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An integral representation of flows near the sound speed

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Abstract

The present paper is concerned with the *Tricomi-equation*

$$\eta\psi_{\theta\theta} + \psi_{\eta\eta} = 0.$$

This differential equation of mixed type is transformed in a formal-hyperbolic equation in the complex plane. The solutions of this equation are calculated by an integral operator. For this we consider a transformation for simplifying the differential equation. The kernel of this transformation can be represented in closed form. The integral operator also provides a way for studying some properties of the solutions. This equation can be solved as well for the subsonic and supersonic zone as for their transonic line $\eta = 0$ which corresponds to the sound speed. Some particular solutions of the *Tricomi-equation* which are already known are special cases of this solution.

1 Introduction

Transitional problems are problems of flow, at which subsonic and supersonic currents appear. Such a problem we have for instance for a wing at subsonic flight near the sound speed because in this case a supersonic zone will appear at the upperside of the wing. Searched are profiles, at which the non-viscose and compressible flow is almost shock free. Plane flows of compressible mediums, which velocities of flow are lower as well as higher than the sound speed can be described by partial differential equations of second order. But it is very difficult to solve such equations because they change their type by passing the *Mach cone*. In the subsonic area they are of elliptical type. In the supersonic region hyperbolic equations must be discussed. It is very difficult to solve such equations. Mostly we need a simplification of them. They become linear if we transfer to the plane of velocity by introducing the components of velocity of flow as independent variables. But we also obtain disadvantages in this transition. Even in the plane of velocity we only can get solutions in special cases and we have to accept further simplifications for solving.

We shall consider one of the simplest examples of differential equations of mixed type. That is the *Tricomi-equation*

$$\eta\psi_{\theta\theta} + \psi_{\eta\eta} = 0, \quad \psi = \psi(\eta, \vartheta), \tag{1}$$

which is of elliptical resp. hyperbolical type for $\eta > 0$ (subsonic zone) resp. $\eta < 0$ (supersonic zone). This equation describes currents which possess velocities near the sound speed. In gasdynamical applications ψ can be the function of the flow and ϑ represents the angle of the flow. η results from a reversible and unique transformation of the velocity of the current. $\psi = const.$ expresses streamlines. Stagnation points correspond to the value $\eta = +\infty$.

2 A derivation of the Tricomi-equation

In this section we give a very simple derivation of (1) by using fundamental equations.

The theorem of conservation of mass implies the equation of continuity

$$\rho_t + div\ (\rho v) = 0,$$

where t is the time, ρ is the density, and v is the velocity. In the stationary case $\rho_t = 0$. If the flow is two-dimensional and irrotational, then a velocity potential ϕ exists. So we obtain by $v = (v_1, v_2) = grad\ \phi$

$$(\rho\phi_{x_1})_{x1} + (\rho\phi_{x_2})_{x_2} = 0.$$

This is the condition of integrability of the system

$$\rho\phi_{x_1} = \psi_{x_2}, \quad \rho\phi_{x_2} = -\psi_{x_1}. \tag{2}$$

If there exist no external forces, then the second Newton law implies that ρ is a function of $q = |v|$. So (2) is not linear. But it can be made linear by using q and $\vartheta = \arctan(v_2/v_1)$. It means that we transfer to the hodographic plane, i.e. $v_1,\ v_2$ - plane. In this way equation (2) becomes

$$\phi_q = \frac{M^2-1}{\rho q}\psi_\vartheta, \quad \phi_\vartheta = \frac{q}{\rho}\psi_q.$$

$M = q/a$ represents the *Mach-number* with $a = \sqrt{dp/d\rho}$ as the local sound speed and p is the pressure. By defining η via $d\eta/dq = -\rho/q$, we transfer from q into η. Furthermore, we eliminate ϕ by differentiation. So we obtain the *Chaplygin-equation*

$$K\psi_{\theta\theta} + \psi_{\eta\eta} = 0 \quad \text{with} \quad K = \frac{(1-M^2)}{\rho^2}.$$

For small values of $|\eta|$, i.e. if η is close to the streamline, the relation $K \sim b\eta$, holds for constant b. By using a suitable linear transformation of independent variables we receive $K \sim \eta$. This implies the Tricomi-equation (1) which can be seen as an approximation equation for the hodograph with flows near the sound speed.

3 The solution of the Tricomi-equation

First of all we want to transform the Tricomi-equation into an elliptical one. Furthermore, this equation will be transformed into a formally hyperbolic equation which possesses a complex form. This equation we shall solve by an integral operator. The real and imaginary part of this integral operator are solutions of (1).

To obtain the elliptical form of (1) we have to employ

$$x = \frac{2}{3}\eta^{3/2}, \quad y = \vartheta, \quad u(x,y) = \eta^{\beta}\psi(\eta,\vartheta), \quad \beta \in \mathbb{R}.$$

If we substitute this into (1) and devide the new equation by $\eta^{\beta-1}$ the Tricomi-equation becomes

$$u_{xx} + u_{yy} + \left(-2\beta + \frac{1}{2}\right)\eta^{-3/2}u_x + \beta(\beta+1)\eta^{-3}u = 0.$$

The requirement $\beta = 1/4$ yields the self adjoint elliptical equation

$$\Delta u + c(x,y)\,u = 0, \quad c = \frac{k}{x^2}, \quad k = \frac{5}{36}. \tag{3}$$

Remark 1 *If we have subsonic resp. supersonic speed the variable x is a real one resp. an imaginary one.*

Now we assume that $c(x,y)$ is any *arbitrary* analytic function of the real variables x, y for $(x,y) \in D \subset \mathbb{R}^2$ (resp. $\mathbb{C}$), where D denotes a subdomain of $\mathbb{R}^2$. This means that for each $(x_o, y_o) \in D$ exists a rectangle $R(x_o, y_o) := \{x_o - h_1 \leq x \leq x_o + h_1,\ y_o - h_2 \leq y_o + h_2\}$ in which the coefficient $c(x,y)$ of equation (3) has a convergent Taylor series expansion: $c(x,y) = \sum_{m\geq 0}\sum_{n\geq 0} c_{m,n}(x - x_o)^m (y - y_o)^n$. However, this means that $c(x,y)$ converges for complex values of x, y contained in the bidisk $(z_o; r) \equiv \left\{ z \,\middle|\, |z - z_o| \leq \sqrt{h_1^2 + h_2^2} \right\}$, i.e., $c(x,y)$ has a holomorphic extension to a complex neighborhood of $D \times \bar{D}$. $\bar{D}$ denotes the mirror domain of D, that is $\bar{D} = \{z \mid \bar{z} \in D\}$. If D is a simple connected domain of $\mathbb{R}^2$ and $c(x,y)$ has a holomorphic extension, then D is called a fundamental domain of (3) (see [8]).

The basic idea for deriving a formally hyperbolic equation from the real elliptic equation (3) with two independent real variables x, y is the following one: We assume x, y to be independent *complex* coordinates, i.e. $x = x_1 + ix_2$, $y = y_1 + iy_2$, $i^2 = -1$. Then we may continue u and c analytically into a complex domain. Now we introduce the new complex variables

$$z = x + iy \quad \text{and} \quad \zeta = x - iy$$

which are independent. Note that $\zeta = \bar{z}$ (the conjugate) if and only if x and y are real variables. Furthermore, we introduce the differential operators of Wirtinger

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \zeta} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

(the Laplace operator $\Delta = 4\partial^2/\partial z\partial\zeta$). The substitution of x, y, $\partial/\partial x$ and $\partial/\partial y$ into (3) implies that this equation becomes

$$Lw = w_{z\zeta} + C(z,\zeta)\, w = 0, \quad w = w(z,\zeta) \tag{4}$$

with

$$C(z,\zeta) = \frac{1}{4}c\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right) = \frac{k}{(z+\zeta)^2}, \quad w(z,\zeta) = u\left(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\right). \tag{5}$$

Note: If the coefficient $c(z,\bar{z})$ is a real function of the real variables x, y then $C(z,\bar{z})$ is real function too.

Further on we will treat $Lw = 0$ in $D \times \bar{D}$. After this extension of (3) into (4) and the solution of (4) in the complex we will make the restriction $\zeta = \bar{z}$ for obtaining the real solutions of (3) resp. (1). If we have the solutions of (4), make the restriction $\zeta = \bar{z}$, we obtain real solutions of (3) by $\mathrm{Re}\{w(z,\bar{z})\}$ and $\mathrm{Im}\{w(z,\bar{z})\}$.

We shall consider the integral operator suggested by M. Eichler [[2]]. It is defined by

$$(Tf)(z,\zeta) := \int_{\Gamma^a}^{z} E(z,\zeta,s)\, f(s)\, ds - f(z), \tag{6}$$

where Γ is an arc in D connecting the arbitrary fixed point a and z. The holomorphic function E is called the kernel, must be determined and depends only on C, whereas the arbitrary holomorphic function $f(z)$ is independent of C and E. By formal substitution of (6) into (4) Eichler has shown that $w(z,\zeta) = Tf$ is a solution of (4) for any arbitrary function $f(z)$ if

$$LE = 0 \text{ for all } s \in \Gamma \text{ on } D \times \bar{D} \times \Gamma \tag{7}$$

and

$$E_\zeta(z,\zeta,s) = C(z,\zeta) \text{ with } s = z. \tag{8}$$

To a given differential equation $Lw = 0$ with arbitrary coefficient C there exist infinitely many kernels $E(z,\zeta,s)$. It is of interest to investigate them, and to determine those which have some interesting properties. But to obtain kernels, i.e. particular solutions of $LE = 0$ (for all $s \in \Gamma$), which satisfy relation (8) is not very easy. The advantage of the integral operator (6) is that we only need *one* particular solution $E \not\equiv 0$ to construct *infinitely* many particular solutions of the form $w = Tf$ of $Lw = 0$. To that end, f has to vary over infinitely many holomorphic functions, for example $f(z) = bz^n$, $b \in \mathbb{C}$, $n = 0, 1, 2, \cdots$.

Remark 2 *A second solution of the differential equation (4) for any arbitrary holomorphic function $C(z,\zeta)$ can be represented by the operator*

$$(T_1 g)(z,\zeta) := \int_{\Lambda\ b}^{\zeta} F(z,\zeta,s)\, g(s)\, ds - g(\zeta).$$

The solution $w = T_1 g$ is independent of $w = Tf$. Here Λ is an arc in $\bar{D}$ which connects the arbitrary fixed point b with ζ. $g(\zeta)$ can be any arbitrary holomorphic function defined in $\bar{D}$. In this case the condition (8) becomes

$$F_z(z,\zeta,s) = C(z,\zeta) \quad \text{with} \quad s = \zeta,$$

whereas the conditional equation (7) stays the same: $LF = 0$ for $s \in \Lambda$ on $D \times \bar{D} \times \Lambda$, e.g. $F \equiv E$. If we search solutions of a differential equation, which is not a self adjoint one, $F \not\equiv E$. But the condition $F_x(z,\zeta,\zeta) = C(z,\zeta)$ does not change. In our case even $E_\zeta(z,\zeta,z) = F_x(z,\zeta,\zeta) = C(z,\zeta)$ holds, because $C(z,\zeta) = C(\zeta,z)$; (see (5)).

The two conditions (7) and (8) for the kernel become much simpler, if we set

$$E(z,\zeta,s) = \frac{2}{z+\zeta}\frac{dp(r)}{dr}, \quad r = \frac{2s-z+\zeta}{z+\zeta}.$$

By substituting this into (7) this conditional equation becomes the ordinary differential equation

$$\left(1-r^2\right)p''' - 4rp'' - (2+k)\,p' = 0.$$

The integration of this equation yields that $p(r)$ has to fulfill the Legendre-equation

$$\left(1-r^2\right)p'' - 2rp' + n\,(n+1)\,p = 0 \quad \text{with} \quad n = -1/6. \tag{9}$$

So $p(r)$ represents the Legendre-function of the first kind $P_{-1/6}(r)$. Equation (8) with $C(z,\zeta) = k/(z+\zeta)^2$ (see (5)) simplifies to the boundary condition

$$p'(1) = -\frac{k}{2} = -\frac{5}{72}. \tag{10}$$

Observe, that the use of the integral operator T_1 with the equations $LF = 0$ and $F_z(z,\zeta,\zeta) = k(z+\zeta)^{-2}$ for the kernel F, the transformation $F(z,\zeta,s) = 2(z+\zeta)^{-1}p'(r)$ but with the variable $r = (2s+z-\zeta)(z+\zeta)^{-1}$ also implies equations (9) and (10).

Remark 3 *The Legendre-function of the 2. kind $Q_{-1/6}(r)$ is also a solution of (9) but it has a singularity in $r = 1$. So it does not satisfy the condition equation (10).*

With the kernel $E(z,\zeta,r) = 2P'_{-1/6}(r)\,/\,(z+\zeta)$ the representation of solutions of (4) is given by

$$w(z,\zeta) = \frac{2}{z+\zeta}\int_{\Gamma\, a}^{z} \frac{d}{ds}P_{-\frac{1}{6}}(s)\,ds - f(z). \tag{11}$$

Now we introduce the new variable of integration t

$$s = \frac{1}{2}(z+\zeta)\,t + \frac{1}{2}(z-\zeta) \quad \text{with real} \quad t : 1 \leq t \leq \infty.$$

If $a \to \infty$ - then the generating function $P'_{-1/6}(r) \to 0$ with $a \to \infty$ and so also the kernel E vanishes - our integral (11) exists with $a = \infty$ by demanding for very great s

$$|f(s)| < c_1 |s|^{\frac{1}{6}-\varepsilon}, \quad \varepsilon > 0, \quad c_1 > 0. \tag{12}$$

After integration by parts and interchanging the limits of the integral, (11) becomes

$$w(z,\zeta) = \int_1^\infty P_{-\frac{1}{6}}(t)\,\frac{d}{dt} f\left(\frac{1}{2}(z+\zeta)\,t + \frac{1}{2}(z-\zeta)\right) dt. \tag{13}$$

So a solution of the Tricomi-equation can be represented by

$$\psi(\eta,\vartheta) = \eta^{-\frac{1}{4}} w(z,\bar{z}) = \left(\frac{3}{2}x\right)^{-\frac{1}{6}} \int_1^\infty P_{-\frac{1}{6}}(t)\,\frac{d}{dt} f(tx+iy)\,dt \tag{14}$$

$$\text{with} \quad x = \frac{2}{3}\eta^{3/2}, \quad y = \vartheta.$$

resp. by real- and imaginary part of (14). Substitution of different functions f yields single particular solutions of (1).

Henceforth we denote by

$$\psi = \psi\,[f(z)],$$

a shorter manner of writing for formula (14).

If $f(z)$ generates ψ then (14) implies that ψ_ϑ can be constructed by $if'(z)$. That means

$$\frac{\partial}{\partial\vartheta}\psi\,[f(z)] = \psi\,[if'(z)],$$

where the sign $'$ means the derivation with respect to $tx + i\vartheta$. Notice, that the behavior of the decay of $|f(z)|$ which we demanded in (12) improves by the differentiation. If $\psi = \psi[f(z)]$ is a solution of $Lw = 0$, then

$$\psi = \psi[f(\zeta)],$$

also satisfies (4); (see *Remark 2*). This can be deduced from the symmetry of $Lw = 0$ in z and ζ which is equivalent to the fact, that the equations (1) and (3) don't change if we replace ϑ by $-\vartheta$.

The Tricomi-equation (1) only contains ϑ in the derivation but does not possess coefficients which are functions of ϑ. So we can imply that with the holomorphic function $f(z)$ also with $f(z + z_o)$, $z_o = constant$, a solution

$$\psi = \psi[f(z + z_o)],$$

can be generated.

Remark 4 *We also can represent solutions of $Lw = 0$ if we use the integral operator of St. Bergman [[1]], which is defined by*

$$(Bf)(z,\zeta) := \int_{\Upsilon -1}^{1} e(z,\zeta,s)\, f\left(\frac{z}{2}\left(1-s^2\right)\right)\frac{ds}{\sqrt{1-s^2}},$$

where the rectifiable path of integration Υ joining $s = -1$ to $s = 1$ lies in the disk $D_o = \{s \,|\, |s| \le 1\} \subset \mathbb{C}$ and usually omits the origin. (If $s^{-1}e_\zeta$ is continuous for $s = 0$ then Υ can pass through $s = 0$). $f(z)$ is an arbitrary holomorphic function in D, independent of e and the coefficients of $Lw = 0$. Bergman has shown that $w = Bf$ is a solution of $Lw = 0$. Bergman has shown that $w = Bf$ is a solution of $Lw = 0$ if the kernel e satisfies the equation

$$\left(1-s^2\right)e_{\zeta s} - \frac{1}{s}e_\zeta + 2zsLe = 0,$$

and fulfills the conditions

$$(I): \left(1-s^2\right)^{1/2} e_\zeta \to 0 \quad as \quad s \to \pm 1 \quad (uniformly\ on\ D \times \bar{D}),$$

and

$$(II): \; z^{-1}s^{-1}e_\zeta \quad is\ a\ continuous\ function\ on\ D \times \bar{D} \times \Upsilon.$$

If we set $X^2 = s^2 z / (z + \zeta)$ we can calculate e and obtain the closed forms

$$\sqrt{1 - X^2} e(z, \zeta, s) = C_1 \left(X + \sqrt{X^2 - 1}\right)^{2\alpha - 1} + C_2 \left(X - \sqrt{X^2 - 1}\right)^{2\alpha - 1}$$
$$\text{if} \quad 2\alpha \neq 1,$$

$$\sqrt{1 - X^2} e(z, \zeta, s) = C_1 + C_2 \arcsin x \quad \text{if} \quad 2\alpha = 1.$$

This Bergman operator can be prefered if 2α is an integer (incompressible axissymmetric flows and problems of the torsion) because in the case of $2\alpha \neq 1$ we obtain the *Tschebyschew-polynomials*

$$\sqrt{1 - X^2} e(z, \zeta, s) = A_1 T^*_{2\alpha - 1}(X) + A_2 U_{2\alpha - 1}(X).$$

4 The solution $\psi(\eta, \vartheta)$ on the sonic line $\eta = 0$

Of special interest in connection with partial differential equations of mixed type is the case where the function values of the equation are searched on the *transonic line*, i.e., the curve which separates the part of the plane in which the equation is elliptic from this domain in which the equation is hyperbolic. In the Tricomi-equation the transition line is given by $\eta = 0$. This line corresponds to the sound speed. For the moment we only obtain undetermined values of (14) since this formula has a singularity at $\eta = 0$. Eichler [2] investigated the behaviour of axissymmetric potential functions at their singularities. Using this method we obtain (see also [1], p. 121)

$$\psi(0, \vartheta) = \frac{\Gamma\left(\frac{1}{3}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{5}{6}\right)} \int_0^\infty f'(\tau + iy) \frac{d\tau}{(3\tau)^{\frac{1}{12}}} \quad \text{with} \quad y = \vartheta.$$

In a similar way we get the values of the derivation on the sonic line

$$\psi_\eta(0, \vartheta) = \frac{\Gamma\left(\frac{1}{3}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{1}{6}\right)} \int_0^\infty f'(\tau + iy) \frac{d\tau}{(3\tau)^{\frac{5}{12}}} \quad \text{with} \quad y = \vartheta.$$

In these two relations Γ represents the *Gamma Function*, also called the *Euler's Integral of 2. Kind*, which is defined by $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$, $\mathrm{Re}\{z\} > 0$. With these results we can continue the solutions ψ from the elliptic domain resp. the subsonic region $\eta > 0$ to the hyperbolic region which presents the supersonic zone $\eta < 0$.

5 Several particular solutions

The result (13) resp. (14) is significant in two different ways: First, by substituting different functions f we obtain particular solutions of the Tricomi-equation which already were considered by *Carrier, Chaplygin, Darboux, Ehlers, Guderley, Tamada* and *Tomotika*.

Remark 5 *But these man had found* **real** *solutions valid in* **one** *domain, the elliptical or hyperbolic one. We obtained new solutions, because our integral operator (11) resp. (13) holds in the elliptical and hyperbolic domain, whereas* $\psi(0, \vartheta)$ *and* $\psi_\eta(0, \vartheta)$ *hold in the parabolic zone. So we have found a solution for all the domain of definition* $D \times \bar{D}$ *of the differential equation. It is most remarkable that the operator approach permits us to derive all these solutions (and infinitely others) from a unique source, namely, from representation (11). This illustrates the unifying power of the theory of integral operators defined on spaces of complex analytic functions.*

Secondly, Eichler's integral operator provides a way of applying function theoretical methods and results to the study of those solutions. The rate depends on the choice and construction of suitable kernels. And we obtained a simple one. The use of special classes of functions f, for instance polynomials, meromorphic or algebraic functions, etc. delivers the investigation of general properties of solutions. This second significance is also valid for the Bergman operator (see *Remark 4*) and other integral operators (see [7]).

From formula (14) we can now derive sets of particular solutions of the Tricomi equation by choosing various classes of functions f. Let us show this for some cases of practical interest. We first choose

$$f_n(z) = c_0 z^{(4n+1)/6}, \quad c_0 \neq 0, \quad n < 0, \quad n \neq -\frac{1}{4}.$$

We now set

$$\xi = -\frac{\vartheta^2}{x^2}, \quad x = \frac{2}{3}\eta^{3/2}.$$

Then relation (14) yields

$$\psi_n(\eta, \vartheta) = \psi[f(z)] = \eta^n h_n(\xi),$$

$$h_n(\xi) = c_{1n} \int_1^\infty P_{-1/6}(t) \left(t + \sqrt{\xi}\right)^{(4n-5)/6} dt, \tag{15}$$

$$c_{1n} = (4n+1) \tfrac{1}{6} \left(\tfrac{2}{3}\right)^{(4n+1)/6} c_0.$$

h_n is complex valued and such that $h_{n1} = \mathrm{Re}\{h_n\}$ and $h_{n2} = \mathrm{Im}\{h_n\}$ constitute a basis of solutions of hypergeometric equation

$$\xi(1-\xi)h'' + \left(\frac{1}{2} + \frac{4-2n}{3}\xi\right)h' - \frac{n(n-1)}{9}h = 0.$$

These ψ_n in (15) are *Guderley's solutions* [3]. Note that we have obtained an *integral representation* of these solutions. This again illustrates the fact that the method of integral operators can produce new results in the *theory of special functions.*

If $n = -1/4$ then $f(z) = const.$ and $\psi \equiv 0$. This trivial solution corresponds to the stagnation point $(\eta = +\infty)$ which can be seen by (14). But this gap can be closed if we chose

$$f_{-1/4} = c_0 \log z, \quad c_0 \neq 0.$$

Then we obtain from (14), a representation similar to (15), namely

$$\psi_{-1/4}(\eta, \vartheta) = \psi[c_0 \log z] = \eta^{-1/4} h_{-1/4}(\xi),$$

$$h_{-1/4}(\xi) = c_0 \int_1^\infty P_{-1/6}(t) \left(t + \sqrt{\xi}\right)^1 dt = 2\pi c_0 P_{-1/6}\left(\sqrt{\xi}\right).$$

It is known that Guderley's solutions can be transformed into *Darboux's solutions.* Hence the latter may also be obtained directly from formula (14).

Similarly, the *Tomotika-Tamada solutions* can be obtained by slight changes in f_n and $f_{-1/4}$

$$f_n(z) = c_0 (z - \vartheta_0)^{(4n+1)/6}, \; n < 0, \, n \neq -\frac{1}{4}, \quad \text{resp.} \quad f(z) = c_0 \log (z - \vartheta_0),$$

where $\vartheta_0 = const.$

Finally, the solutions of *Tschaplygin* result from the functions

$$f_m(z) = c_0 \exp\left(-\frac{m\pi}{\vartheta_0} z\right), \quad \mathrm{Re}\left\{\frac{m}{\vartheta_0} z\right\} \geq 0.$$

It is interesting that for this choice of f, the integral (14) represents essentially a modified Bessel function

$$K_\nu(z) = \frac{\pi}{2 \sin \nu\pi}\left[e^{i\nu\pi/2} J_{-\nu}(iz) - e^{-i\nu\pi/2} J_\nu(iz)\right],$$

of order $\nu = 1/3$. Indeed, we have ([6], p. 85)

$$\int_1^\infty e^{-zt} P_{\nu-1/2}(t)\, dt = \sqrt{\frac{2}{\pi z}} K_\nu(z), \qquad \mathrm{Re}\{z\} > 0.$$

Substituting $f = f_m$ into (14) yields the result

$$\psi_m(\eta, \vartheta) = c_2 \eta^{1/2} \exp\left(-i\frac{m\pi}{\vartheta_0}\vartheta\right) K_{1/3}\left(\frac{2m\pi}{3\vartheta_0}\eta^{3/2}\right),$$

where

$$c_2 = -2c_0 \left(\frac{m}{3\vartheta_0}\right)^{1/2}.$$

This Chapligin solutions have the important property that they vanish at infinity, as is needed in various applications.

To gain full appreciation of present results, we should recall that in transonic flow, almost all specific results have been obtained by the so-called *indirect method* or *inverse method*, that is, by searching large classes of special solutions for members that may correspond to practically relevant flows. On the other hand, this accounts for the importance of special solutions and explains why they have been investigated in great detail. On the other hand it shows that *integral operators are particularly suitable for the indirect method*, for the simple reason that, by taking a sufficiently large class holomorphic functions we may readily obtain a large supply of special solutions having similar properties.

We finally want to mention that the indirect method and its variants are also useful in elasticity theory.

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Integralna prezentacija strujanja u okolini brzine zvuka

Rad se bavi Trikomijevom jednačinom

$$\eta\psi_{\theta\theta} + \psi_{\eta\eta} = 0.$$

Ova diferencijalna jednačina mešovitog tipa transformisana je u formalno hiperboličnu jednačinu. Rešenja jednačine odredjena su pomoću integralnog operatora. Za to je razmatrana transformacija koja pojednostavljuje diferencijalnu jednačinu. Jezgro te transformacije se može izraziti u zatvorenoj formi. Integralni operator se takodje može koristiti i za ispitivanje svojstava rešenja. Jednačina se može rešiti kako za subsonične tako i za supersonične zone, a najzad i za transsoničnu liniju $\eta = 0$, koja odgovara brzini zvuka. Neka partikularna rešenja Trikomijeve jednačine koja su već poznata su specijalni slučajevi ovde dobijenog rešenja.