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Rheonomic tangent and cotangent bundles and applications

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Abstract

Let us consider an extended configuration manifold $\mathcal{M} = M^n \times x^0$, where M is Lagrange's configurations n-dimensional manifold and $x^0 = \tau(\aleph t)$ is taken from the given relations generating the manifold $\mathcal{M}$. In some special cases $x^0 = t$ can be taken, but $x^0 \equiv t$ can not. The extended tangent bundle has dimension $2n+2$. Metric on $\mathcal{M}$ is given by $ds^2 = g_{ij}(x)dx^i dx^j + 2g_{0i}dx^0 dx^i + g_{00}dx^0 dx^0$. The set of all covectors $p_i = g_{ij}\dot{x}^j + g_{i0}\dot{x}^0$ and $p_0 = g_{0j}\dot{x}^j + g_{00}\dot{x}^0$ at $x = (x^0, x^1, ..., x^n)$ forms an extended cotangent bundle $T^*\mathcal{M}$ having a natural symplectic structure. The adding coordinates x^0 to M, $\dot{x}^0$ to TM or p_0 to T^*M is not simply algebraic extension respectively of M, TM and T^*M, but have geometric and physical meaning. The geometry of such spaces is not considered so far and we modified the whole standard Lagrange's and Hamilton's mechanics systems on such a geometric base. The D'Alembert-Lagrange's principle as well as Hamiltonian principle or the least action principle are appropriately restated. There are more independent Lagrange's and Hamilton's equations and more variations of rheonomic constraints ($x_0 \neq 0, \delta t = 0$). Poincare's integral invariant for nonautonomous systems is generalized.

1 Introduction

Symplectic geometry adequately describes the movement of mechanical systems with scleronomic constraints. For such systems symplectic geometry is identified with Hamiltonian mechanics on $2n$-dimension cotangent bundles (see for example [3]). Taking into consideration the system of N material points with constant masses m_v, $(v = 1, ..., N)$ scleronomic geometric constraints

$$f_\mu \left(y^1, ..., y^{3N}\right) = 0, \quad \mu = 1, ..., k < 3N, \quad y \in E^{3N}, \tag{1}$$

generate an n-dimensional differential metric manifold $M := M^n$ where is $n = 3N - k$. The metric on manifolds M is Riemannian

$$ds^2 = a_{ij}(x)\, dx^i dx^j, \quad x \in M, \quad (i, j = 1, ..., n), \tag{2}$$

and the metric tensor $a_{ij} = a_{ji}(x^1, ..., x^n)$ is identified with the inertia tensor of the mechanical system. N-dimensional tangent vector $\dot{x} = (\dot{x}^1, ..., \dot{x}^n)^T$ can exist in any point of M.

Coordinates of the impulse covector

$$p_i = a_{ij}(x)\, \dot{x}^j, \tag{3}$$

belong to $2n$-dimensional cotangent bundles T^*M. On them the first and the second form

$$\omega := pdx, \quad \Omega := dp \wedge dx, \tag{4}$$

are defined and the integral invariant relation

$$\int_{\partial C} \omega = \iint_C \Omega, \tag{5}$$

is proved.

The mechanical motion on T^*M is described by Hamiltonian function (see for example [2])

$$H = a_{ij}(x)\, p_i p_j + \Pi(x), \tag{6}$$

and the system of $2n$ differential equations

$$\dot{p}_i = -\frac{\partial H}{\partial x^i}, \quad \dot{x}^i = \frac{\partial H}{\partial p_i}. \tag{7}$$

However, if Eqs. (1) are non-autonomous, i.e. if the system has time dependent constraints such geometry cannot thoroughly describe the mechanical motion. In difference from scleronomic constraints (1) rheonomic constraints contain an independent parameter t, and in standard analytical mechanics are written as in differential geometry in the form

$$f_\mu (y, t) = 0, \quad y \in E^{3N}, \quad x \in M. \tag{8}$$

The metric of manifolds generated in that way considerably differs from the metric (2) because

$$ds^2 = a_{ij} (x, t) \, dx^i dx^j + 2b_i (x, t) \, dx^i dt + c (x, t) \, dt^2. \tag{9}$$

The difference between that metric tensor and the tensor in metric (2) is very important as regarding the number of dimensions so regarding the physical nature. In metric (2) all coordinates are of geometric nature and in metric (9) a differential of kinematic parameter t appears. Therefore symplectic geometry has met with difficulties at first application in the mechanics of rheonomic systems.

In the case of metric form (2) the kinetic energy E of the metric system appears as a homogeneous square form of the generalized velocities $\dot{x}$

$$2E_k = \sum_{v=1}^{N} m_v v_v^2 = a_{ij} (x) \, \dot{x}^i \dot{x}^j, \quad v_v \in T_x M, \quad \dot{x} \in TM, \tag{10}$$

but for the metric form (9) there is another case

$$2E_k = a_{ij} (x, t) \, \dot{x}^i \dot{x}^j + b_i (x, t) \, \dot{x}^i + c (x, t) \, . \tag{11}$$

Differences are obvious, significant and far-reaching. N coordinates of a tangent vector act in each of the two formulas (10) and (11). In mechanics it is a vector of velocity $v(x)$.

If the constraints are written in the form $f_\mu(\mathbf{r}_1, ..., \mathbf{r}_N) = 0$, then the position vectors $\mathbf{r}$; $\mathbf{r} \in \mathbb{R}^3$, appear as vector functions $\mathbf{r} = \mathbf{r}(x^1, ..., x^n)$. The vector of velocity of the point v-th is by definition

$$\mathbf{v}_v := \frac{d\mathbf{r}_v}{dt} = \frac{\partial \mathbf{r}_v}{\partial x^i}\dot{x}^i. \tag{12}$$

This is a tangent vector which in any point on manifolds M^n has n base vectors

$$\frac{\partial \mathbf{r}(x)}{\partial x^i} := g_i(x) \in T_x M,$$

as well as n coordinates $\dot{x} \in T_x M$. However, in the case of time dependent constraints $f_\mu(\mathbf{r}, t) = 0 \rightarrow \mathbf{r} = \mathbf{r}(x, t)$ the vectors of velocity $\mathbf{v}_v = (\partial \mathbf{r}_\nu / \partial x^i)\, \dot{x}^i + \partial \mathbf{r}_\nu / \partial t$ have, as it can be seen, the $n+1$ vector component, and in the formula (8), like in other relevant relations, mechanics acts only as the n coordinates $\dot{x}_i$ of the $(n+1)$ dimensional vector field $\mathbf{v}$. In order to overcome that discord a series of papers [2], [4], [5], [6], [7], [11] were written at the end of the 1920s and especially the 1930s. A good ground for the solution of that problem, in my opinion, provided the papers of Zdenek Horak. Although Horak had earlier been quoted I saw his papers as late as in October 1994 at the Faculty of Mechanics and Mathematics of the Moscow University. I did not have the opportunity to get the earliest Horak's paper until the 6th Conference on Differential Geometry in Brno in 1995. I hope I would get the opportunity to write at length about his papers on some other occasion. Unfortunately another standpoint known as "homogeneous formalism" gained the advantage.

According to that formalism time t has been taken for $(n+1)$ coordinate, $x_{n+1} \equiv t$ and by it the configuration manifold M has been extended to M^{n+1}. The negative Hamiltonian $p_{n+1} = -H$, the physical dimensions of which cannot be brought into harmony with Descartes's and Newton's definition, contained in the relations (3) has by definition been taken for the corresponding $(n+1)$ coordinate of the covector impulse (3). That point of view of homogeneous formalism has not brought about any results in classical mechanics. On the contrary, it has brought into analytical dynamics disorder of invariance, of the variational principles and of the integral invariance $\int \omega^1$, which are the generating foundations of the theory of mechanics. Some contemporary authoritative and popular authors (see, for example [1]) having noticed difficulties with manifolds in the case of the non-autonomous system go back to the spaces E^{3N} with the vector orhonormal base $e = (e_1, e_2, e_3)$. However, by a more serious analysis of the geometric and dynamic nature of the non-autonomous constraints (8) a conclusion can be made that in fact they generate natural extended manifolds M^{n+1}.

2 Extended configuration manifolds

We observed N constrained masses points the positions of which were determined by $3N$ rectilinear rectangular coordinates $y = (y_v^1, y_v^2, y_v^3) = (y^1, ..., y^{3N}) \in E^{3N}$. The relations (8) can be written in a vectorial form

$$f_\mu(\mathbf{r}_1, ..., \mathbf{r}_N, t) = 0, \tag{13}$$

where t is an independent variable, $t \in [-\infty, \infty]$, $[dim\ t] = T$. In mechanics and physics t is a symbol of time by which the change of the position of points M_v is described. Accordingly, all coordinates y or vectors $\mathbf{r}$ act as functions of t and some parameter $\aleph$ which the relations (13) contain. Since the coordinates y, $[\dim\ y] = L$ and the variable t have different physical dimensions there is always a parameter $\aleph$ in the relations (8) or (13) which makes them dimensionally homogeneous. For example, the constant parameter l in the relation $f(y, t) = y^1 + y^2 + y^3 - 4l^2(2 - cos^2 \omega t) = 0$ must have a dimension of the length L, and $[\dim\ \omega] = T^{-1}$. It is therefore more appropriate to write down the relations in the form

$$f\left(y^0, y^1, ...y^{3N}\right) = 0, \quad y^0 = \tau(\aleph t). \tag{14}$$

We called the coordinate y^0 rheonomic coordinate [10] because it generally speaking differs from other coordinates y only in that y^0 is a known function of the time t and the known parameter $\aleph$ while other coordinates y are unknown functions of t and of some other parameter. The function $y^0(\aleph, t)$ is not arbitrary but is taken from one of the equations (13). In the given example it could be one of the following functions: $y^0 = l(2-\cos\omega t)$, $y^0 = l\cos\omega t$, $y^0 = \cos\omega t$, $y^0 = \omega t$ and $y^0 = \alpha t$, $\alpha = 1$; $[\dim\alpha] = LT^{-1}$, but cannot be $y^0 \equiv t$.

Introducing the coordinate $x^0 = \tau(\aleph, t)$ which is differentiable according to the variant t and variable according to the parameter $\aleph$, like other dependent y and independent x coordinates, the problem of non-autonomous constraints (13) is reduced to the case (1) and of non-homogeneous square forms (11) to the homogeneous form (10) whereby the number of dimensions of manifold M is increased by one coordinate x^0, x^0 & $x \in M^{n+1} =: \mathcal{M}$. Really on the basis of the theorem on implicit functions and on condition that $rank\,\{\partial y/\partial t\} = k$, it is possible to determine coordinates y from k equations (14) by means of $3N-k+1$ independent coordinates x & x^0; $y = y(x^0, x)$ or

$$\mathbf{r}_v = \mathbf{r}_v\left(x^0, x\right). \tag{15}$$

In that way the from of all relations from (1) to (7), as well as (10), (12) is helpful for non-autonomous systems too whereby indexes take the values of $0, 1, ..., n$ instead of $1, ..., n$ for non-autonomous systems. That statement can simply and quickly be proved. If we start from (15) we see that the vectors of velocity

$$\mathbf{v}_v = \frac{\partial \mathbf{r}_v}{\partial x^0}\dot{x}^0 + \frac{\partial \mathbf{r}_v}{\partial x^i}\dot{x}^i = \frac{\partial \mathbf{r}_v}{\partial x^\alpha}\dot{x}^\alpha, \quad \alpha = 0, 1, ..., n, \tag{16}$$

have each $n+1$ vector components and the same number of coordinates $\dot{x}^\alpha$. In every v-th point there is $(n+1)$-dimensions bases $(\partial\mathbf{r}_v/\partial x^0, \partial\mathbf{r}_v/\partial x^1, ..., \partial\mathbf{r}_v/\partial x^n)$ which generate the tangent bundle $\mathbf{T}_x \in R^{n+1}$ and adequately $(2n+2)$ tangent bundle $T\mathcal{M}$ to which generalized velocities $x = (\dot{x}^0, \dot{x}^1, ..., \dot{x}^n)^T$ belong.

Since the summary of the vectors $\mathbf{v}_\nu$ square with added scalar-masses m_ν can be written in the form

$$\sum_{v=1}^{N} m_v v_v^2 = \sum_{v=1}^{N} m_v \left(\frac{d\mathbf{r}_v}{dt}\right)^2 = \sum_{v=1}^{N} m_v \frac{\partial \mathbf{r}_v}{\partial x^\alpha} \cdot \frac{\partial \mathbf{r}_v}{\partial x^\beta}\dot{x}^\alpha \dot{x}^\beta,$$

it follows that the non-homogeneous square form of energy (11) obtains on $T\mathcal{M}$ a homogeneous form

$$2E_k = \sum_{v=1}^{N} m_v \mathbf{v}_v^2 = g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta, \tag{17}$$

and that the metric of manifolds $\mathcal{M}$ is

$$ds^2 = 2E_k dt^2 = g_{\alpha\beta} dx^\alpha dx^\beta, \tag{18}$$

where

$$g_{\alpha\beta} = \sum_{v=1}^{N} m_v \frac{\partial \mathbf{r}_v}{\partial x^\alpha} \cdot \frac{\partial \mathbf{r}_v}{\partial x^\beta} = g_{\alpha\beta}, \quad \alpha, \beta = 0, 1, ..., n. \tag{19}$$

is the metric tensor. In mechanics that tensor represents characteristics of inertia and is therefore called inertia tensor [10].

3 Rheonomic cotangent bundles

According to Descartes and Newton the impulse of the ν-th material point is

$$\mathbf{p}_v = m_v \mathbf{v}_v, \tag{20}$$

and the system impulse should be a summary of impulses of the material points. In order to avoid difficulties of summarizing constrained vectors due to indispensable parallel motion one resorts to projection of all vectors (20) on coordinate lines and to the subsequent summarizing of projections. Taking into account that $\partial \mathbf{r}_v / \partial x^\beta =: \mathbf{g}_{(v)\beta}$ are basic coordinate vectors of tangent bundles of the extended configuration manifolds $\mathcal{M}$, $\mathbf{g}_v \in \mathbf{T}_x \mathcal{M}$ operations of scalar multiplication of relations (20) by corresponding vectors $\mathbf{g}_v$ and of summarizing by index bring about generalized impulses

$$p_\beta = \sum_{v=1}^{N} m_v \frac{\partial \mathbf{r}_v}{\partial x^\alpha} \cdot \frac{\partial \mathbf{r}_v}{\partial x^\beta} \dot{x}^\beta = g_{\alpha\beta} \left(x^0, x\right) \dot{x}^\alpha. \tag{21}$$

Therefore, without introduction of new definitions by means of simple algebraic relations $(n+1)$-dimensional covector (21) is obtained or in a more elaborate form (21)

$$p_i = g_{ij} \left(x^0, x\right) \dot{x}^j + g_{i0} \left(x^0, x\right) \dot{x}^0, \tag{22}$$

$$p_0 = g_{0j} \left(x^0, x\right) \dot{x}^j + g_{00} \left(x^0, x\right) \dot{x}^0. \tag{23}$$

The last relation explicitly shows how much (23) differs from definition $p_{n+1} := -H$. The covector $p = (p_0, p_1, ..., p_n)$ on manifold $\mathcal{M}$ produces extended cotangent bundles $T^*\mathcal{M}$.

The bundle T^*M appears as a cotangent subbundle $T^*\mathcal{M} \hookrightarrow T^*M$ so that it is possible to follow the motion on T^*M and on $T^*\mathcal{M}$ but naturally portraits and physical attributes are different in these bundles.

The motion of the mechanical system with non-autonomous constraints on extended cotangent bundles is described with mechanical energy function

$$E = \frac{1}{2} g^{\alpha\beta}\left(x^0, x\right) p_\alpha p_\beta + \Pi\left(x^0, x\right) + P\left(x^0\right); \quad P := -\int R_0 dq^0,$$

and the system of $2n+2$ differential equations in Hamiltonian form

$$\dot{p}_\alpha = -\frac{\partial E}{\partial x^\alpha} \text{ and } \dot{x}^\alpha = \frac{\partial E}{\partial p_\alpha}. \tag{24}$$

With the aim to make a comparison with the equations (7), the equations (24) can be written in the form

$$\dot{p}_i = -\frac{\partial H}{\partial x^i}, \quad \dot{x}^i = \frac{\partial H}{\partial p_i} \tag{25}$$

$$\dot{p}_0 = -\frac{\partial H}{\partial x^0} + R_0, \quad \dot{x}^0 = \frac{\partial H}{\partial p_0}, \tag{26}$$

where $H = g^{\alpha\beta} p_\alpha p_\beta / 2 + \Pi$ is Hamiltonian and R_0 the power of non-autonomous constraints [9,10].

If the constraints are autonomous there is no rheonomic coordinate x^0 and according to that the bundles $T^*\mathcal{M}$ are reduced to T^*M and the equations (26), as well as the function $P(x^0)$ disappear. In that case the equations (24) are reduced to the system of $2n$ of Hamiltonian equations (7).

4 Integral invariance

It is proved similarly as in section 2 that the relation (3) remains preserved on $T^*\mathcal{M}$ whereby each of the forms ω and Ω is extended by one summand, i.e.

$$\omega = p_i dx^i + p_0 dx^0,$$

$$\Omega = dp \wedge dx + dp_0 \wedge dx^0.$$

Instead of the invariance $I = \oint_c p_i \delta x^i$ the invariance

$$J = \oint_{\bar{c}} p_i \delta x^i = \oint_{\bar{c}} \left(p_i \delta x + p_0 \delta x^0\right),$$

appears on $T^*\mathcal{M}$ where $\delta x^0, \delta x^1, ..., \delta x^n$ are the variations of generalized coordinates x^0 & x.

It gain one's ends: on such naturally extended $(2n+2)$-dimensional cotangent bundles $T^*\mathcal{M}$ it is possible to describe the motion of the non-autonomous systems which by its form and way is the same as the motion of autonomous systems on $2n$-dimensional cotangent bundles T^*M. The motion differs by the number of dimensions and the physical attributes of motion.

Example 1 *([1], p. 87). Consider the motion of a bead with mass m along a vertical circle of radius r which rotates with angular velocity ω around the vertical axis passing through the center O of the circle. Let x be the angular coordinate on the circle, measured from the highest point.*

Let y_1, y_2, y_3 be Cartesian coordinates in E^3 with origin O and vertical axis y_3. Let $x_0 := \varphi$ be the angle of the plane of the circle with the plane $y_1 O y_3$. The bead is constrained by the relations

$$f_1 = r - const. = 0, \quad f_2 = \varphi - \omega t = 0,$$

or

$$y_1 = r \sin x \cos x_0, \quad y_2 = r \sin x \sin x_0, \quad y_3 = r \cos x.$$

Let $x_0 = \omega t$.

The manifold M is circle, but manifold $\mathcal{M}$ is sphere.

The kinetic energy has the form

$$T = \frac{m}{2}\left(\dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2\right) = \frac{mr^2}{2}\left(\dot{x}^2 + \dot{x}_0^2 \sin^2 x\right) = \frac{1}{2mr^2}\left(p^2 + \frac{p_0^2}{\sin^2 x}\right),$$

because the impulses are $p = mr^2\dot{x}$ and $p_0 = mr^2\dot{x}_0 \sin^2 x$.

The potential energy is $\Pi = mgr(1 + \cos x)$. Hamiltonian is

$$H = E_k + \Pi = \frac{1}{2mr^2}\left(p^2 + \frac{p_0^2}{\sin^2 x}\right) + mgr(1 + \cos x).$$

Differential equations of motion (25) and (26) are

$$\dot{p} = \frac{p_0^2}{2mr^2} \frac{\cos x}{\sin^3 x} - mgr \sin x, \quad \dot{x} = \frac{p}{mr^2},$$

$$\dot{p}_0 = \mathcal{R}_0, \quad \dot{x}_0 = \frac{p}{mr^2 \sin^2 x}.$$

It follows $R_0 = m\omega r^2 x \sin^2 x$; in this case R_0 in Mechanics is the momentum of the Coriolis force.

The hash line on carte (p, x) are

$$p^2 = 2m^2r^3 \int (r\omega^2 \cos x + g) \sin x dx =$$

$$= m^2r^3 \left(r\omega^2 \cos^2 x - 2g \cos x\right) + const.$$

Remark 1 *There are two main, but contradictory, complains. One of them says that all that is not possible; it would mean that the results are incorrect. The other one says that it is well known, pointing to numerous papers of different authors. The first complain is answered in monograph [10] and in papers [13-15]. For the theory of homogeneous formalism one can consult [16]. But, to make things clearer, let us compare our approach to one of:*

4.1 The pointed papers (For example [12, p. 923-925])

- "(2) Author's approach is based on a doubling of time variable. Thus one introduced "two time" denoted t and τ, playing different roles; the time t plays the role of geometrical time corresponding to the control of the system, while the other τ, which parametrizes the evolution plays the role of a dynamical time."

- "(3):(a) "The state space of a dynamical system with time- dependent constraints and n degrees of freedom is given by a $(2n+1)$-dimensional differentiable manifold W admitting an interesting structure."

- "(b) ...We say that a dynamical system has time-dependent constraints if its state space can be defined by a product canonical manifold (W, x, t, E)."

- "(d) Consider the $2(n+1)$-manifold $\tilde{W} = W \times R$ and let $\pi : \tilde{W} \to W$ to the corresponding projection."

- "(5) (a) Dynamics is determined on the state space by a function $H \in N(W)$, the classical (time-dependent) Hamiltonian of a system. The *Hamiltonian* determines on W a vector field

$$Y_H = E + [\Lambda, E].\text{"} \tag{27}$$

- "Denote by $\{\ ,\ \}$ the Poisson bracken of the state space and introduce a canonical chart $\{q^0 = t, q, p\}$ of domain U of this space. For a motion $c(t)$, the proposition can be translated by

$$\frac{dq^0}{dt}(c(t)), \tag{28}$$

and

$$\frac{dq^\alpha}{dt}(c(t)) = \frac{\partial H}{\partial p_\alpha}(c(t)) = \{H, q^\alpha\}(c(t)),$$
$$\frac{dp_\alpha}{dt}(c(t)) = -\frac{\partial H}{\partial q^\alpha}(c(t)) = \{H, p_\alpha\}(c(t)), \tag{29}$$

or

$$\frac{dq^{\alpha}}{dt}=\frac{\partial H}{\partial p_{\alpha}},\quad \frac{dp_{\alpha}}{dt}=-\frac{\partial H}{\partial q^{\alpha}}."$$

- "(b) Introduce Poisson bracket { , } of the symplectic manifold $\left(\widetilde{W},\widetilde{\Lambda}\right)$."
- "The Hamiltonian vector field of $\left(\widetilde{W},\widetilde{\Lambda}\right)$ corresponding to $(p_0+H)\in N\left(\widetilde{W}\right)$:

$$\widetilde{Y}_H=\left[\widetilde{\Lambda},p_0+H\right],$$

is such that $\pi:\widetilde{Y}_H=Y_H$ admits the component

$$\widetilde{Y}_H^0=-\frac{\partial H}{\partial t}.$$

Therefore, the projection by π of the integral curves of $\widetilde{Y}_H$ are the integral curves of Y_H and Hamilton's equations can be completed by

$$\frac{dp_0}{dt}=-\frac{\partial H}{dt}." \tag{30}$$

- "(c) It can be easily verified that if $u\in N(W)$, its total derivative with respect to t, along the orbits of Y_H, is given by

$$\frac{du}{dt}=\frac{\partial u}{\partial t}+\{H(t),u(t)\}". \tag{31}$$

- "If $\bar{u}\in N\left(\widetilde{W}\right)$, its total derivative with respect to t, along the orbits of $\widetilde{Y}_H$, is given by

$$\frac{d\tilde{u}}{dt}=\{p_0+H,\tilde{u}\} \tag{32}$$

If $\tilde{u}=u$, (32) reduced to (31). Also, if we take $\tilde{u}=p_0+H$, the relation (32) gives

$$p_0+H=const. \tag{33}$$

Thus one can roughly say that, *along the orbits, p_0 is equal to the negative of energy up to an additive constant.*"

4.2 Our approach

- Our approach is based only on time t as independent variable. Time t has physical dimension (dim $t = T$) and plays kinematical time. Our auxiliary coordinate $q^0 = \tau(\aleph, t)$ is a faction of a geometrical or kinematical parameter $\aleph$ and the time, from the equations of time-dependent constraints. The coordinate q^0 can have meaning of angle of phase. It can linearly dependent on time ($q^0 = \aleph t,\ \aleph \in R$) but $q^0 \not\equiv t$.

- The state space of a mechanical system with time-dependent constraints and $n+1$ degrees of freedom is given by a $(2n+2)$-dimensional differentiable manifold $T^*\mathcal{M}$ admitting an symplectic structure.

- We say that state space of a system of material points can be defined by a $(2n+2)$-dimensional canonical cotangent manifold $T^*\mathcal{M}$ if there exist time-dependent constraints.

- Dynamics on the state space is determine by the Hamilton's function
$$H = a^{ij} p_i p_j + V(q) = H(p, q),$$
where $q := q_0, q_1, ..., q_n)$ & $p := (p_0, p_1, p_n) \in T^*\mathcal{M}$, and change constraints force R_0.

- Denote by $\{\ ,\ \}$ the Poisson bracket of the state space (q&p) $T^*\mathcal{M}$. For the motion $p(t)$ & $q(t)$ of a mechanical system can be translated by
$$\frac{dq^i}{dt} = \left\{H, q^i\right\}, \tag{34}$$
$$\frac{dp_i}{dt} = \{H, p_i\} + Q_i, \qquad (i = 0, 1, ..., n), \tag{35}$$
or more explicit
$$\frac{dq^0}{dt} = \frac{\partial H}{\partial p_0} = a^{00} p_0 + Q^{0\beta} p_\beta,$$
$$\frac{dp_0}{dt} = -\{H, p_0\} + R_0 = -\frac{\partial H}{\partial q^0} + R_0, \tag{36}$$

$$\frac{dq^0}{dt} = \{H, q^\alpha\} = \frac{\partial H}{\partial p_\alpha},$$
$$\frac{dp_\alpha}{dt} = \{H, p_\alpha\} = -\frac{\partial H}{\partial q^\alpha}, \quad (\alpha = 1, 2, ..., n). \tag{37}$$

- If $\tilde{u} \in T^*\mathcal{M}$, its total derivative with respect to t in the sense of the equations (34) and (35) is given by

$$\frac{d\tilde{u}}{dt} = \{H, \tilde{u}\} + R_0 \frac{\partial \tilde{u}}{\partial p_0}. \tag{38}$$

At the other hand, the composition of the covector (35) and vector (24) reduces to

$$\frac{\partial H}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial H}{\partial q^i}\frac{dq^i}{dt} = Q_i \frac{dq^i}{dt}, \quad \frac{dH}{dt} = R_0 \frac{dq^0}{dt}. \tag{39}$$

The equation (38) as well as the equation (39) gives

$$H = \int R_0 dq^0 + const. \tag{40}$$

Comment: The relation (33) and (40) clearly illustrate the qualitative difference of the two approaches. In our approach, as it can be seen from (3) and (6) p can never be equal to the negative of energy up to an additive constant. As in Descartes's and Newton's approach, p_0 is generalized momentum (impulse), and Hamilton's function can not linearly depend on p (see, for example, (20), (23) and (26) or [10]).

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Reonomni tangentni i kotangentni snopovi i primene

Posmatra se proširena mnogostrukost $\mathcal{M} = M^n \times x^0$, gde je M Lagranževa konfiguracija n-dimenzionalne mnogostrukosti a $x^0 = \tau(\aleph t)$ je uzeto iz zadatih relacija koje generišu mnogostrukost $\mathcal{M}$. U nekim specijalnim slučajevima može se uzeti da je $x^0 = t$ ali ne i $x^0 \equiv t$. Prošireni tangentni snop ima dimenziju $2n+2$. Metrika na $\mathcal{M}$ je data sa $ds^2 = g_{ij}(x)dx^i dx^j + 2g_{0i}dx^0 dx^i + g_{00}dx^0 dx^0$. Skup svih kovektora $p_i = g_{ij}\dot{x}^j + g_{i0}\dot{x}^0$ i $p_0 = g_{0j}\dot{x}^j + g_{00}\dot{x}^0$ u $x = (x^0, x^1, ..., x^n)$ čini prošireni kotangentni snop simplektičke strukture $T^*\mathcal{M}$. Dodavanje koordinate x^0 u M, $\dot{x}^0$ u TM ili p_0 u T^*M ne predstavlja prosto algebarsko proširenje M, TM i T^*M, već ima geometrijsko i fizičko značenje. Geometrija dobijenih prostora nije razmatrana već je na takvoj geometrijskoj bazi modifikovana standardna Lagranževa i Hamiltonova mehanika, a takodje i Dalamber-Lagranžev princip, Hamiltonov princip i princip najmanje prinude. U ovom pristupu ima više nezavisnih Lagranževih i Hamiltonovih jednačina i više varijacija reonomnih ograničenja ($x_0 \neq 0$, $\delta t = 0$). Poenkareova integralna invarijanta za neautonomne sisteme je uopštena.