P \cdot \mathbf{H} + \mathbf{H} \cdot \mathbf{E}_P) + \gamma_9(\mathbf{E}_P^2 \cdot \mathbf{H} + \mathbf{H} \cdot \mathbf{E}_P^2). \tag{43}$$

The relationships between sets $\{c_1, ..., c_9\}$ and $\{\gamma_1, ..., \gamma_9\}$ can be found as follows. Let us multiply (36) as well as (42) by the tensors $\mathbf{1}$, $\mathbf{E}_P$ and $\mathbf{E}_P^2$ finding traces of both sides. If we introduce notations:

$$\begin{aligned} s_1 &= tr\mathbf{T},\ s_2 = tr\{\mathbf{E}_P \cdot \mathbf{T}\},\ s_3 = tr\{\mathbf{E}_P^2 \cdot \mathbf{T}\}, \\ s_4 &= tr\{\mathbf{T}^2\},\ s_5 = tr\{\mathbf{E}_P \cdot \mathbf{T}^2\},\ s_6 = tr\{\mathbf{E}_P^2 \cdot \mathbf{T}^2\}, \\ s_7 &= tr\{\mathbf{H}^2\},\ s_8 = tr\{\mathbf{E}_P \cdot \mathbf{H}^2\},\ s_9 = tr\{\mathbf{E}_P^2 \cdot \mathbf{H}^2\}. \end{aligned} \tag{44}$$

then such a procedure will make possible finding relationships between $\{c_1, ..., c_6\}$ and $\{\gamma_1, ..., \gamma_6\}$. Of course, the same procedure applied to (37) as well as (43) would connect sets $\{c_7, ..., c_9\}$ and $\{\gamma_7, ..., \gamma_9\}$.

Similarly, the evolution equations for plastic strain rate and residual magnetization rate are explicitly stated by the following formulae:

$$\begin{aligned} D\mathbf{E}_P &= < f > [d_1\mathbf{1} + d_2\mathbf{E}_e + d_3\ (\mathbf{E}_e{\cdot}\mathbf{E}_P + \mathbf{E}_P \cdot \mathbf{E}_e) + \\ &\quad d_4\ (\mathbf{E}_e{\cdot}\mathbf{E}_P^2 + \mathbf{E}_P^2 \cdot \mathbf{E}_e) + d_5\mathbf{E}_P + d_6\mathbf{E}_P^2 + \\ &\quad d_7(\mathbf{M}_r \cdot \mathbf{E}_P - \mathbf{E}_P \cdot \mathbf{M}_r) + d_8(\mathbf{M}_r \cdot \mathbf{E}_P^2 - \mathbf{E}_P^2 \cdot \mathbf{M}_r) + \\ &\quad d_9(\mathbf{M}_R \cdot \mathbf{E}_P - \mathbf{E}_P \cdot \mathbf{M}_R) + d_{10}(\mathbf{M}_R \cdot \mathbf{E}_P^2 - \mathbf{E}_P^2 \cdot \mathbf{M}_R) + \\ &\quad d_{11}(\mathbf{E}_P \cdot \mathbf{M}_r \cdot \mathbf{E}_P^2 - \mathbf{E}_P^2 \cdot \mathbf{M}_r \cdot \mathbf{E}_P) + \\ &\quad d_{12}(\mathbf{E}_P \cdot \mathbf{M}_R \cdot \mathbf{E}_P^2 - \mathbf{E}_P^2 \cdot \mathbf{M}_R \cdot \mathbf{E}_P)], \end{aligned} \tag{45}$$

$$\begin{aligned} D\mathbf{M}_R &= e_1\mathbf{M}_r + e_2(\mathbf{M}_r \cdot \mathbf{E}_P + \mathbf{E}_P \cdot \mathbf{M}_r) + e_3\left(\mathbf{M}_r \cdot \mathbf{E}_P^2 + \right. \\ &\quad \left. \mathbf{E}_P^2 \cdot \mathbf{M}_r\right) + e_4\mathbf{M}_R + e_5(\mathbf{M}_R \cdot \mathbf{E}_P + \mathbf{E}_P \cdot \mathbf{M}_R) + \\ &\quad e_6(\mathbf{M}_R \cdot \mathbf{E}_P^2 + \mathbf{E}_P^2 \cdot \mathbf{M}_R) + e_7(\mathbf{E}_P \cdot \mathbf{E}_E - \mathbf{E}_E \cdot \mathbf{E}_P) + \\ &\quad e_8(\mathbf{E}_P^2 \cdot \mathbf{E}_E - \mathbf{E}_E \cdot \mathbf{E}_P^2) + e_9(\mathbf{E}_P^2 \cdot \mathbf{E}_E \cdot \mathbf{E}_P - \mathbf{E}_P \cdot \mathbf{E}_E \cdot \mathbf{E}_P^2). \end{aligned} \tag{46}$$

It should be noted here that all the scalar coefficients in above relationships (36)-(37) and (3)-(46) are functions of the principal invariants of the plastic strain tensor $\mathbf{E}_P$. Of course, if plastic strain itself is small, then the corresponding complete linearization of constitutive and evolution equations is straightforward which might be of interest especially if dynamic effects are considered i.e. wave equations of the linearized problem written (cf. [6]). Evolution equations then would reduce to Onsager-Casimir reciprocity relations.

4 Generalized normality applied to small magnetoelastic-viscoplastic strains of isotropic insulators

At the end of this paper let us see what consequences could have an introduction of a generalized loading function Ω with the following

orthogonality properties (cf. [1])

$$D\mathbf{E}_P = D\Lambda\frac{\partial\Omega}{\partial\mathbf{H}} \quad \text{and} \quad D\mathbf{M}_R = D\Lambda\frac{\partial\Omega}{\partial\mathbf{H}}. \tag{47}$$

where the material time rate of a scalar function Λ vanishes if the yield functions f as well as f_0 are either negative or zero (cf. (24)).Suppose, for simplicity that the assumption *(A4)* holds whereas the assumption *(A3)* is replaced by means of:

(A5) elastic and plastic strain, reversible and irreversible magnetization, as well as plastic strain rate and irreversible magnetization rate are all small of the same order.

Then we may assume the loading function in the following polynomial form

$$\Omega = \frac{1}{2}\omega_1\ s_1^2 + \frac{1}{2}\omega_2\ s_4 + \frac{1}{2}\omega_3\ s_7, \tag{48}$$

leading by means of (48) into the following two evolution equations

$$D\mathbf{E}_P = D\Lambda\ [\ \omega_1\mathbf{1}\ tr\mathbf{T} + \omega_2\ \mathbf{T}\ ], \tag{49}$$

$$D\mathbf{M}_R = D\Lambda\ \omega_3\ \mathbf{H}, \tag{50}$$

whose simplicity follows from the above very special loading scalar function Ω. In addition, the free energy function (40) might be reduced into:

$$F = \frac{1}{2}c_1\ i_1^2 + c_4\ i_4 + \frac{1}{2}c_7\ i_7 + F^*(\mathbf{E}_P, \mathbf{M}_R) \tag{51}$$

where F^* would depend proper and mixed invariants of $\mathbf{E}_P$ and $\mathbf{M}_R$. Such a function allows the following specialized constitutive equations:

$$\mathbf{T} = c_1\mathbf{1}\ tr\mathbf{E}_e + 2c_4\ \mathbf{E}_e, \tag{52}$$

$$\mathbf{H} = c_7\mathbf{M}_r, \tag{53}$$

whose material constants are easily recognized to be *Lame constants*

$$c_1 \equiv \lambda \quad \text{and} \quad c_2 \equiv \mu, \tag{54}$$

as well as the constant of *magnetic susceptibility* (cf. [19]) is identified with inverse of c_7 i.e.

$$\chi \equiv \frac{1}{c_7}. \tag{55}$$

It should be noted that if the tensor of magnetostriction constants $\mathcal{L}$ is introduced into (52) then magnetostriction process can be shown explicitly.

The situation described in this section could correspond to piezomagnetism induced by low-cycle fatigue of ferromagnetics. Such a process was investigated in the paper [27]. A cylindrical specimen of AISI 1018 was uniaxially treated by push-pull tests on MTS-810 servohydraulic testing machine such that total strain was periodic and triangularly shaped $\|\mathbf{E}\| \in \{0,\ 0.009\}$ with cycle duration of 2 s. Magnetic induction due to piezomagnetism effect was also almost periodic with very slight changes with increase of relative number of cycles N/N_f and cumulation of phase delay with respect to strain with growth of accumulated plastic strain. Maxima and minima of $\mathbf{E}$ are almost coincident with minima and maxima of the magnetic induction $\mathbf{B}$. Thus, if plastic strain accumulation is calculated by means of

$$\pi(t) := \int_0^t \|D\mathbf{E}_P(\tau)\| \ d\tau \tag{56}$$

then if uniaxial components of $\mathbf{E}$ as well as $\mathbf{B}, \mathbf{M}_r, \mathbf{M}_R$ are denoted by means of E_{11} as well as B_{11},M_{r11}, M_{R11} the following memory-type equation

$$B_{11}(t) := \int_0^t J(\pi, t-\tau)\ DE_{11}(\tau)\ d\tau, \tag{57}$$

would describe fairly well the above explained experimental situation. Time differentiation of the above relationship gives rise to the expression:

$$DB_{11}(t) := J(\pi, 0)\ DE_{11}(t) + \int_0^t \frac{\partial}{\partial t} J(\pi, t-\tau)\ DE_{11}(\tau)\ d\tau. \tag{58}$$

In the above integro-differential equation the second term on the right hand side is responsible for the above mentioned change of time delay and the deflection of pure periodicity of $B_{11}(t)$. Therefore, it is much smaller than the first part. On the other hand, if the constitutive equation $B_{11} = \mu\ H_{11}$ (where μ is *magnetic permeability*) is used, then we have

$$DB_{11} = \frac{\mu}{\chi}(DM_{11} - DM_{R11}) \quad \text{as well as} \quad DE_{11} = DE_{E11} + DE_{P11}. \tag{59}$$

Since in the paper [27] such a splitting has not been made, a more specific comment on simultaneous zeros of $D\mathbf{E}_P$ and $D\mathbf{M}_R$ (following from (49) and (50)) is not possible.

5 Conclusions

Concluding this paper it is inevitable to compare the foregoing results with existing achievements in the field. The main contributions to viscoplasticity of ferromagnetic materials have been given by Maugin and his collaborators in [1, 19]. The principal assumptions accepted in our work are closer to the scope of the first of these two papers where

1. small strain case together with absence of exchange forces and gyromagnetic effects has been assumed
2. the accent on hysteresis effects has been given and
3. evolution equations derived by normality of plastic strain rate and residual magnetization rate onto a loading surface.

The main results of this paper might be summarized as follows:

1. in the case of finite plastic strains magnetic anisotropy induced by plastic strain is predicted by (46) where development of residual magnetization by mechanical terms is also evident;
2. the influence of magnetization on plastic strain rate is obtained even in the case of isotropic ferromagnetic materials;

3. the extended thermodynamics procedure allows for more general history effects with inhomogeneities of magnetization taken into account;

4. the obtained relationships with couplings allow for magnetic measurements of inelastic phenomena but the measurements will show their order of magnitude and practical measurability of these phenomena;

5. the developed theory is of non-associate type for plastic strain rate and residual magnetization rate are not perpendicular to the yield surface;

6. although a generalized normality is much simpler with smaller number of material constants, a careful examination of the experiments on piezomagnetism and magnetostriction processes would give the final judgement which theory should be applied.

Acknowledgment The author is grateful to Prof. V. Babović for valuable suggestions concerning some aspects of electromagnetism dealt with in this paper.

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O viskoplastičnosti feromagnetika
UDK 539.374; 537.622.4

U radu se razmatraju viskoplastični feromagnetski materijali. Evolucione jednačine se izvode iz specijalnog slučaja materijala diferencijalnog tipa sa neelastičnom bledećom memorijom ili pomoću generalisane ortogonalnosti i funkcije opterećenja. U oba slučaja se na skup konstitutivnih i evolucionih jednačina primenjuje tenzorska reprezentacija. Ograničenja na jednačine polja se uspostavljaju pomoću proširene termodinamike ireverzibilnih procesa. Mala magnetoelastična deformacija u dva specijalna slucaja: (a) konačne plastične deformacije kao i (b) male plastične deformacije izotropnih izolatora se analizira u detalje sa osvrtom na neke eksperimentalne rezultate gde se piezomagnetizam pojavljuje tokom niskocikličnog zamora.