THEORETICAL AND APPLIED MECHANICS
25, pp. 33-59, 1999, UDK 531.01

Isolated Hopf bifurcation of symmetric weakly damped systems

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Submitted 15 November 1998

Abstract

This paper deals with the *dynamic stability* of autonomous weakly damped symmetric (potential) systems. Conditions for the occurrence of a *limit cycle mode* of instability are established via a through discussion of the *effect* of the *damping matrix* on the *Jacobian eigenvalues.* It was found that such a response may occur through *a new type* of local dynamic bifurcation identified as *an isolated* Hopf bifurcation as well as through a *double zero* (eigenvalue) local dynamic bifurcation. As a consequence of this, *undamped stable* symmetric systems may become *unstable* with the inclusion of *damping.* Numerical results confirm the validity of the theoretical findings presented herein.

1 Introduction

Early studies have shown [1,2] the importance of *damping* on the *elastic stability* of flexurally vibrating systems particularly of *non-self adjoint* (nonconservative) *asymmetric* systems. However, this effect was usually *ignored* in case of *potential* (conservative) systems. For such *undamped symmetric* systems, if they are *stable* it was widely accepted [3] that the addition of *damping does not change* their stability. However, two recent studies [4,5] have shown that the previous result under certain special conditions *may be not true* for 2-DOF models.

During the last decades, the *qualitative topological* and *geometrical* approach revealed new phenomena which revolutionalize the mathematical theory of *dynamic bifurcations.* The majority of these are associated with *geometrical nonlinearities* and *interactions* of them with other *control parameters* [6-9]. However, strangely enough several types of *local dynamic bifurcations* which could be explored via a classical (linear) analysis escaped the attention of eminent researchers of the past. The present analysis motivated by this fact shows that there is still room for further pertinent investigation in the framework of *linear* (classical) analysis.

The *objective* of this work is to discuss the conditions under which *autonomous weakly damped symmetric* (potential) systems may exhibit a *limit cycle mode of instability,* and moreover if such a mode may occur *prior* to *divergence* instability.

The analysis that follows is also addressed to scientists not necessarily familiar with the rudiments of the mathematical theory of *dynamical systems.* Because of this, some care has been taken to avoid technicalities for the purpose of making the analysis more comprehensible.

2 Mathematical analysis

Consider a general N-DOF vibrating *symmetric weakly damped* system under *step* loading of *infinite duration* and *constant direction.* The response of such an *autonomous potential* system can be described in the context of *linear* analysis in terms of generalized coordinates by the following matrix-vector differential equation

$$[\alpha_{ij}]\,\ddot{\mathbf{q}} + [c_{ij}]\,\dot{\mathbf{q}} + [V_{ij}]\,\mathbf{q} = 0, \tag{1}$$

where the dots denote derivatives with respect to time t, while $\mathbf{q}\,(t)$ is an n-dimensional *state* vector (column matrix) with coordinates $q_i\,(t)$ $(i = 1, ..., n)$; $[\alpha_{ij}]$, $[c_{ij}]$ and $[V_{ij}]$ are in general *real* $n \times n$ matrices. More specifically, matrix $[\alpha_{ij}]$ is *positive definite* associated with the quadratic form of the *total kinetic energy* with elements α_{ij} being functions of concentrated masses (i.e. $[\alpha_{ij}] = [\alpha_{ij}(m_i)]$; matrix $[c_{ij}]$ is symmetric *positive definite*, *positive semi-definite* [e.g. in case of pervasive damping [11,12] or *indefinite* associated with velocity-dependent dissipation forces [13]; finally, the *stiffness* matrix $[V_{ij}]$ is *symmetric* with real *distinct* eigenvalues and a *complete* set of eigenvectors, whose elements V_{ij} are assumed to be functions of the *loading* λ, i.e. $V_{ij} = V_{ij}(\lambda)$, where λ is the main *control parameter.* Setting $\det[V_{ij}(\lambda)] = 0$ we obtain the *buckling* equation from which we get the *critical buckling* (divergence) loads $\lambda^C_{(i)}$ $(i = 1, ..., n)$.

The present work aims at discussing the effect of *slight damping* on the response of the above autonomous symmetric (potential) systems. More specifically, the *objective* of this analysis is to discuss via a linear analysis the conditions under which these systems may exhibit local dynamic bifurcations associated either with a Hopf bifurcation (existence of a pair of purely imaginary eigenvalues) or a double zero eigenvalue bifurcation.

In view of the above, attention is focused on seeking *steady-state* solutions of *symmetric* weakly damped systems governed by eq. (1) (i.e. equilibria and periodic motions) which may be associated with *periodic attractors.*

We can seek solutions of eqs. (1) in the form

$$\mathbf{q} = \mathbf{r}e^{\rho t}, \tag{2}$$

where ρ is, in general, a *complex* eigenvalue and $\mathbf{r}$ the corresponding *complex vector*, independent of t. Substituting this expression of $\mathbf{q}$ into eq. (1) leads to

$$\mathbf{L}(\rho)\,\mathbf{r} = \left([\alpha_{ij}]\,\rho^2 + [c_{ij}]\,\rho + [V_{ij}]\right)\mathbf{r} = 0, \tag{3}$$

where $\mathbf{L}(\rho) = [\alpha_{ij}]\,\rho^2 + [c_{ij}]\,\rho + [V_{ij}]$ is a *matrix-valued* function. Solutions of eq. (1) are intimately related to the algebraic properties of $\mathbf{L}(\rho)$, and more specifically to the nature of *Jacobian* eigenvalues ρ_i $(i = 1, ..., 2n)$ obtained by solving the *characteristic* (secular) equation

$$\det \mathbf{L}(\rho) = \left|[\alpha_{ij}]\,\rho^2 + [c_{ij}]\,\rho + [V_{ij}]\right| = 0, \tag{4}$$

which guarantees the existence of *nontrivial* solutions of eq. (1) or (3). Expansion of eq. (4) leads to the following equation [7]

$$\rho^{2n}+\alpha_1\rho^{2n-1}+\alpha_2\rho^{2n-2}+...+\alpha_{2n-1}\rho+\alpha_{2n}=0, \tag{5}$$

where $\alpha_i\,(i=1,...,2n)$ are determined through Bocher recurrence formulae.

As stated above, the eigenvalues (roots) of eq. (4) occur in *complex* conjugate pairs $\rho_i=v_i+\mu_i j$ (where $j=\sqrt{-1}, i=1,...,n$ and μ_i and v_i are real numbers) with corresponding *complex conjugate* eigenvectors. Hence, the solutions of eq. (1) associated with eq. (2) are of the form

$$Ae^{v_i t}\cos\mu_i t \quad \text{and} \quad Be^{v_i t}\sin\mu_i t, \tag{6}$$

where A and B are *constants* determined from the initial conditions. Clearly, solutions (6) are *bounded*, tending to zero as $t\to\infty$, if all eigenvalues of eq. (5) have *negative* real parts, i.e. $v_i<0$ for all i [10]. According to *Routh-Hurvitz stability criteria* a *necessary* condition in order that all eigenvalues have *negative* real parts is $\alpha_i>0$ for all i, while a *sufficient* condition assuring this is all Routh-Hurvitz determinants Δ_i of even (or odd) order to be *positive* [10]. Moreover, a *necessary* and *sufficient* condition for all eigenvalues to lie in the left-hand side of the ρ-complex plane is $\Delta_i>0$ for all i. Then, eq. (9) has complex conjugate eigenvalues of the above form, i.e. $\rho_i=v_i\pm\mu_i j$, where $v_i<0$ and $\mu_i>0\;(i=1,...,n)$.

Premultiplying eq. (3) by $\overline{\mathbf{r}}^{\mathrm{T}}$, the *conjugate transpose* of $\mathbf{r}$, gives the following equation

$$\overline{\mathbf{r}}^{\mathrm{T}}\left([\alpha_{ij}]\,\rho^2+[c_{ij}]\,\rho+[V_{ij}]\right)\mathbf{r}=0, \tag{7}$$

in which all *quadratic* forms are real *scalar* quantities. Eq. (7) is a 2^{nd} degree algebraic polynomial with respect to ρ from which we get

$$\rho=\frac{1}{2\overline{\mathbf{r}}^{\mathrm{T}}[\alpha_{ij}]\,\mathbf{r}}\left[-\overline{\mathbf{r}}^{\mathrm{T}}[c_{ij}]\,\mathbf{r}\pm\sqrt{(\overline{\mathbf{r}}^{\mathrm{T}}[c_{ij}]\,\mathbf{r})^2-4(\overline{\mathbf{r}}^{\mathrm{T}}[\alpha_{ij}]\,\mathbf{r})(\overline{\mathbf{r}}^{\mathrm{T}}[V_{ij}]\,\mathbf{r})}\right], \tag{8}$$

where the corresponding *real* and *imaginary* parts are

$$v=-\frac{\overline{\mathbf{r}}^{\mathrm{T}}[c_{ij}]\,\mathbf{r}}{2\overline{\mathbf{r}}^{\mathrm{T}}[\alpha_{ij}]\,\mathbf{r}},\quad \mu=\frac{\left[4\left(\overline{\mathbf{r}}^{\mathrm{T}}[\alpha_{ij}]\,\mathbf{r}\right)\left(\overline{\mathbf{r}}^{\mathrm{T}}[V_{ij}]\,\mathbf{r}\right)-\left(\overline{\mathbf{r}}^{\mathrm{T}}[c_{ij}]\,\mathbf{r}\right)^2\right]^{1/2}}{2\overline{\mathbf{r}}^{\mathrm{T}}[\alpha_{ij}]\,\mathbf{r}}.$$

Let the complex conjugate eigenvectors be

$$\mathbf{r} = \mathbf{x}+j\mathbf{y}, \quad \bar{\mathbf{r}} = \mathbf{x}-j\mathbf{y}, \quad j=\sqrt{-1}, \tag{9}$$

where $\mathbf{x} = (x_1, ..., x_n)^{\mathrm{T}}$ and $\mathbf{y} = (y_1, ..., y_n)^{\mathrm{T}}$. Introducing now $\rho = v+j\mu$ and $\mathbf{r} = \mathbf{x}+j\mathbf{y}$ into eq. (3) and setting real and imaginary parts equal to zero we obtain

$$\left.\begin{aligned} \left([\alpha_{ij}]\left(v^2-\mu^2\right)+v\left[c_{ij}\right]+\left[V_{ij}\right]\right)\mathbf{x} &= \mu\left(\left[c_{ij}\right]+2v\left[\alpha_{ij}\right]\right)\mathbf{y} \\ \left([\alpha_{ij}]\left(v^2-\mu^2\right)+v\left[c_{ij}\right]+\left[V_{ij}\right]\right)\mathbf{y} &= -\mu\left(\left[c_{ij}\right]+2v\left[\alpha_{ij}\right]\right)\mathbf{x} \end{aligned}\right\}. \tag{10}$$

This is a *homogeneous* system with respect to $\mathbf{x}$ and $\mathbf{y}$, whose *determinant* for a *nontrivial solution* must be zero.

For given matrices $[\alpha_{ij}]$ and $[c_{ij}]$, as the loading increases gradually from zero, the variation of the matrix $[V_{ij}(\lambda)]$ influences the values of the complex conjugate *eigenvalues* ρ [i.e. $v = v(\lambda)$ and $\mu = \mu(\lambda)$] and the corresponding *complex eigenvectors* $\mathbf{r}$ (i.e. $\mathbf{r} = \mathbf{r}(\lambda)$). The stiffness matrix $[V_{ij}(\lambda)]$ is *positive definite* if $\lambda < \lambda^C_{(1)}$ (1^{st} buckling load) and *positive semi-definite* if $\lambda = \lambda^C_{(1)}$.

Assuming, due to physical considerations *weak damping* [2], the response of the system will be discussed in connection with the algebraic structure of the damping matrix $[c_{ij}]$. Moreover, regardless of the structure of matrix $[c_{ij}]$ one can assume that the *initial* equilibrium path (for small values of loading) is *stable.*

3 The effect of matrix $[c_{ij}]$ on the eigenvalues

Subsequently, the effect of *damping* on the *real* part of the eigenvalues of the above symmetric systems will be studied with the aid of eq. (8).

There is a variety of kinds of damping in physical problems. However, we do not intend to examine their nature (i.e. sources) or role on the dynamics of vibrating systems [2,14] but confine our attention to damping matrices $[c_{ij}]$ which are *positive definite* or *positive semi-definite.*

3.1 Case (a): $[c_{ij}]$ positive definite

If matrix $[c_{ij}]$ is *positive definite*, the quadratic form $\bar{\mathbf{r}}^T [c_{ij}] \mathbf{r}$ is always a real positive quantity for $\mathbf{r} \neq 0$, while the quantity under the radical in eq. (8), due to weak damping and the positive definiteness of matrix $[\alpha_{ij}]$ is *negative* if $\lambda < \lambda^C_{(1)}$ (implying $\det [V_{ij}] > 0$). Then, eq. (8) yields complex conjugate eigenvalues with *negative* real parts as long as $\lambda < \lambda^C_{(1)}$. At the *static* (divergence) *critical state C*, occurring for $\lambda = \lambda^C_{(1)}$, we have, as stated above, $\det \left[V_{ij} \left(\lambda^C_{(1)} \right) \right] = \det [V_{ij}]^C = 0$. Then, eq. (8) has one *zero* eigenvalue and one *negative* eigenvalue. Hence, if the nondissipative system is *stable*, the dissipative system is *asymptotically* stable provided that the *damping* matrix $[c_{ij}]$ is *positive definite* [3].

In the above type of *divergence* instability (occurring at $\lambda = \lambda^C$), as the loading λ increases slowly from zero, at least a *pair* of *complex* conjugate eigenvalues follows in the ρ-complex plane the *path* shown in Fig. 1.

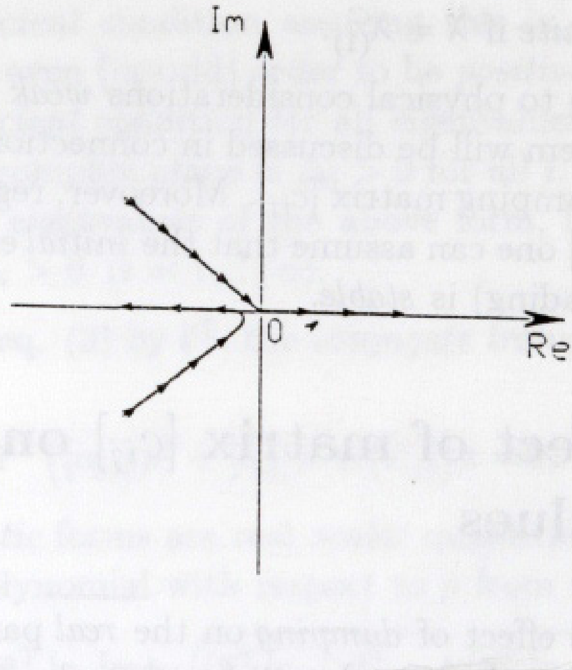

Fig. 1. The λ-dependent paths of a pair of complex conjugate eigenvalues transformed into a zero and into a zero and into a negative eigenvalues at $\lambda = \lambda^C_{(1)}$ which for $\lambda > \lambda^C_{(1)}$ are moving in opposite directions along the real axis (divergence instability).

From eq. (8) it is deduced that this pair at $\lambda = \lambda^C_{(1)}$ (where the Jacobian becomes singular) is transformed into a *zero* eigenvalue (becoming positive and increasing for $\lambda > \lambda_{(1)}^{C}$) yielding *divergence instability* and into a *negative* eigenvalue (decreasing for $\lambda > \lambda_{(1)}^{C}$); these two real eigenvalues for $\lambda > \lambda_{(1)}^{C}$ are moving in opposite directions in the *real* axis.

3.2 Case (b): $[c_{ij}]$ positive semi-definite

This case, according to which det $[c_{ij}] = 0$, will be examined in connection with the algebraic form of the stiffness matrix $[V_{ij}]$. Thus, two subcases related to a *positive definite* and then to a *positive semi-definite* matrix will $[V_{ij}]$ be discussed.

We consider first the case for which $[V_{ij}]$ is a *positive definite* matrix. As stated above, if $\lambda < \lambda_{(1)}^{C}$, matrix $[V_{ij}(\lambda)]$ is *positive definite*, and then $\alpha_{2n} = \det[V_{ij}(\lambda)] / \det[\alpha_{ij}] > 0$. The eigenvalues of eq. (5) (depending on λ) and the corresponding eigenvectors are associated, in general, with $\bar{\mathbf{r}}^{\mathrm{T}}[c_{ij}]\,\mathbf{r} > 0$. Hence, from eq. (8) follows that *all* eigenvalues have *negative* real parts. However, at a *certain* value of λ (> 0), the corresponding *quadratic* quantity may become equal to zero, i.e.

$$\bar{\mathbf{r}}^{\mathrm{T}}[c_{ij}]\,\mathbf{r} = 0, \quad (i, j = 1, ..., n). \tag{11}$$

Since matrix $[c_{ij}]$ is *positive semi-definite*, eq. (11) implies

$$[c_{ij}]\,\mathbf{r} = 0. \tag{12}$$

Introducing eq. (12) into eq. (7) implies that $\mathbf{r}$ is also an *eigenvector* of the (potential) *conservative* system [13]

$$\left([\alpha_{ij}]\,\rho^2 + [V_{ij}(\lambda)]\right)\mathbf{r} = 0. \tag{13}$$

Clearly, if $\mathbf{r}$ is an eigenvector of the *nondissipative* system (13) satisfying eq. (12), then $\mathbf{r}$ is also an eigenvector of the *dissipative* system of eq. (3) with corresponding eigenvalues (resulting from eq. (8)) which are *purely imaginary.* Setting $\rho = \pm j\mu$ into eq. (13), one can determine the corresponding eigenvector $\mathbf{r}$ which is *real.* Apparently, since $[c_{ij}]$ is a *positive semi-definite* matrix one of its eigenvalues is *zero*, and hence

$$([c_{ij}] - 0\mathbf{I}_n)\,\mathbf{r} = 0. \tag{14}$$

From this equation [or eq. (12)] one can determine the *real* eigenvector $\mathbf{r}$ for a given damping matrix $[c_{ij}]$. Then, eq. (8) gives

$$-\rho^2 = \mu^2 = \frac{\mathbf{r}^{\mathrm{T}}\left[V_{ij}(\lambda)\right]\mathbf{r}}{\mathbf{r}^{\mathrm{T}}\left[\alpha_{ij}\right]\mathbf{r}}, \tag{15}$$

from which μ can be established *implicitly* as a function of the load λ for a given matrix $[\alpha_{ij}(m_i)]$.

Introducing $\mathbf{r}$ (obtained from eq. (14)) into eq. (13), after eliminating ρ^2, one can determine a *critical* (dynamic) load, defined as a *Hopf critical load* $\lambda = \lambda_H$, provided that $\lambda_H > 0$. Clearly, λ_H is a function of the *damping coefficients* c_{ij} and *masses* m_i (> 0), i.e. $\lambda_H = \lambda_H(c_{ij}, m_i)$. Inserting λ_H into eq. (15) we obtain the pair of conjugate *purely imaginary* eigenvalues. From a practical viewpoint, it is more important the case of *dynamic instability* occurring *prior* to *divergence*, i.e. when

$$0 < \lambda_H < \lambda_{(1)}^{C}. \tag{16}$$

Analytical evaluation of λ_H is given in the Appendix.

Besides this *general* method for the evaluation of λ_H, one can also apply another procedure for determining it, in case where the coefficients α_i in eq. (5) are known or can be readily obtained. To this end, after introducing $\rho = \pm j\mu$ into eq. (5) and setting the *real* and *imaginary* parts equal to zero, we get from the latter an algebraic equation of order $(2n-2)$ in μ^2, whose coefficients α_i are all of *odd* order. Combining this equation with eq. (13), one can establish the Hopf bifurcation load λ_H.

In case such a load exists (i.e. satisfying inequality (16)), we have established the *unexpected* result that a *symmetric* (potential) *weakly damped* system may exhibit an *isolated* Hopf bifurcation at a load λ_H *smaller* that the *divergence* (static) buckling load $\lambda_{(1)}^{C}$. Note also that the corresponding to this situation pair of *purely imaginary* eigenvalues is associated with a *real* eigenvector which is in *common* to the *nondissipative* system (see eq. (13)) and to the *damping* matrix $[c_{ij}]$ (see eq. (14)). This can also be shown via eqs. (10). Inserting $v = 0$

into eqs. (10), we obtain

$$\left.\begin{aligned} ([V_{ij}] - \mu^2 [\alpha_{ij}])\, \mathbf{x} &= \mu [c_{ij}]\, \mathbf{y} \\ ([V_{ij}] - \mu^2 [\alpha_{ij}])\, \mathbf{y} &= -\mu [c_{ij}]\, \mathbf{x} \end{aligned}\right\}, \tag{17}$$

or

$$\begin{bmatrix} [V_{ij}] - \mu^2 [\alpha_{ij}] & -\mu [c_{ij}] \\ \mu [c_{ij}] & [V_{ij}] - \mu^2 [\alpha_{ij}] \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = 0. \tag{17'}$$

For a *non-trivial* solution the determinant of this *homogeneous* system must be *zero* yielding the characteristic (secular) eq. (5) after setting $\rho = \mu j$.

Given that the matrix $[c_{ij}]$ is positive semi-definite, the system of eqs. (17) is satisfied by setting

$$\mathbf{x} = \kappa \mathbf{y} \quad (\kappa \neq 0), \tag{18}$$

where κ is a real number. Indeed, introducing this expression of $\mathbf{x}$ into both eqs. (17) it follows that

$$\left(\frac{1}{\kappa} + \kappa\right) [c_{ij}]\, \mathbf{y} = 0, \quad \text{or} \quad [c_{ij}]\, \mathbf{y} = 0, \tag{19}$$

which holds for $\mathbf{y} \neq 0$, since $[c_{ij}]$ is positive semi-definite. Note that the matrix $([V_{ij}] - \mu^2 [\alpha_{ij}])$ is also *positive semi-definite.*

The important conclusion which can be drawn is that the *real* and *imaginary* parts $\mathbf{x}$ and $\mathbf{y}$ of the eigenvector $\mathbf{r}$ are *linearly dependent.* Moreover, due to eqs. (9) and (18) the eigenvector $\mathbf{r}$ and its conjugate $\bar{\mathbf{r}}$ are *linearly dependent* since the latter is a *multiple eigenvector* of the former one, and vice-versa. Thus, *one* eigenvector $\mathbf{r}$ corresponds to both eigenvalues $+\mu j$ and $-\mu j$.

The last results hold provided the matrix $[V_{ij}(\lambda)]$ is *positive definite* (i.e. when $\lambda < \lambda^C_{(1)}$). Let us now consider the *special* case, where, in addition to the damping matrix $[c_{ij}]$, the *stiffness* matrix $[V_{ij}(\lambda)]$ is also *positive semi-definite.* If the real eigenvector of the *matrix* $\left[V\left(\lambda^C_{(1)}\right)\right]$ happens to be also an eigenvector of the matrix $[c_{ij}]$, then eq. (8) has a double zero eigenvalue ($-\rho^2 = \mu^2 = 0$) and inequality (16) becomes

$$0 < \lambda_H \equiv \lambda^C_{(1)}. \tag{20}$$

Contrary to the previous case of a *Hopf bifurcation*, the *critical* (dynamic) load $\lambda_H \equiv \lambda^C_{(1)}$ corresponding to a *double zero* eigenvalue is *independent* of the matrix $[\alpha_{ij}(m_i)]$ (and thus of the masses m_i). Note that in this case the system exhibits a *limit* cycle response although the corresponding bifurcation occurs at a *static* (divergence) critical state called Arnold-Bogdanov bifurcation [15].

This *phenomenon*, associated with a *couple flutter-divergence instability*, was also shown using a different procedure *valid*, however, only for the simple case of 2-DOF symmetric damped systems [5].

In view the above, one can draw the following *important findings* for the occurrence of a *double zero* eigenvalue: *the necessary* condition for such a situation is the *positive semi-definiteness* of both, the *damping* and the *stiffness* matrices, while the *sufficient condition* is the corresponding *eigenvector* to be in *common* to the last two matrices.

The above two types of *dynamic* mode of instability can also be established in terms of *eigenvalue paths* in the ρ-complex plane as λ varies slowly in the vicinity of the critical state. Fig. 2 shows the type of instability related to an *isolated Hopf* bifurcation, while Fig. 3 that of a *typical* Hopf bifurcation [9].

Clearly, as shown from Fig. 2, λ as increases slowly, a pair of complex conjugate eigenvalues approaches from the left half-plane the *imaginary* axis and at $\lambda = \lambda_H$ the corresponding paths become tangential to this axis; thus for $\lambda > \lambda_H$ these paths *remain* in this plane yielding complex conjugate eigenvalues with *negative* real part.

At $\lambda = \lambda_H$ a *family of limit cycles* may *branch off* this critical equilibrium state associated with an *isolated* Hopf bifurcation since for $\lambda \neq \lambda_H$ the system is *asymptotically stable.* Hence, the system assumes at $\lambda = \lambda_H$ a *dynamic mode* (limit cycle) instability. This is a *new type* of local dynamic bifurcation identified as an *isolated* Hopf bifurcation.

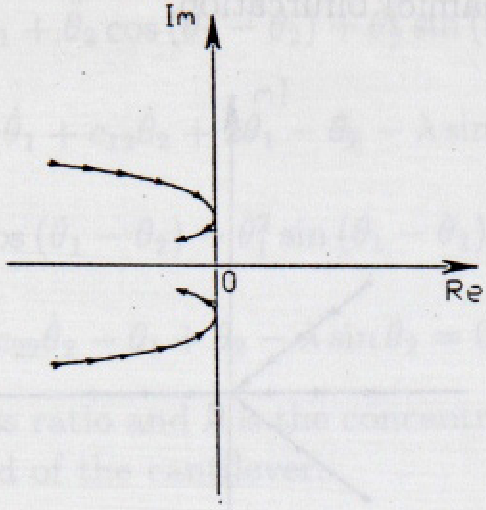

Fig. 2. The λ-dependent paths of a pair of complex conjugate eigenvalues being tangential to the imaginary axis (isolated Hopf bifurcation).

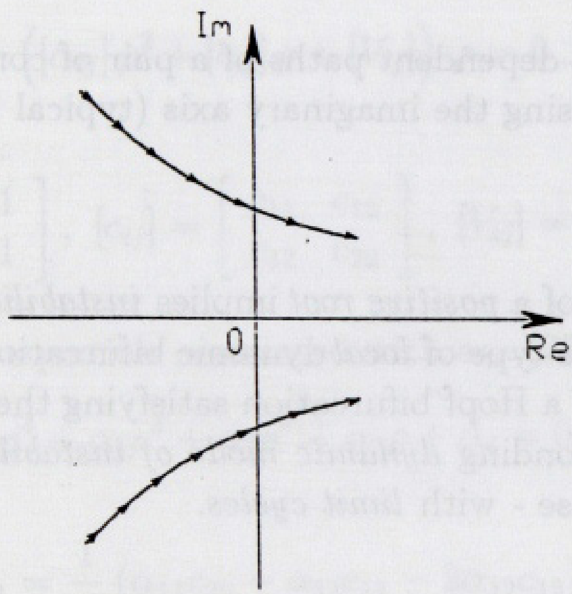

Fig. 3. The λ-dependent paths of a pair of complex conjugate eigenvalues passing for $\lambda = \lambda^C_{(1)}$ through the origin (double zero eigenvalue bifurcation).

Fig. 4 shows the λ-dependent paths of a pair of complex conjugate eigenvalues which pass through the origin at $\lambda = \lambda^C_{(1)}$, while for $\lambda > \lambda^C_{(1)}$

proceed in opposite directions along the real axis. This is a *double zero eigenvalue* local (dynamic) bifurcation.

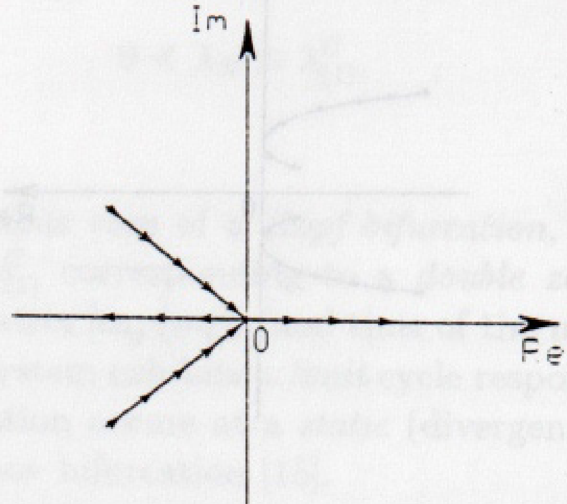

Fig. 4. The λ-dependent paths of a pair of complex conjugate eigenvalues crossing the imaginary axis (typical Hopf bifurcation).

The existence of a *positive root* implies *instability* of the *trivial* equilibrium state. This type of *local* dynamic bifurcation - as shown above - is a *special* case of a Hopf bifurcation satisfying the pertinent equation. Thus, the corresponding *dynamic mode of instability* is associated - as in the previous case - with *limit cycles.*

4 Numerical examples

Consider the 2-DOF Ziegler's cantilever under a concentrated compressive tip load as a model for which a rich variety of numerical results, based on linear and recently nonlinear analyses, are available [8,9]. The governing nonlinear equations of motion for the *perfect* bifurcation model can be written in dimensionless form as follows

$$(1+m)\,\ddot{\theta}_1+\ddot{\theta}_2\cos(\theta_1-\theta_2)+\dot{\theta}_2^2\sin(\theta_1-\theta_2)+$$
$$c_{11}\dot{\theta}_1+c_{12}\dot{\theta}_2+2\theta_1-\theta_2-\lambda\sin\theta_1=0, \tag{21}$$
$$\ddot{\theta}_2+\ddot{\theta}_1\cos(\theta_1-\theta_2)-\dot{\theta}_1^2\sin(\theta_1-\theta_2)+c_{12}\dot{\theta}_1+$$
$$c_{22}\dot{\theta}_2-\theta_1+\theta_2-\lambda\sin\theta_2=0,$$

where m denotes mass ratio and λ is the concentrated compressive load acting on the free end of the cantilever.

Linearizing eqs. (21) and then setting

$$\boldsymbol{\theta}=\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}=e^{\rho t}\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}=e^{\rho t}\boldsymbol{\varphi}, \tag{22}$$

one can obtain

$$\left([\alpha_{ij}]\,\rho^2+[c_{ij}]\,\rho+[V_{ij}]\right)\boldsymbol{\varphi}=0, \tag{23}$$

where

$$[\alpha_{ij}]=\begin{bmatrix}1+m & 1\\ 1 & 1\end{bmatrix},\ [c_{ij}]=\begin{bmatrix}c_{11} & c_{12}\\ c_{12} & c_{22}\end{bmatrix},\ [V_{ij}]=\begin{bmatrix}2-\lambda & -1\\ -1 & 1-\lambda\end{bmatrix}. \tag{24}$$

The *characteristic* equation associated with eq. (23) is

$$\rho^4+\alpha_1\rho^3+\alpha_2\rho^2+\alpha_3\rho+\alpha_4=0, \tag{25}$$

where

$$\left.\begin{aligned}
\alpha_1&=\frac{1}{m}\left(\alpha_{11}c_{22}+\alpha_{22}c_{11}-2\alpha_{12}c_{12}\right)\\
\alpha_2&=\frac{1}{m}\left(\alpha_{11}V_{22}+\alpha_{22}V_{11}-2V_{12}\alpha_{12}+|c_{ij}|\right)\\
\alpha_3&=\frac{1}{m}\left(c_{11}V_{22}+c_{22}V_{11}-2V_{12}c_{12}\right)\\
\alpha_4&=\frac{1}{m}\left(V_{11}V_{22}-V_{12}^2\right)
\end{aligned}\right\}. \tag{26}$$

The critical *static* (divergence) state (related to det $[V_{ij}] = 0$) is governed by

$$\lambda^2 - 3\lambda + 1 = 0, \tag{27}$$

from which we get

$$\left.\begin{array}{l} \lambda^c_{(1)} = 0.5\left(3 - \sqrt{5}\right) = 0.381966011 \\ \lambda^c_{(2)} = 0.5\left(3 + \sqrt{5}\right) = 2.618033989 \end{array}\right\}. \tag{28}$$

Subsequently we consider models corresponding to the following t w o different types of the damping matrix $[c_{ij}]$:

$$[c_{ij}] = \begin{bmatrix} 0.01 & 0.002 \\ 0.002 & 0.0004 \end{bmatrix}, \ [c_{ij}] = \begin{bmatrix} 0.080 & 0.0494427 \\ 0.0494427 & 0.0305572572 \end{bmatrix}. \tag{29}$$

Clearly, both damping matrices are *positive semi-definite* (i.e. det $[c_{ij}] = |c_{ij}| = 0$).

Depending on the nature of the matrix $[V_{ij}]$ we consider *two subcases*: (a) an *isolated* Hopf bifurcation (if $[V_{ij}]$ is positive definite) and (b) a *double zero* eigenvalue bifurcation (if $[V_{ij}]$ is positive semi-definite).

Isolated Hopf bifurcation ($m = 10$)

In this case eqs (12) and (13) are written as follows

$$[c_{ij}]\,\varphi = 0, \quad \left([\alpha_{ij}]\,\rho^2 + [V_{ij}(\lambda)]\right)\varphi = 0. \tag{30}$$

From the first of these equations we get

$$0.01\varphi_1 + 0.002\varphi_2 = 0 \quad \text{or} \quad \varphi_1/\varphi_2 = -0.20. \tag{31}$$

From the second of eqs. (30) one can obtain

$$\rho^2 = \frac{1 - (2-\lambda)(\varphi_1/\varphi_2)}{\alpha_{11}(\varphi_1/\varphi_2) + \alpha_{12}} = \frac{\varphi_1/\varphi_2 - 1 + \lambda}{\alpha_{12}(\varphi_1/\varphi_2) + \alpha_{22}}. \tag{32}$$

Introducing $\alpha_{11} = 11, \alpha_{12} = \alpha_{22} = 1$ and $\varphi_1/\varphi_2 = -0.20$ into these equations we obtain

$$\lambda_H = \frac{0.32}{1.04} = 0.307692307 < \lambda^C_{(1)}. \tag{33}$$

Then, from eq. (32) follows also that $\mu^2 = -\rho^2 = 1.115384615$ and $\mu = 1.056117709$. The corresponding conjugate *purely imaginary* eigenvalues are $\rho_{3,4} = \pm 1.056117709j$, while the other *two complex* conjugate eigenvalues are $\rho_{1,2} = -0.00052 \pm 0.124034j$.

Figs. 5 and 6 show the corresponding to $\lambda = 0.29 < \lambda_H$ and $\lambda = 0.33 > \lambda_H$ phase-plane portraits with the associated complex conjugate eigenvalues. Clearly, in both cases the system exhibits a *point attractor*.

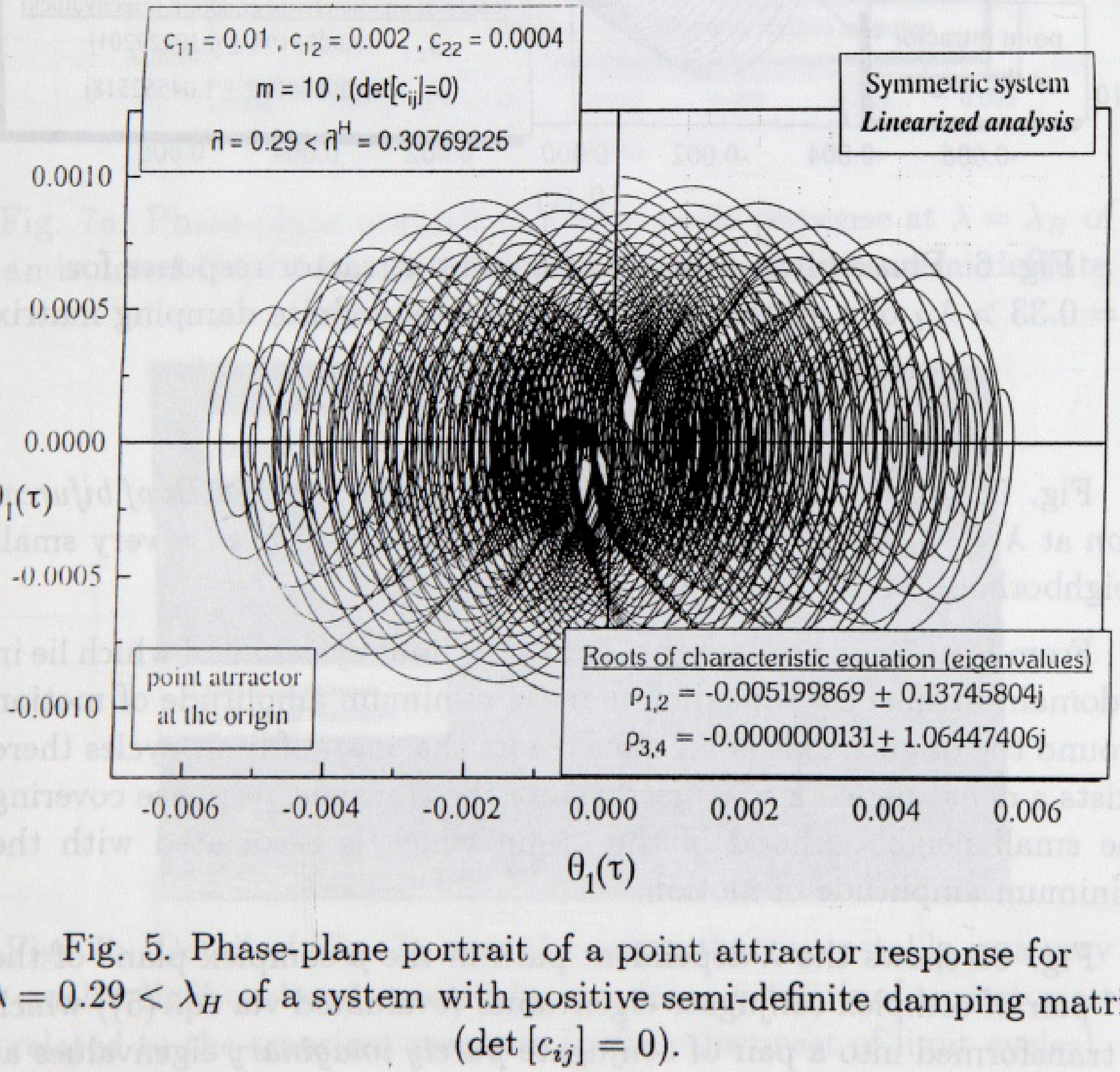

Fig. 5. Phase-plane portrait of a point attractor response for $\lambda = 0.29 < \lambda_H$ of a system with positive semi-definite damping matrix ($\det [c_{ij}] = 0$).

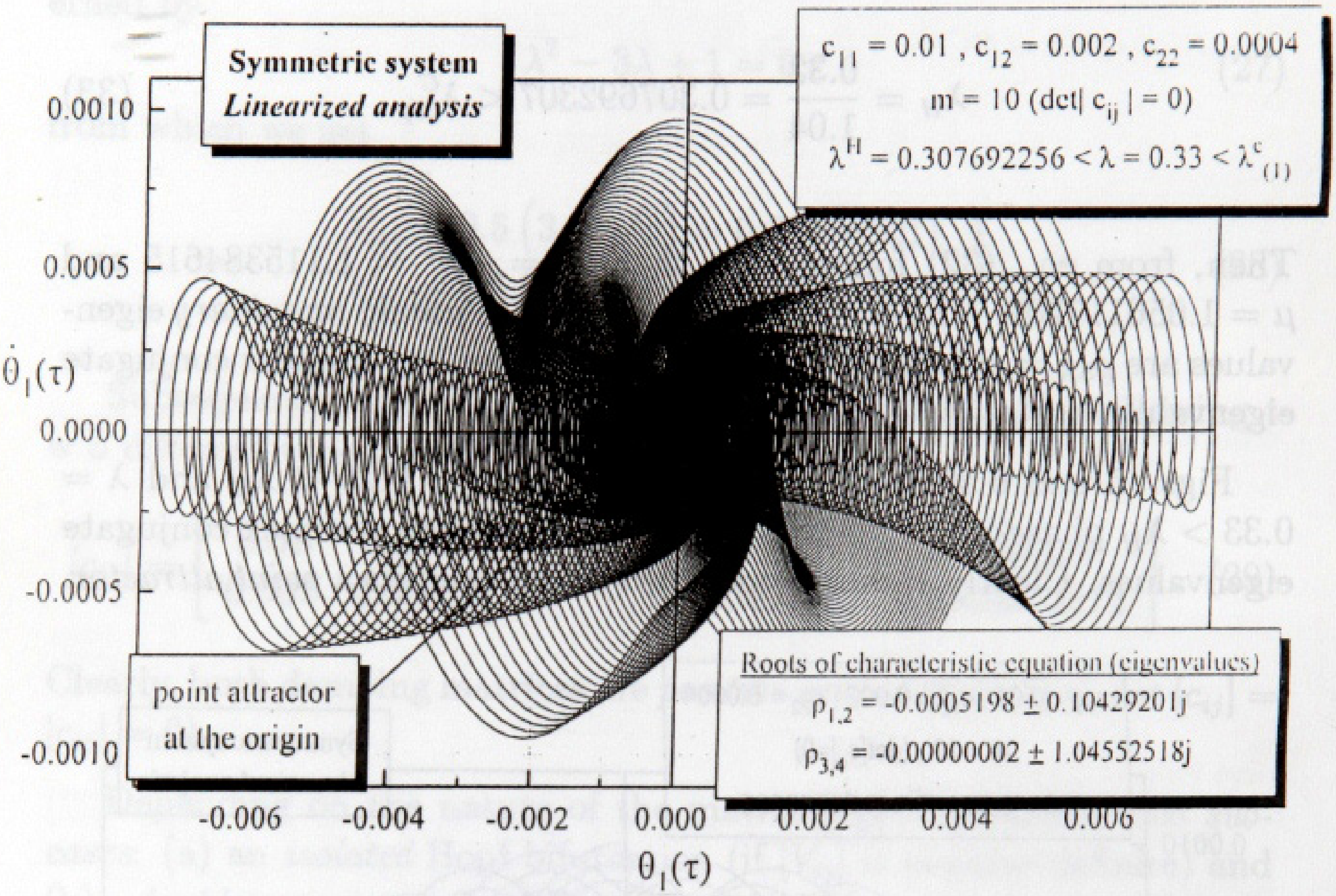

Fig. 6. Phase-plane portrait of a point attractor response for $\lambda = 0.33 > \lambda_H$ of a system with positive semi-definite damping matrix ($\det [c_{ij}] = 0$).

Fig. 7a shows the phase-plane portrait of the *isolated Hopf bifurcation* at $\lambda = \lambda_H$ associated with limit cycles, starting from a very small neighborhood of the (unstable) origin (Fig. 7b).

From Fig. 7b one cannot see clearly the limit cycles all of which lie in a domain defined by a maximum and a minimum amplitude of motion around the origin. This is so, since before the onset of limit cycles there exists a dense network of trajectories of the transient response covering the small neighbourhood of the origin which is associated with the minimum amplitude of motion.

Fig. 8a shows the λ-dependent path in the ρ-complex plane of the 2^{nd} pair of complex conjugate eigenvalues (evaluated via eq. (5)) which is transformed into a pair of conjugate *purely imaginary* eigenvalues at

$\lambda = \lambda_H$. Details of both paths at $\lambda = \lambda_H$ are depicted in Fig. 8b.

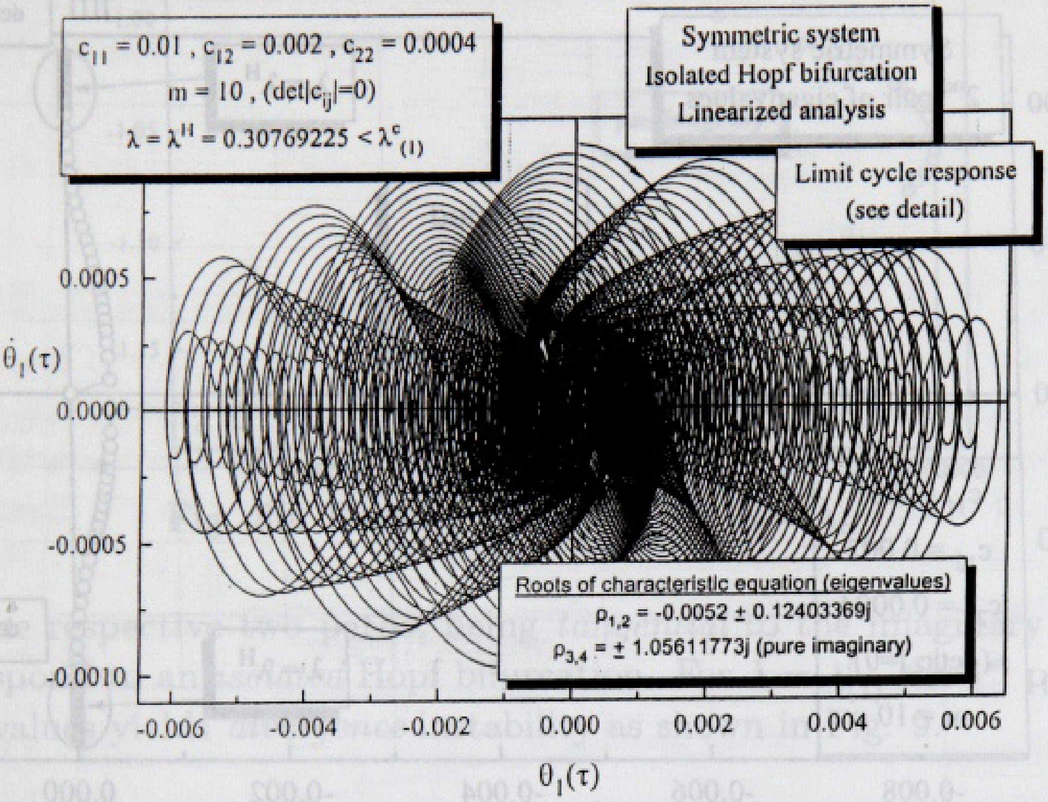

Fig. 7a. Phase-plane portrait of a limit cycle response at $\lambda = \lambda_H$ of an isolated Hopf bifurcation of a system with positive semi-definite damping matrix ($\det[c_{ij}] = 0$).

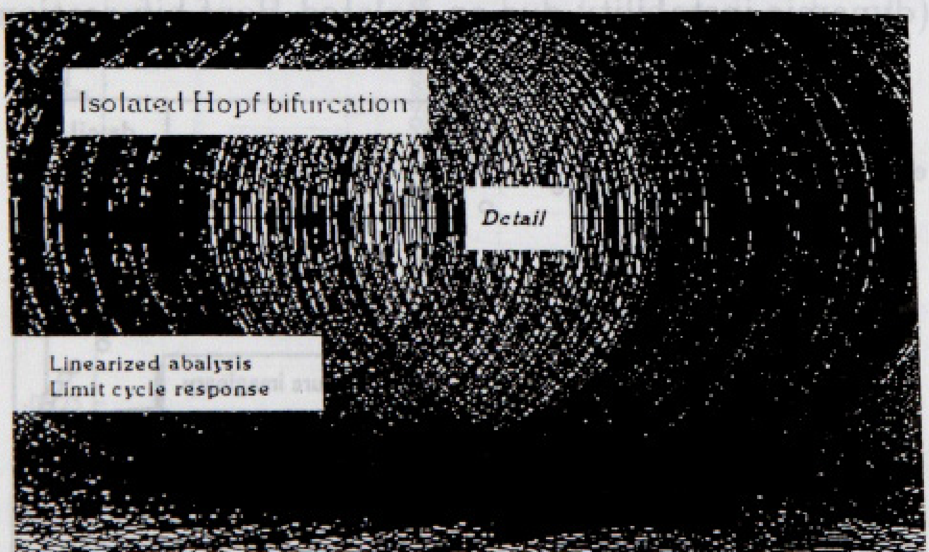

Fig. 7b. Detail of Fig. 7a near the origin that is unstable in a very small neighbourhood covered by the dense network of trajectories related to the transient response (before the onset of limit cycles).

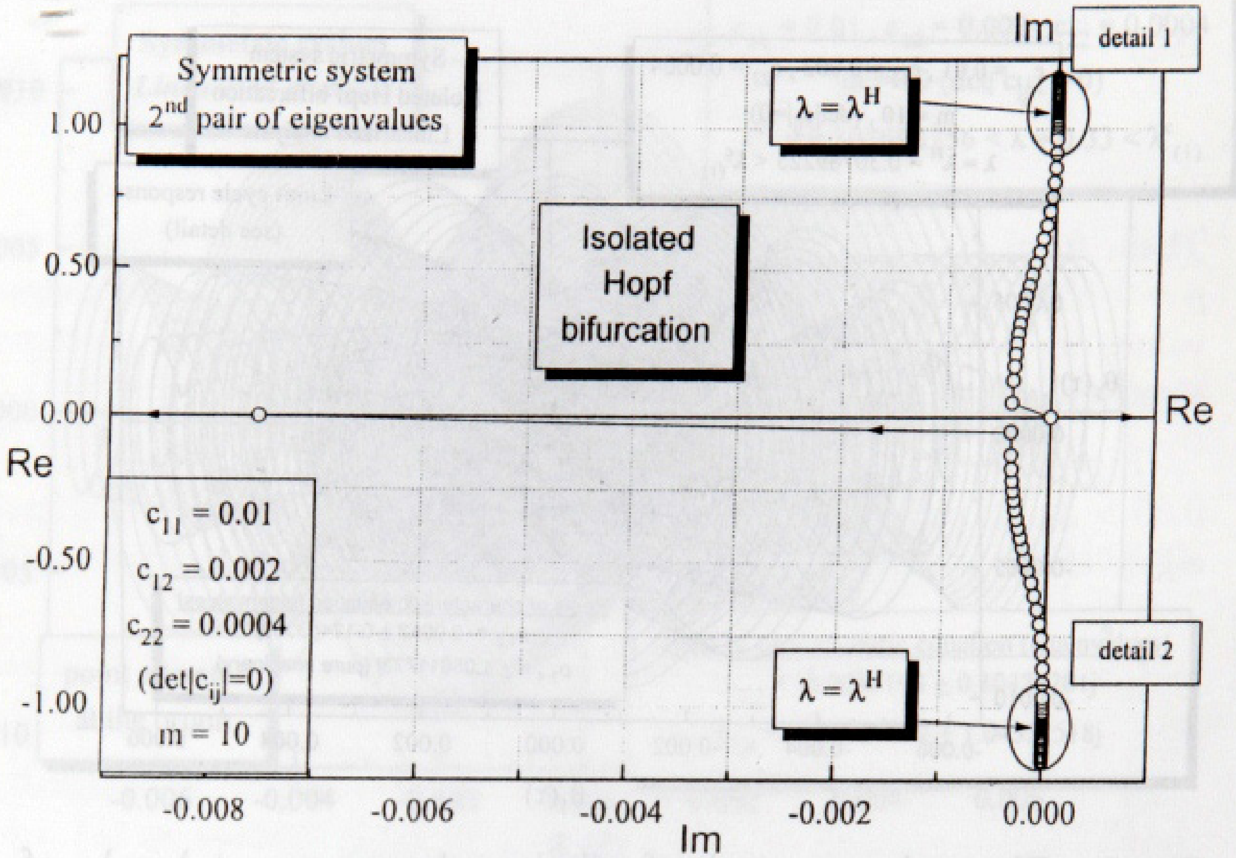

Fig. 8a. The λ-dependent paths (of the 2[nd] pair of complex conjugate eigenvalues) which are tangetial to the imaginary axis at $\lambda = \lambda_H$ (dynamic instability due an isolated Hopf bifurcation).

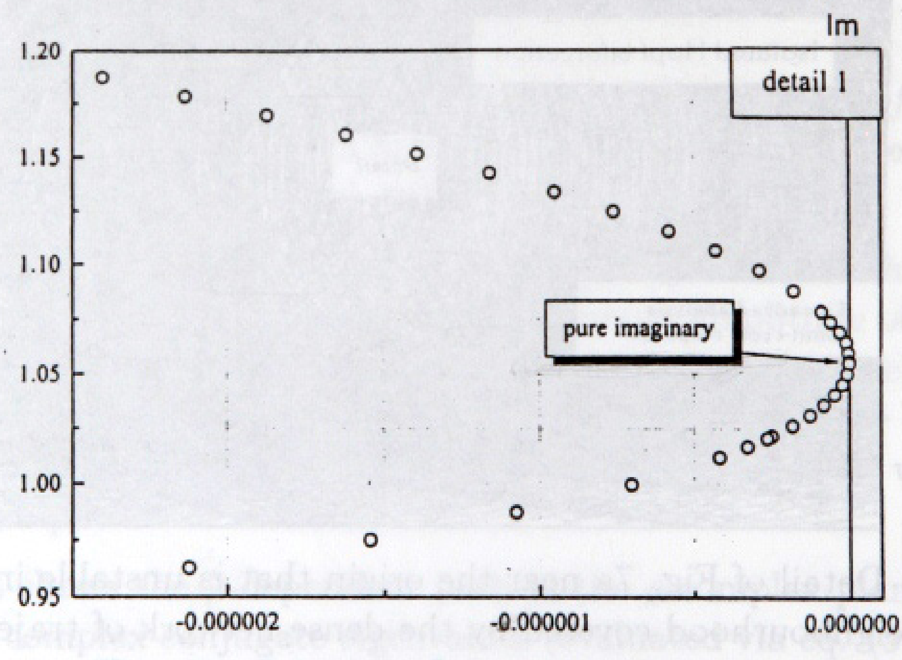

Fig. 8b. Detail 1 of the path at $\lambda = \lambda_H$.

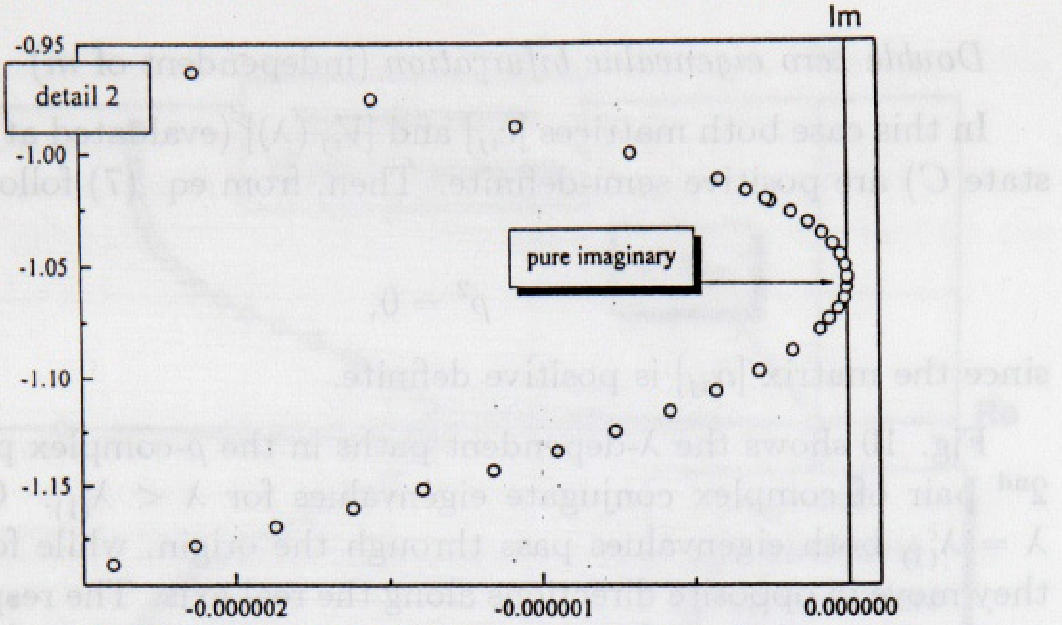

Fig. 8b. Detail 2 of the path at $\lambda = \lambda_H$.

The respective two paths, being *tangential* to the imaginary axis, correspond to an *isolated* Hopf bifurcation. For $\lambda = \lambda_{(1)}^c$ the 1st pair of eigenvalues yields *divergence* instability as shown in Fig. 9.

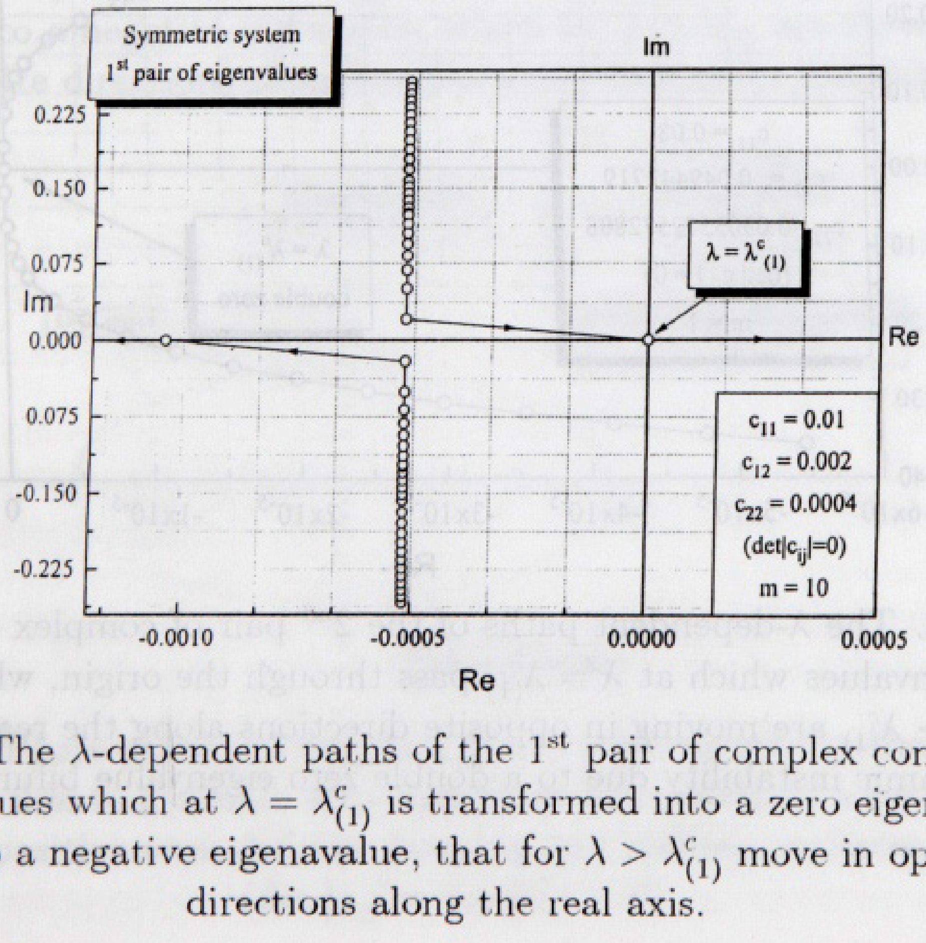

Fig. 9. The λ-dependent paths of the 1st pair of complex conjugate eigenvalues which at $\lambda = \lambda_{(1)}^c$ is transformed into a zero eigenvalue and into a negative eigenavalue, that for $\lambda > \lambda_{(1)}^c$ move in opposite directions along the real axis.

Double zero eigenvalue bifurcation (independent of m)

In this case both matrices $[c_{ij}]$ and $[V_{ij}(\lambda)]$ (evaluated at the critical state C) are positive semi-definite. Then, from eq. (7) follows that

$$\rho^2 = 0, \tag{34}$$

since the matrix $[\alpha_{ij}]$ is positive definite.

Fig. 10 shows the λ-dependent paths in the ρ-complex plane of the 2nd pair of complex conjugate eigenvalues for $\lambda < \lambda^c_{(1)}$. Clearly, for $\lambda = \lambda^c_{(1)}$ both eigenvalues pass through the origin, while for $\lambda > \lambda^c_{(1)}$ they move in opposite directions along the real axis. The respective two paths of the 1st pair of eigenvalues associated with the 2nd divergence load $\lambda = \lambda^c_{(2)}$ are shown in Fig. 11.

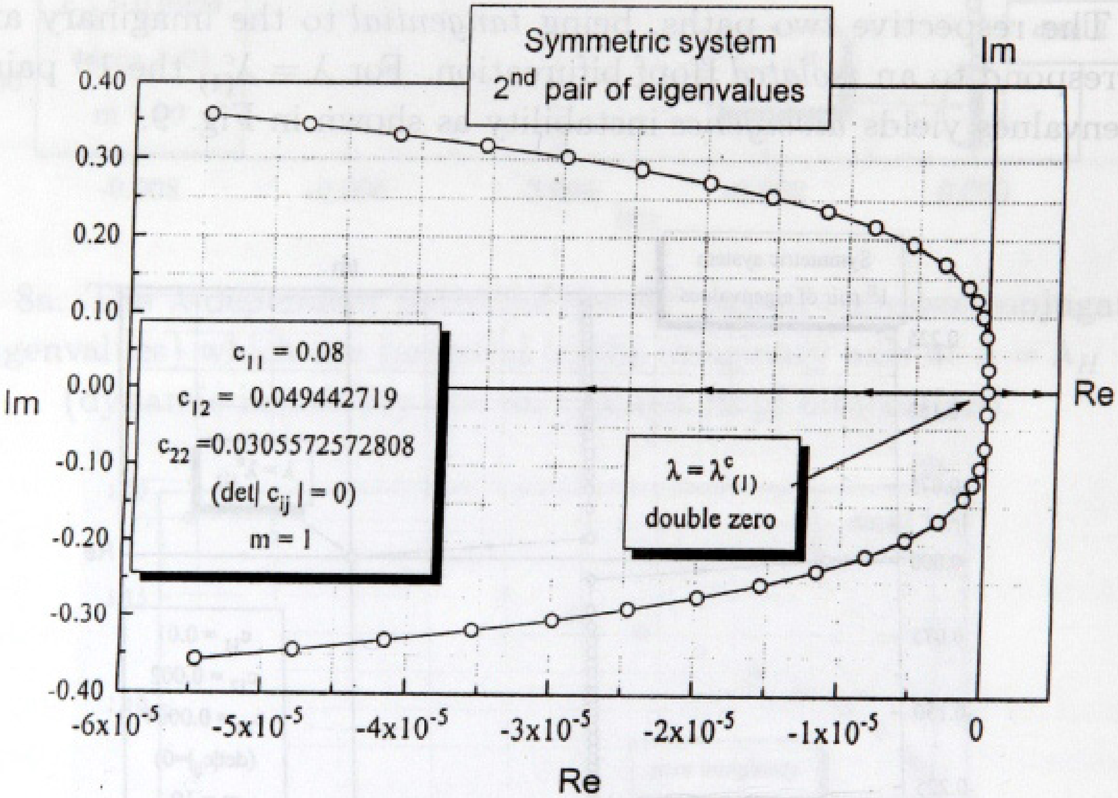

Fig. 10. The λ-dependent paths of the 2nd pair of complex conjugate eigenvalues which at $\lambda = \lambda^c_{(1)}$ pass through the origin, while for $\lambda > \lambda^c_{(1)}$ are moving in opposite directions along the real axis (dynamic instability due to a double zero eigenvalue bifurcation).

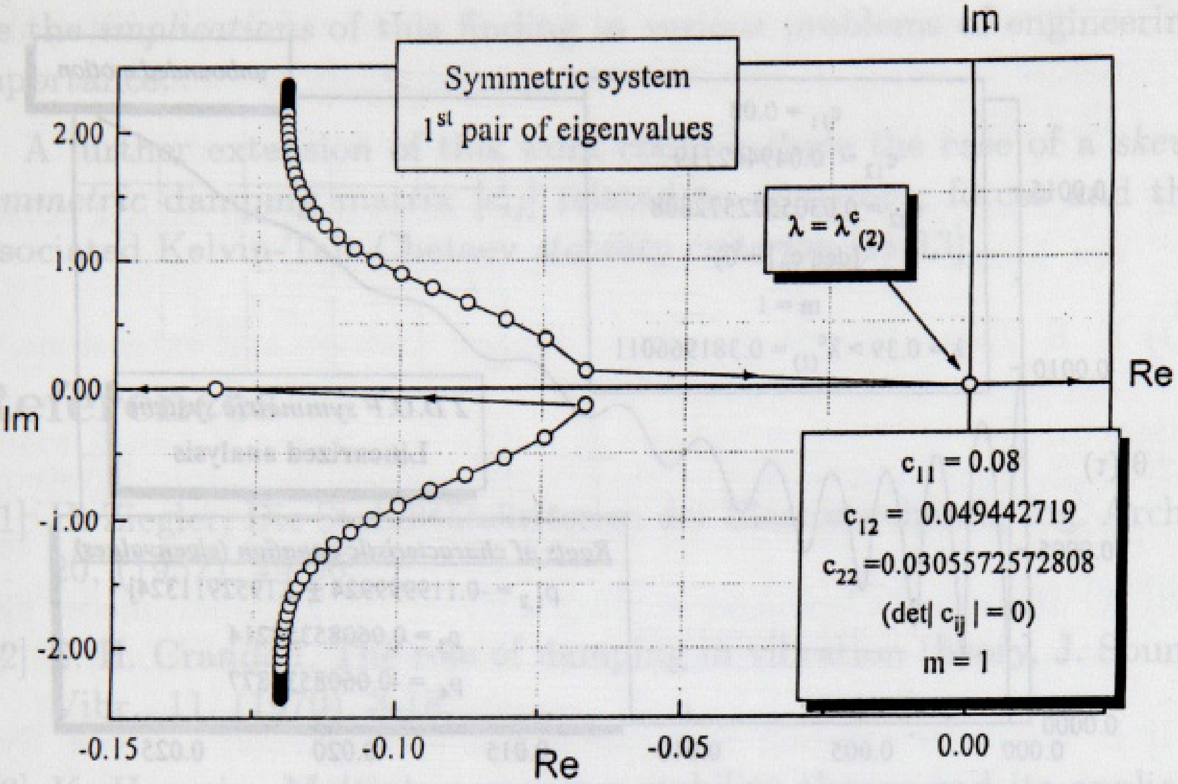

Fig. 11. The λ-dependent paths (of the 1st pair of complex conjugate eigenvalues) which at $\lambda = \lambda^c_{(2)}$ is transformed into a zero eigenvalue and into a negative eigenvalue, which for $\lambda > \lambda^c_{(1)}$ are moving in opposite directions along the real axis (divergence instability).

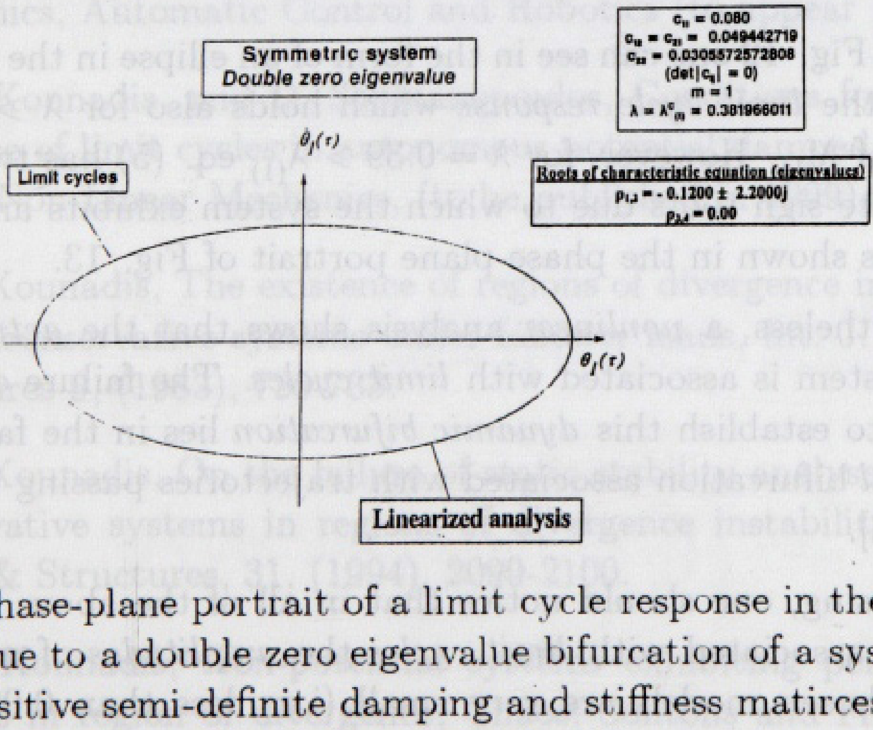

Fig. 12. Phase-plane portrait of a limit cycle response in the form of an ellipse due to a double zero eigenvalue bifurcation of a system with positive semi-definite damping and stiffness matirces (det $[c_{ij}] =$ det $[V_{ij}] = 0$).

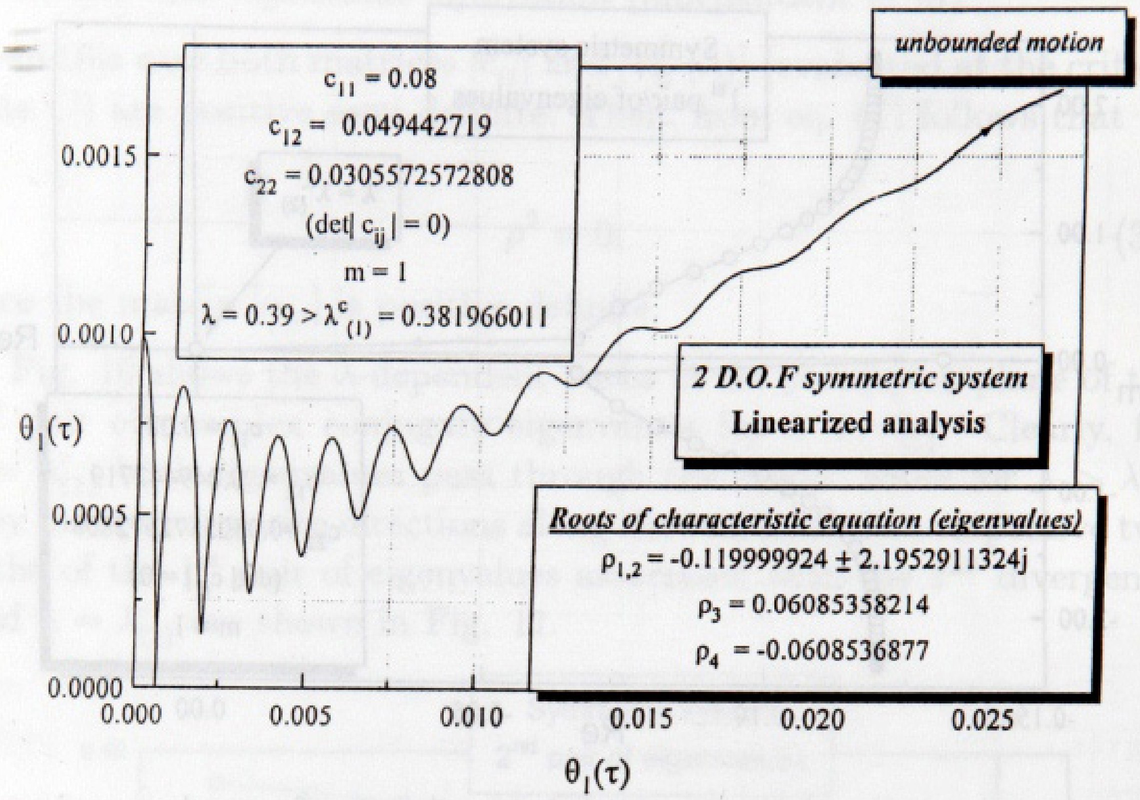

Fig. 13. The phase-plane portrait of the previous damped system at $\lambda = 0.39 > \lambda^c_{(1)}$ showing an unbounded motion (erronous result due to linearization).

From Fig. 12 one can see in the form of an ellipse in the phase-plane portrait the *limit cycle response* which holds also for $\lambda > \lambda^c_{(1)}$ in the vicinity of $\lambda^c_{(1)}$. However, for $\lambda = 0.39 > \lambda^c_{(1)}$ eq. (5) has two *equal* but of opposite sign roots due to which the system exhibits an *unbounded* motion as shown in the phase-plane portrait of Fig. 13.

Nevertheless, a *nonlinear* analysis shows that the *actual* response of the system is associated with *limit cycles.* The failure of the *linear* analysis to establish this *dynamic bifurcation* lies in the fact that this is a *global* bifurcation associated with trajectories passing through the origin [16].

In closing, one should notice that in all of the above phase-plane portraits associated with *limit cycles* the *amplitudes* of motion $\theta_1(t)$ for the chosen models are very small (i.e. less than 0.001 rad). A thorough quantitative discussion should be required in order to examine the *implications* of this finding in various problems of engineering importance.

A further extension of this work could include the case of a *skew-symmetric* damping matrix $[c_{ij}]$ related to *gyroscopic* forces and the associated Kelvin-Tait-Chetaev *stability criterion* [11-13].

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Appendix

The critical load λ_H corresponding to a *Hopf bifurcation* of a (weakly) damped system is obtained as follows:

Eq. (10) can be written in a *partitioned* matrix form as follows:

$$\begin{bmatrix} \mathbf{C} & \mathbf{C} \\ \mathbf{C}_1^{\mathrm{T}} & c_{nn} \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ r_n \end{bmatrix} = 0, \quad (r_n \neq 0) \,, \tag{a}$$

where $\mathbf{C}$ is an $(n-1) \times (n-1)$ *symmetric* matrix obtained from the *damping* matrix $[c_{ij}]$ by deleting its last line and its last column; $\mathbf{C}_1$ is an $(n-1) \times 1$ column matrix related to the last column of matrix $[c_{ij}]$

by deleting its element c_{nn}; $\mathbf{r}$ is an $(n-1)\times 1$ column matrix (vector) obtained from $\mathbf{r}$ by deleting its last element r_n.

The symmetric matrices $[\alpha_{ij}]$ and $[V_{ij}]$, can also be written in a similar *partitioned* form, as follows i.e.

$$[\alpha_{ij}] = \begin{bmatrix} \mathbf{A} & \mathbf{A}_1 \\ \mathbf{A}_1^{\mathrm{T}} & \alpha_{nn} \end{bmatrix}, \quad [V_{ij}] = \begin{bmatrix} \mathbf{V} & \mathbf{V}_1 \\ \mathbf{V}_1^{\mathrm{T}} & V_{nn} \end{bmatrix}, \tag{b}$$

where $\mathbf{A}$ and $\mathbf{V}$ are $(n-1)\times(n-1)$ *symmetric* positive definite matrices obtained from the matrices $[\alpha_{ij}]$ and $[V_{ij}]$ by deleting their respective last lines and last columns; $\mathbf{A}_1$ and $\mathbf{V}_1$ are $(n-1)\times 1$ column matrices related to the last columns of the matrices $[\alpha_{ij}]$ and $[V_{ij}]$ by deleting their last elements (i.e. α_{nn} and V_{nn} respectively).

From eq. (a) one can establish

$$\mathbf{r} = -r_n \mathbf{C}^{-1}\mathbf{C}_1, \tag{c}$$

where det $\mathbf{C} > 0$ since the matrix $[c_{ij}]$ is *positive semi-definite.*

Eq. (13). written also in a *partitioned* form, with the aid of relations (b) yields

$$\left(\rho^2\mathbf{A}_1^{\mathrm{T}} + \mathbf{V}_1^{\mathrm{T}}\right)\mathbf{r} + \left(\rho^2\alpha_{nn} + V_{nn}\right)r_n = 0. \tag{d}$$

Substituting $\mathbf{r}$ from eq. (c) into eq. (d) leads to

$$\left(\rho^2\mathbf{A}_1^{\mathrm{T}} + \mathbf{V}_1^{\mathrm{T}}\right)\mathbf{C}^{-1}\mathbf{C}_1 = \rho^2\alpha_{nn} + V_{nn}. \tag{e}$$

On the other hand, since $\mathbf{r}^{\mathrm{T}} = [\mathbf{r}\ r_n]$, eq. (15) with the aid of relations (c) and (b) gives

$$-\rho^2 = \mu^2 = \frac{\mathbf{C}_1^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{V}\mathbf{C}^{-1}\mathbf{C}_1 - 2\mathbf{V}_1^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{C}_1 + V_{nn}}{\mathbf{C}_1^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{A}\mathbf{C}^{-1}\mathbf{C}_1 - 2\mathbf{A}_1^{\mathrm{T}}\mathbf{C}^{-1}\mathbf{C}_1 + \alpha_{nn}}. \tag{f}$$

Note that each term in the numerator and denominator of this ratio is a *scalar* quantity, and thus $\rho^2\,(=-\mu^2)$ is a *linear* function of the load λ.

Combining eqs. (e) and (f) we find the *necessary* condition for a Hopf bifurcation in a *symmetric weakly damped system*

$$\begin{aligned}&\Big\{\Big[\Big(\mathbf{C}_1^T\mathbf{C}^{-1}\mathbf{A}\mathbf{C}^{-1}\mathbf{C}_1 - 2\mathbf{A}_1^T\mathbf{C}^{-1}\mathbf{C}_1 + \alpha_{nn}\Big)\mathbf{V}_1^T - \\ &\Big(\mathbf{C}_1^T\mathbf{C}^{-1}\mathbf{V}\mathbf{C}^{-1}\mathbf{C}_1 - 2\mathbf{V}_1^T\mathbf{C}^{-1}\mathbf{C}_1 + V_{nn}\Big)\mathbf{A}_1^T\Big]\Big\}\mathbf{C}^{-1}\mathbf{C}_1 = \\ &\Big(\mathbf{C}_1^T\mathbf{C}^{-1}\mathbf{A}\mathbf{C}^{-1}\mathbf{C}_1 - 2\mathbf{A}_1^T\mathbf{C}^{-1}\mathbf{C}_1 + \alpha_{nn}\Big)V_{nn} - \\ &\Big(\mathbf{C}_1^T\mathbf{C}^{-1}\mathbf{V}\mathbf{C}^{-1}\mathbf{C}_1 - 2\mathbf{V}_1^T\mathbf{C}^{-1}\mathbf{C}_1 + V_{nn}\Big)\alpha_{nn}.\end{aligned} \tag{g}$$

For given matrices $[\alpha_{ij}(m_i)]$ (or masses m_i) and $[c_{ij}]$, one can readily obtain a load $\lambda = \lambda(c_{ij}, m_i)$ which is defined as a *Hopf bifurcation* load $\lambda = \lambda_H$ provided that $\lambda_H > 0$. The implications of this load if inequality (16) is satisfied (i.e. if dynamic instability due to a Hopf bifurcation occurs prior to divergence) are very important in engineering practice [7].

In case of a *double zero* eigenvalue the numerator of eq. (f) becomes zero and eq. (g) is reduced to the following simple form

$$\mathbf{V}_1^T\mathbf{C}^{-1}\mathbf{C}_1 = V_{nn}. \tag{h}$$

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Izolovane bifurkacije Hopfovog tipa simetričnih slabo prigušenih sistema

U radu se razmatra dinamička stabilnost autonomnih slabo prigušenih simetričnih (potencijalnih) sistema. Uslovi za pojavu nestabilnosti oblika graničnog ciklusa su postavljeni kroz diskusiju efekata matrice prigušenja na sopstvene vrednosti Jakobijana. Pokazano je da se takav

odziv može pojaviti u vidu novog tipa lokalne dinamičke bifurkacije koja je identifikovana kao izolovana bifurkacija Hopfovog tipa kao i u vidu lokalne dinamičke bifurkacije sa dvostrukom nultom sopstvenom vrednosti. Kao posledica neprigušeni stabilni sistemi mogu postati nestabilni sa uključenjem prigušenja. Numerički rezultati potvrdjuju validnost ovde prezentovanih teorijskih otkrića.