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Stochastic stability of a viscoelastic beam under time and space-dependent loading

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Abstract

In this paper uniform stochastic stability of a viscoelastic Voigt-Kelvin beam is investigated. The beam is simply supported and subjected to time and space-dependent wide-band, Gaussian process. Using Liapunov direct method and functional Ito lemma uniform stochastic stability condition is obtained. Stability regions are given as function of retardation time, intensity of stochastic loading and spatial correlation radius.

1 Introduction

The dynamic stability of continuous systems under time-dependent deterministic or stochastic loading was studied by many authors in last thirty years. The problem has been solved not only for a simple elastic beams subject to time-dependent force but also for archers, panels, plates and shells. In most papers the dissipation of energy was described by an external viscous model of damping. The "best" Liapunov functional suitable for studying the almost sure asymptotic stability of elastic beams and panels axially compressed by zero mean stationary ergodic forces, whose samples are continuous with probability one, was introduced by Kozin [1].

One of the first analyses of the dynamic stability of viscoelastic columns has been made by Plaut [3], where the Liapunov method is used to determine stability criteria. The stability problem associated with an Euler-Bernoulli beam made of an arbitrary linear viscoelastic material in presence of constant as well as periodic loads is analyzed by Shirahatti and Sinha [4]. The application of a finite time stability concept is shown for the constant loading case when the traditional stability criterion fails to make sense. For the case of a periodic loading, the stability diagrams are obtained through an application of Floquet theory.

Tylikowski [5] studied linear viscoelastic systems subjected to time-dependent deterministic or stochastic parametric excitations. Using appropriate Liapunov functional general sufficient conditions for the asymptotic stability, the almost sure asymptotic stability as well as the uniform stochastic stability are obtained.

The Voigt-Kelvin column with random initial geometrical imperfections is considered by Tylikowski [6]. Sufficient criteria for stability of viscoelastic composite columns via direct Liapunov method is obtained by the same author [7]. The influence of standard model parameters and eigenfrequencies on stability domains is shown.

In [2] the direct Liapunov method is used for investigation of the stochastic stability of viscoelastic beams including effect of the rotator inertia. Using suitable functional obtained by the Parks-Pritchard method the sufficient conditions for almost sure asymptotic stability as function of a reduced retardation time and geometric parameters is derived.

The purpose of the present paper is investigation of the dynamic stability of viscoelastic beams subjected to time and space-dependent axial forces. Using recent results of the stochastic processes theory we can apply Liapunov method for obtaining sufficient criteria for uniform stochastic stability in terms of retardation time, intensity and spatial correlation radius.

2 Problem formulation

We consider a uniform viscoelastic beam of length l, simply supported, subjected to uniform distributed stochastic axial time and space-dependent loading $F_a(\omega, T, Z)$. The deflected shape of a typical element of a beam is shown in Figure 1. F_T is the shear force due to bending, M is the bending moment, A is the cross-sectional area, I_x is the cross-section moment of inertia, w is the transverse displacement in the Y direction T is the time and ρ is the mass density.

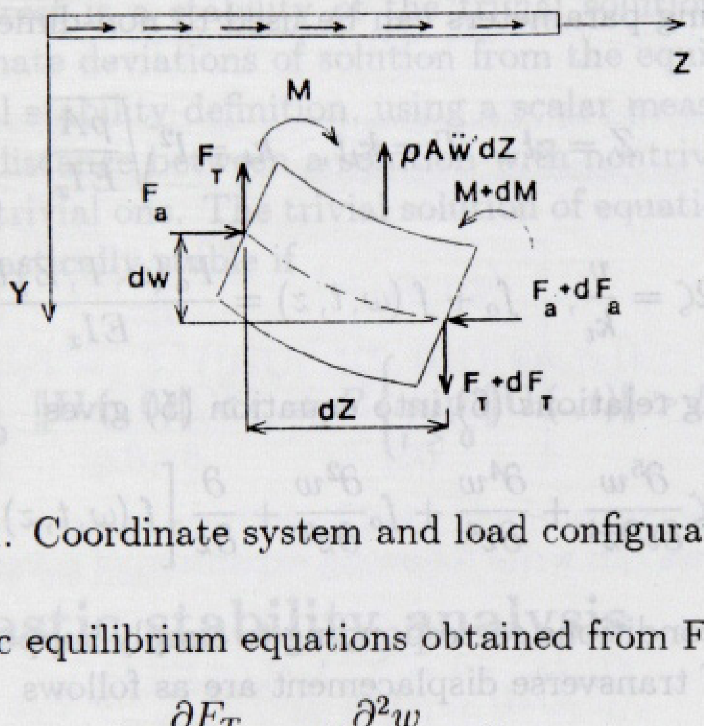

Fig. 1. Coordinate system and load configuration.

The dynamic equilibrium equations obtained from Fig. 1 are

$$\frac{\partial F_T}{\partial Z} - \rho A \frac{\partial^2 w}{\partial T^2} = 0, \tag{1}$$

$$F_T + F_a \frac{\partial w}{\partial Z} - \frac{\partial M}{\partial Z} = 0, \tag{2}$$

where shear deformation and rotatory inertia are neglected.

We assume a Voigt-Kelvin type of viscoelasticity where stress-strain relation has the form

$$\sigma = E\left(\varepsilon + v\frac{d\varepsilon}{dT}\right) \tag{3}$$

where E is Young modulus and v is so called retardation time.

Euler-Bernoulli moment curvature relationship for small deflections, with respect to (3), gives

$$M = -EI_x\left(\frac{\partial^2 w}{\partial Z^2} + v\frac{\partial^3 w}{\partial Z^2 \partial T}\right) \tag{4}$$

F_T and M can be eliminated from equations (1), (2) and (4). The governing equation of motion for the beam is

$$\rho A\frac{\partial^2 w}{\partial T^2} + vI_x\frac{\partial^5 w}{\partial T \partial Z^4} + EI_x\frac{\partial^4 w}{\partial Z^4} + \frac{\partial}{\partial Z}\left[F_a(\omega, T, Z)\frac{\partial w}{\partial Z}\right] = 0. \tag{5}$$

The following parameters can be used to non-dimensionalize equation (5)

$$Z = zl, \quad T = k_t t, \quad k_t = l^2\sqrt{\frac{\rho A}{EI_x}}$$
$$2\zeta = \frac{v}{k_t}, \quad f_o + f(\omega, t, z) = \frac{F_a(\omega, T, Z)\, l^2}{EI_x}. \tag{6}$$

Substituting relations (6) into equation (5) gives

$$\frac{\partial^2 w}{\partial t^2} + 2\zeta\frac{\partial^5 w}{\partial t \partial z^4} + \frac{\partial^4 w}{\partial z^4} + f_o\frac{\partial^2 w}{\partial z^2} + \frac{\partial}{\partial z}\left[f(\omega, t, z)\frac{\partial w}{\partial z}\right] = 0. \tag{7}$$

Boundary conditions corresponding to simply supported edges written in terms of transverse displacement are as follows

$$\left.\begin{array}{l} z = 0 \\ z = 1 \end{array}\right\} w = 0, \quad \frac{\partial^2 w}{\partial z^2} = 0. \tag{8}$$

Denoting

$$v = \frac{\partial w}{\partial t}, \quad \mathbf{U} = \left\{\begin{array}{c} w \\ v \end{array}\right\}, \quad \{\xi\} = \frac{d\mathbf{W}}{dt} = \left\{\begin{array}{c} f(\omega, t, z) \\ \dfrac{\partial f(\omega, t, z)}{\partial z} \end{array}\right\}, \tag{9}$$

we can rewrite equation (7) in the form

$$\frac{d\mathbf{U}}{dt} = \mathbf{AU} + [\mathbf{BU}]\{\xi\}, \tag{10}$$

where $\{\xi\}$ is a generalized "white" noise belonging to a Hilbert space, and

$$\mathbf{A}=\begin{bmatrix} 0 & 1 \\ -\frac{\partial^4}{\partial z^4}-f_o\frac{\partial^2}{\partial z^2} & -2\zeta\frac{\partial^4}{\partial z^4}\end{bmatrix},\quad [\mathbf{BU}]=\begin{bmatrix} 0 & 0 \\ -\frac{\partial^2 w}{\partial z^2} & -\frac{\partial w}{\partial z}\end{bmatrix}. \tag{11}$$

Equation (10) is a stochastic evolution of Ito type in Hilbert space X with inner product $\langle .,.\rangle=\int_0^1 (.,.)\,dz$ over a probability space $(\Omega,\mathcal{B},P)$.

Equation (10) has the trivial solution $\mathbf{U}_o=(w_o\ v_o)^T=(0\ 0)^T$. The question of interest is a stability of the trivial solution (equilibrium point). To estimate deviations of solution from the equilibrium we introduce a formal stability definition, using a scalar measure (distance) $\|\cdot\|$, which is a distance between a solution with nontrivial initial conditions and the trivial one. The trivial solution of equation (7) is called uniformly stochastically stable if

$$\bigwedge_{\varepsilon>0}\ \bigwedge_{\delta>0}\ \bigvee_{r>0}\ \|\mathbf{U}(.,0)\|<r\Rightarrow P\left\{\sup_{t\geq 0}\|\mathbf{U}(.,t)\|>\delta\right\}<\varepsilon. \tag{12}$$

3 Stochastic stability analysis

The time dependent component of axial force is assumed to be a wide-band process for any $z\in(0,1)$. In such case we represent it as time derivative of normal Wiener process in Hilbert space Y with expectation equal to zero, $E\{\xi\}=0$, and a covariance operator

$$E\left[\{\xi(\omega,t,z)\}\otimes\{\omega,s,z'\}\right]=\mathbf{Q}(z,z')\,\delta(t-s), \tag{13}$$

where $\mathbf{Q}(z,z')$ is a self-adjoint positive definite nuclear operator, δ - "ordinary" Dirac function, and $\otimes$ denotes the sign of tensor product.

In order to obtain the elements of operator $\mathbf{Q}$ we assume the spatial covariance function of loading $Q_{11}(z_1, z_2)$ to be known

$$Q_{11}(z_1,z_2)\,\delta(t-s) = E\left[f(\omega,t,z_1)\;f(\omega,s,z_2)\right],$$

$$Q_{12}(z_1,z_2)\,\delta(t-s) = E\left[f(\omega,t,z_1)\;\frac{\partial f(\omega,s,z)}{\partial z}\Big|_{z=z_2}\right],$$

$$Q_{21}(z_1,z_2)\,\delta(t-s) = E\left[\frac{\partial f(\omega,s,z)}{\partial z}\Big|_{z=z_1}\;f(\omega,s,z_2)\right], \tag{14}$$

$$Q_{21}(z_1,z_2)\,\delta(t-s) = E\left[\frac{\partial f(\omega,s,z)}{\partial z}\Big|_{z=z_1}\;\frac{\partial f(\omega,s,z)}{\partial z}\Big|_{z=z_2}\right].$$

Using properties of covariance function derivatives we have the explicit form of the covariance operator

$$\mathbf{Q}(z_1,z_2)\,\delta(t-s) = \begin{bmatrix} Q_{11}(z_1,z_2) & \dfrac{\partial Q_{11}(z_1,z_2)}{\partial z_2} \\ \dfrac{\partial Q_{11}(z_1,z_2)}{\partial z_1} & \dfrac{\partial^2 Q_{11}(z_1,z_2)}{\partial z_1 \partial z_2} \end{bmatrix} \delta(t-s). \tag{15}$$

Therefore, we can write following expansion of covariance functions:

$$Q_{ij}(z_1,z_2) = \sum_{k=1}^{\infty} \lambda_k e_k^i(z_1)\, e_k^j(z_2), \tag{16}$$

where λ_k are eigenvalues of operator $\mathbf{Q}$, and $\{e_i(z)\} = (e_i^1(z)\; e_i^2(z))^T$ is an orthonormal basis for Hilbert space Y (this space is separable).

We choose Liapunov functional in the form:

$$V = \int_0^1 \left[\left(v + \zeta\frac{\partial^4 w}{\partial z^4}\right)^2 + \zeta^2\left(\frac{\partial^4 w}{\partial z^4}\right)^2 - f_o\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial^2 w}{\partial z^2}\right)^2\right] dz. \tag{17}$$

To obtain the differential dV along sample solutions of equation (10) it is necessary apply functional Ito lemma in a version derived by

Curtain and Falb

$$dV = \langle V'_U, \mathbf{AU}\rangle\, dt + \frac{1}{2} Tr\left([\mathbf{BU}]^T V''_{UU} [\mathbf{BU}]\,\mathbf{Q}\right) dt + \langle V'_U, [\mathbf{BU}]\, d\mathbf{W}\rangle\,, \tag{18}$$

where $Tr(.)$ denotes a trace of operator, V'_U and V''_{UU} denote the first and the second Freshet derivative of functional, respectively.

Substituting evolution equation (10) into the first term of differential (18) and using boundary conditions leads to

$$\langle V'_U, \mathbf{AU}\rangle = -2\zeta \int_0^1 \left[\left(\frac{\partial^2 v}{\partial z^2}\right)^2 + \left(\frac{\partial^4 w}{\partial z^4}\right)^2 - f_o \left(\frac{\partial^3 w}{\partial z^3}\right)^2\right] dz. \tag{19}$$

Using the trace definition we can rewrite the second term of differential (18) in the form

$$d_T V = \frac{1}{2} \sum_{i=1}^{\infty} \left\langle [\mathbf{BU}]^T V''_{UU} [\mathbf{BU}]\,\mathbf{Q} e_i, e_i \right\rangle. \tag{20}$$

In accordance with properties of both the nuclear operator $\mathbf{Q}$ and the inner product we have

$$d_T V = \frac{1}{2} \sum_{i=1}^{\infty} \lambda_i \left\langle V''_{UU} [\mathbf{BU}]\, e_i, [\mathbf{BU}]\, e_i \right\rangle. \tag{21}$$

After calculating the trace term of differential (18) can be written in the form

$$d_T V = \int_0^1 \left[Q_{11}(z,z)\left(\frac{\partial^2 w}{\partial z^2}\right)^2 + 2Q_{12}(z,z)\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2} + Q_{22}(z,z)\left(\frac{\partial w}{\partial z}\right)^2\right] dz. \tag{22}$$

On integrating differential (18) with respect to t from s to $\tau_\delta(t)$, where $\tau_\delta(t) = \min\{\tau_\delta, t\}$, $\tau_\delta = \inf\{t : \|\mathbf{U}\| > \delta > 0\}$ and rearranging the integrand it follows that

$$V_{[\tau_\delta(t)]} = V(s) - \int_s^{\tau_\delta(t)} \int_0^1 \left[2\zeta \left(\frac{\partial^2 v}{\partial z^2}\right)^2 + 2\zeta \left(\frac{\partial^4 w}{\partial z^4}\right)^2 - 2\zeta f_o \left(\frac{\partial^3 w}{\partial z^3}\right)^2 \right.$$

$$-Q_{11}(z,z)\left(\frac{\partial^2 w}{\partial z^2}\right)^2 - 2Q_{12}(z,z)\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2} -$$

$$\left. Q_{22}(z,z)\left(\frac{\partial w}{\partial z}\right)^2 \right] dz + \int_0^{\tau_\delta(t)} \langle V'_U, [\mathbf{BU}]\, d\mathbf{W} \rangle . \tag{23}$$

Now we take the conditional average of equation (23) remembering that the second integral is a stochastic one, so the conditional average of it is equal to zero

$$EV_{[\tau_\delta(t)]} = V(s) - E \int_s^{\tau_\delta(t)} \int_0^1 \left[2\zeta \left(\frac{\partial^2 v}{\partial z^2}\right)^2 + 2\zeta \left(\frac{\partial^4 w}{\partial z^4}\right)^2 - 2\zeta f_o \left(\frac{\partial^3 w}{\partial z^3}\right)^2 \right.$$

$$\left. -Q_{11}(z,z)\left(\frac{\partial^2 w}{\partial z^2}\right)^2 - 2Q_{12}(z,z)\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2} - Q_{22}(z,z)\left(\frac{\partial w}{\partial z}\right)^2 \right] dzdt. \tag{24}$$

We see that functional V is a supermartingale, i.e. $EV_{[\tau_\delta(t)]} \leq V(s)$, if the integrand of equation (24) is nonnegative. Then we have the following chain of inequalities

$$V(s) \geq EV_{[\tau_\delta(t)]} = \int_\Omega V_{[\tau_\delta(t)]} P(d\omega) \geq$$

$$\int_{\left\{\omega:\ \sup_{t \geq s} \|\mathbf{U}\| > \delta\right\}} V_{[\tau_\delta(t)]} P(d\omega) \geq \delta^2 P \left\{ \sup_{t \geq s} \|\mathbf{U}\| > \delta \right\}. \tag{25}$$

Neglecting the first positive term, integrand is nonnegative if

$$\Bigg[\int_0^1 2\zeta\left(\frac{\partial^4 w}{\partial z^4}\right)^2 - 2\zeta f_o\left(\frac{\partial^3 w}{\partial z^3}\right)^2 - Q_{11}(z,z)\left(\frac{\partial^2 w}{\partial z^2}\right)^2 - \qquad (26)$$

$$2Q_{12}(z,z)\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2} - Q_{22}(z,z)\left(\frac{\partial w}{\partial z}\right)^2\Bigg] dz \geq 0.$$

In the case when the field of the loading is statistically uniform, $Q_{ij}(z_1,z_2) = Q_{ij}(z_1 - z_2)$, then is

$$Q_{11}(z,z) = Q_{11}(z-z) = Q_{11}(0),$$

$$\left.\begin{array}{l} Q_{12}(z,z) = Q_{12}(z-z) = Q_{12}(0) = 0, \\ Q_{21}(z,z) = Q_{21}(z-z) = Q_{21}(0) = 0, \end{array}\right\rangle \text{(normal processes)}$$

$$Q_{22}(z,z) = Q_{22}(z-z) = Q_{22}(0). \qquad (27)$$

Using the extremum property of the minimal eigenvalue

$$\int_0^1 \left(\frac{\partial^4 w}{\partial z^4}\right)^2 dz \geq \pi^2 \int_0^1 \left(\frac{\partial^3 w}{\partial z^3}\right)^2 dz \geq \qquad (28)$$

$$\geq \pi^4 \int_0^1 \left(\frac{\partial^2 w}{\partial z^2}\right)^2 dz \geq \pi^6 \int_0^1 \left(\frac{\partial w}{\partial z}\right)^2 dz,$$

we obtain from (26) and (27) the following the uniform stability condition

$$\left[2\zeta\pi^2\left(\pi^2 - f_o\right) - Q_{11}(0)\right]\pi^2 - Q_{22}(0) \geq 0. \qquad (29)$$

4 Numerical results and discussion

Let the space correlation of statistically uniform field be described by a formula

$$Q_{11}(z-y)=\sigma^2\exp\left[-\frac{(z-y)^2}{l_z^2}\right], \tag{30}$$

where l_z is spatial correlation radius, it follows from (15) and30 that

$$Q_{11}(0)=\sigma^2, \quad Q_{22}(0)=\frac{2\sigma^2}{l_z^2}. \tag{31}$$

Now uniform stability condition takes a form:

$$2\zeta \geq \frac{1+\dfrac{2}{\pi^2 l_z^2}}{\pi^4\left(1-\dfrac{f_o}{\pi^2}\right)}\sigma^2. \tag{32}$$

It should be noted that when correlation radius tends to infinity, critical retardation time has a form given in [5].

The inequality (32) gives us the possibility to obtain regions of the uniform stability as a function of the retardation time of viscoelastic beam, intensity of deterministic loading (f_o), intensity (σ) and correlation radius (l_z) of stochastic process.

The boundaries of uniform stability regions in $\sigma-\zeta$ plane, as a function of intensity of deterministic loading are plotted in Fig. 2, and as a function of correlation radius are presented in Fig. 3. The uniform stability region is defined as the set where retardation times are greater than critical retardation time.

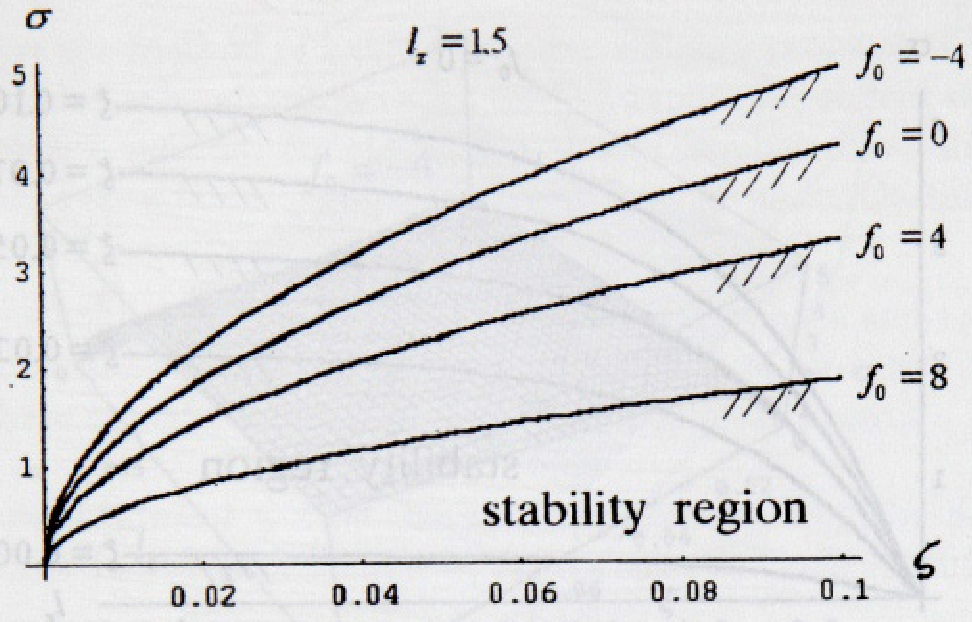

Fig. 2. Stability regions in $\sigma - \zeta$ plane as a function of loading intensity f_o.

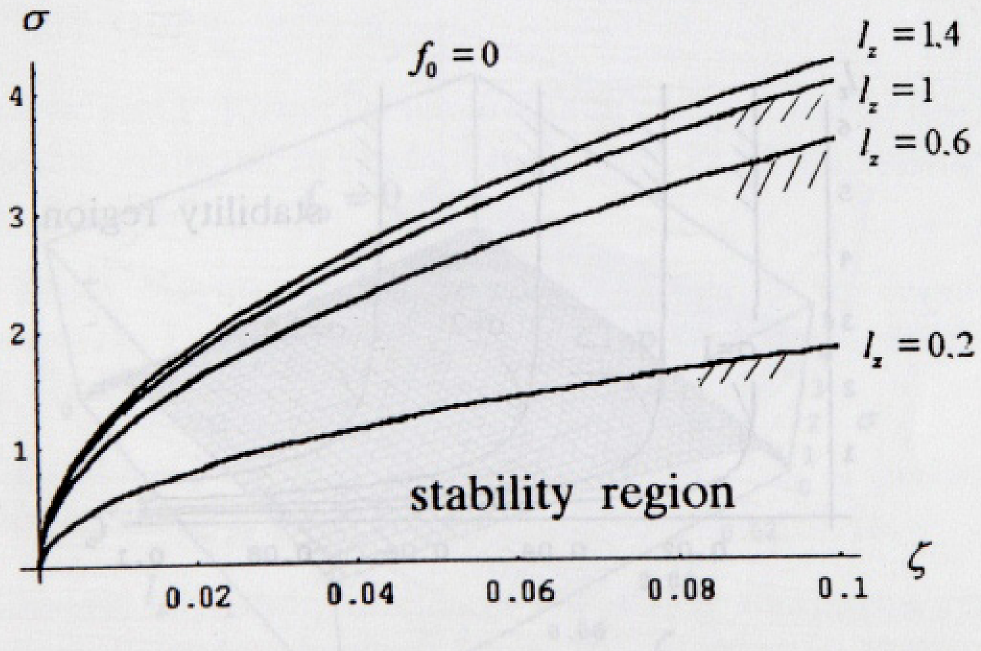

Fig. 3. Stability regions in $\sigma - \zeta$ plane as a function of correlation radius l_z.

The boundaries of uniform stability in $\sigma - l_z$ plane as a function of retardation time are shown in Fig. 4, and in $l_z - \zeta$ plane as a function of intensity of the stochastic loading in Fig. 5.

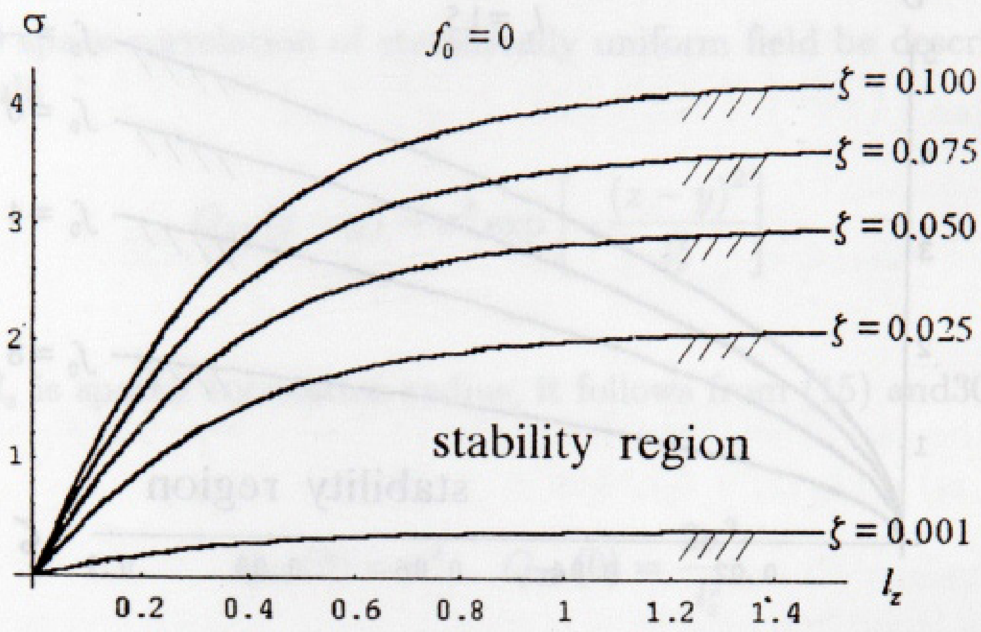

Fig. 4. Stability regions in $\sigma - l_z$ plane as a function of retardation time ζ.

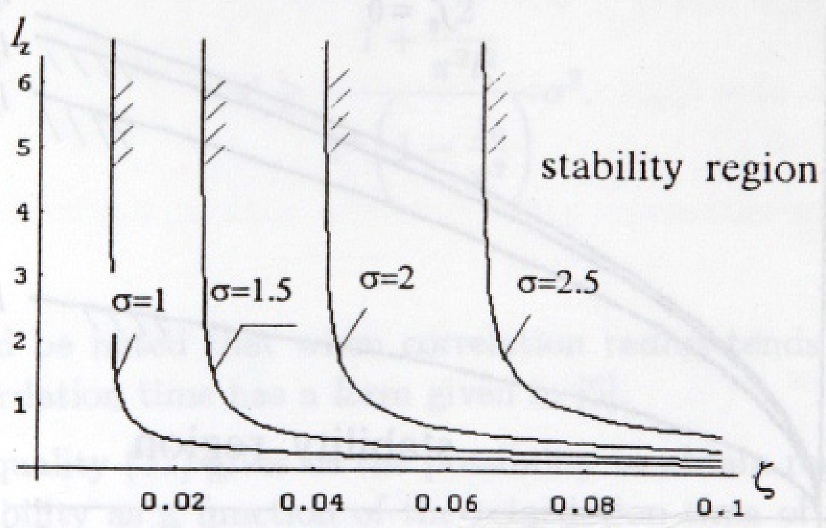

Fig. 5. Stability regions in $l_z - \zeta$ plane as a function of stochastic process intensity σ.

In Fig. 6, 7, 8 and 9 the three-dimensional stability surfaces in space σ, ζ and l_z for different values of the intensity of deterministic load f_o

are given. Stability regions are below these surfaces, and we see that increase of pressure load, leads to their shrinkage.

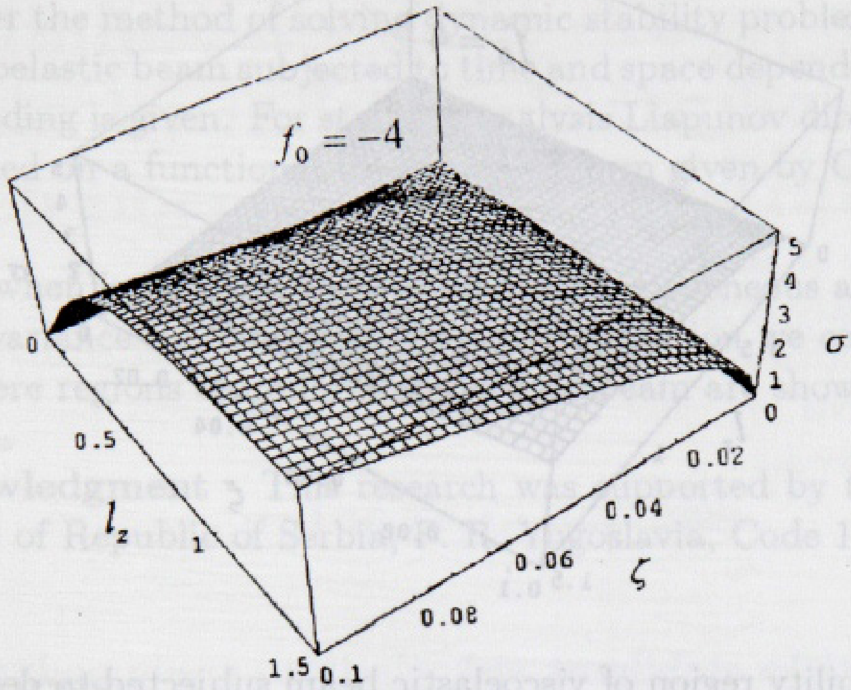

Fig. 6. Stability region of viscoelastic beam in $\sigma - l_z - \zeta$ space subjected to deterministic tension ($f_o = -4$) and stochastic loading.

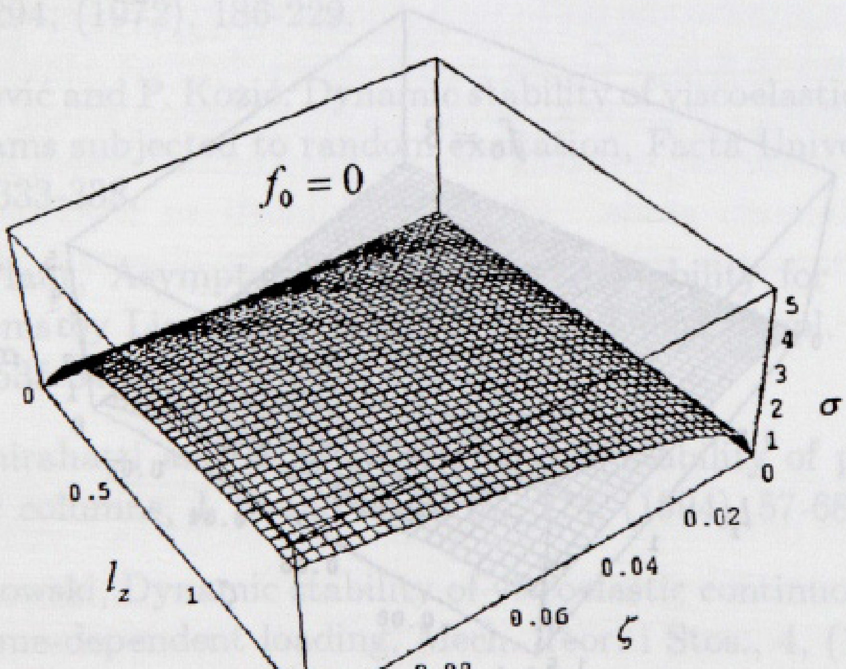

Fig. 7. Stability region of viscoelastic beam subjected only to stochastic loading.

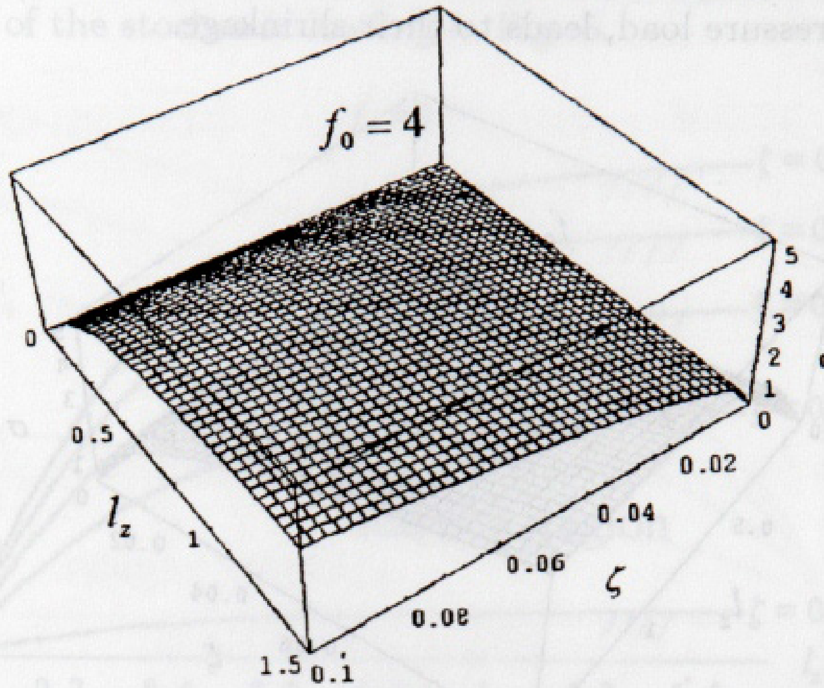

Fig. 8. Stability region of viscoelastic beam subjected to deterministic pressure ($f_o = 4$) and stochastic loading.

Fig. 9. Stability region of viscoelastic beam subjected to deterministic pressure ($f_o = 8$) nearly to Euler critical force ($f_{ocr} = -\pi^2$).

5 Conclusions

In this paper the method of solving dynamic stability problem of Voigt-Kelvin viscoelastic beam subjected to time and space dependent stochastic axial loading is given. For stability analysis Liapunov direct method is used, based on a functional Ito lemma in form given by Curtain and Falb.

In case when is random field statistically homogeneous and approximating covariance function with normal distribution we construct diagrams where regions of uniform stability of beam are shown.

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Stohastička stabilnost viskoelastične grede podvrgnute dejstvu vremenski i prostorno promenljivog opterećenja

U radu je proučavana uniformna stohastička stabilnost viskoelastične Voigt-Kelvin-ove grede. Greda je slobodno oslonjena i podvrgnuta dejstvu vremenski i prostorno promenljivog širokopojasnog Gauss-ovog procesa. Korišćenjem direktne metode Ljapunova dobijen je uslov uniformne stabilnosti grede. Oblasti stabilnosti su izražene u funkciji vremena retardacije intenziteta i poluprečnika disperzije stohastičkog opterećenja.