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A generalized Greenhill problem with shear deformation, compressibility and imperfections

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Abstract

The classical Greenhill problem of a column loaded by a force and a couple is treated in a generalized version. Namely we assume that, besides a concentrated force and couple the rod is loaded by a small concentrated load at its upper end. It is assumed that the rod has Haringx's type of constitutive equation that allows for the influence of both shear deformation and axial compressibility.

1 Introduction

Consider an elastic rod, fixed at the lower end **O** and fixed to a rigid movable plate at the point **K**. The plate can freely slide in a plane parallel to the $O\ \bar{y}\ \bar{z}$ - plane. Suppose that the rod is loaded by a compressive force and a couple at its end **K** (see Fig. 1).

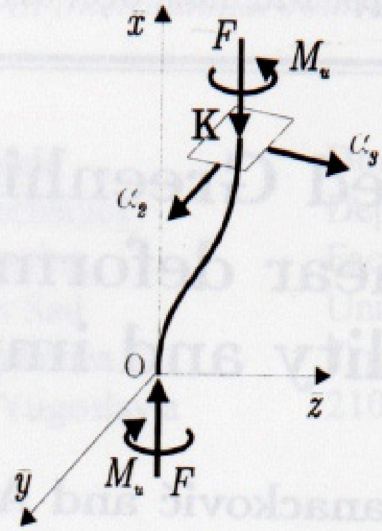

Fig. 1. Coordinate system and load configuration.

We assume that the rod is initially straight so that its axis coincides with the $\bar{x}$ - axis of a rectangular Cartesian coordinate system $O\ \bar{x}\ \bar{y}\ \bar{z}$. The action line of the compressive force F is always parallel with the $\bar{x}$ - axis and the couple of intensity M_u acts in a plane perpendicular to the $\bar{x}$ - axis.

The problem of determining the values of the load (F, M_u) so that the rod loses stability, i.e., it has a nontrivial equilibrium configuration in which its axis is not straight, constitutes the so called Greenhill problem, [1]. The problem and its generalizations have been treated by many authors. We mention works of Beck [2], Antman and Kenny [3], Atanackovic [4], Biezeno and Grammel [5] and Atanackovic and Glavardanov [6].

Our intention in this note is to use constitutive equations of the Haringx's type to analyse the generalised Greenhill problem with load imperfections. The constitutive equations that we use are linear in the selected strain measures. This will allow us to obtain expressions for the critical load in closed form. The imperfections consist in a small concentrated force acting at the point **K** and having action line in a plane perpendicular to the $\bar{x}$ - axis. To describe the configuration of the rod in the deformed state we shall use a special type of Euler angles the so called ship angles (nautical angles). They have important property that for small deformation the transformation matrix connecting quantities in initial and deformed configuration is nonsingular, [4].

2 Equilibrium equations

We begin with the problem of describing the deformation of the rod. In general the axis of the rod in the natural state is a space curve of length L that we denote by C_0. The position vector of an arbitrary point on C_0 is denoted by $\mathbf{r}_0 = \mathbf{r}_0(S)$ where $S \in [0, L]$ is the arc-length of C_0. With each point of the rod in the natural state we associate the coordinate system $\bar{x}_{10}$, $\bar{x}_{20}$ and $\bar{x}_{30}$ oriented along the tangent of rod axis and along the principal directions of the cross-section, respectively. The unit vectors along the axis $\bar{x}_{10}$, $\bar{x}_{20}$ and $\bar{x}_{30}$ are denoted by $\mathbf{e}_{10}$, $\mathbf{e}_{20}$ and $\mathbf{e}_{30}$, respectively and may be taken as directors. In the deformed configuration the position vector of the point that was at $\mathbf{r}_0$ in the natural state is $\mathbf{r} = \mathbf{r}(S)$. Let $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$ be the axes oriented along the normal to the cross-section and along the principal directions of the cross-section in the deformed state, respectively. We assume that $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$ form an orthogonal coordinate system. Thus, the model that we describe falls into the class of constrained Cosserat models. Let $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ be the unit vectors oriented along the axes $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$, respectively (directors in the deformed state). To describe the orientation of the system $\bar{x}_{10}$, $\bar{x}_{20}$, $\bar{x}_{30}$ or $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$ with respect to fixed coordinate system $O\,\bar{x}\,\bar{y}\,\bar{z}$ we use the special type of Euler angles the so-called ship angles [4]. For example the system of axes $\bar{x}$, $\bar{y}$ and $\bar{z}$ could be transformed into $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$ by three rotations. The sequence of rotation starts with the rotation of amount ϑ_1 about $\bar{x}$ - axis. The resulting system is $\bar{x}$, η, ζ. The next rotation is about ζ - axis over an angle ϑ_3 (see Fig. 2). Finally, the rotation for amount ϑ_2 is performed about axis $\bar{x}_2$. All rotations are performed right-handed. With these rotations the "angular velocity vector" denoted by ω becomes

$$\boldsymbol{\omega} = \vartheta_1'\mathbf{i} + \vartheta_3'\mathbf{k}_1 + \vartheta_2'\mathbf{e}_2, \tag{1}$$

where $(\cdot)' = d(\cdot)/dS$ and $\mathbf{k}_3$ is the unit vector along the ζ - axis. By using simple trigonometric relations we obtain the following expressions for the components of $\boldsymbol{\omega}$ in the coordinate system with unit vectors

$\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$

$$\omega_1 = \vartheta_1' \cos\vartheta_2 \cos\vartheta_3 - \vartheta_3' \sin\vartheta_2,$$

$$\omega_2 = \vartheta_2 - \vartheta_1' \sin\vartheta_3, \tag{2}$$

$$\omega_3 = \vartheta_1' \cos\vartheta_3 \sin\vartheta_2 + \vartheta_3' \cos\vartheta_2.$$

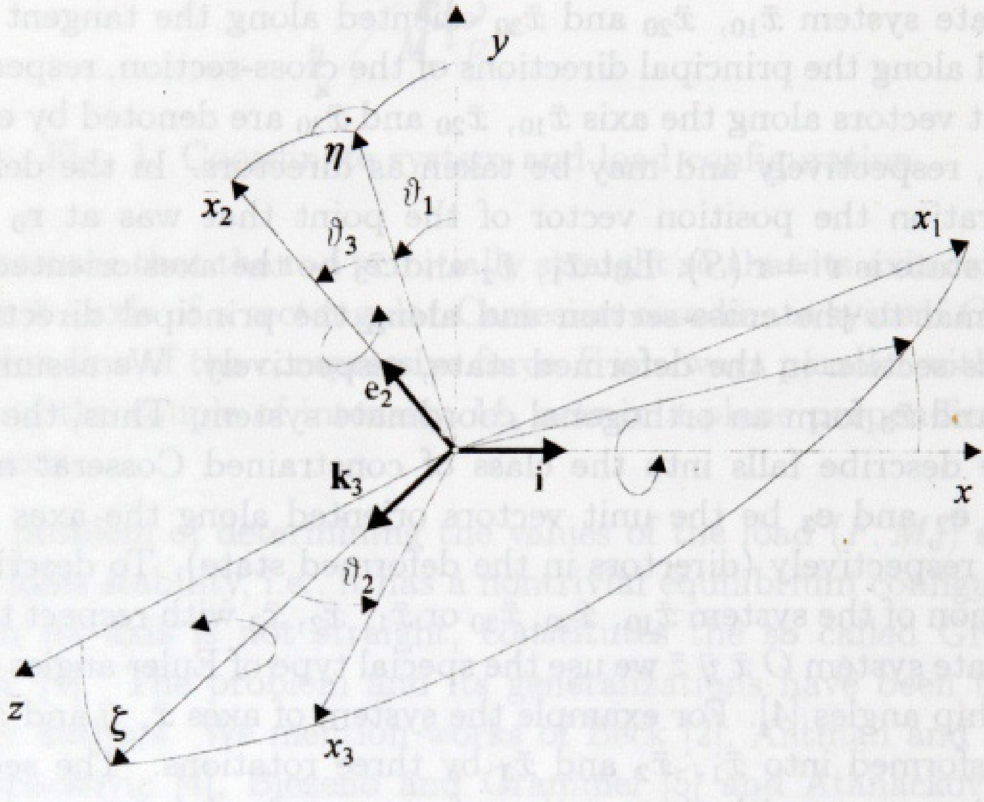

Fig. 2. Ship angles and rotation sequence.

Finally, we give the transformation matrix $\mathbf{U}$ connecting the components of an arbitrary vector $\mathbf{c}$ in the coordinate systems $\bar{x}$, $\bar{y}$, $\bar{z}$ and $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$. If the vector $\mathbf{c}$ is given in both coordinate systems as $\mathbf{c} = \bar{c}_1\mathbf{i} + \bar{c}_2\mathbf{j} + \bar{c}_3\mathbf{k} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3$, then $\bar{c}_i$ and c_i are connected by the matrix whose components are direction cosines, $u_{11} = \mathbf{e}_1 \cdot \mathbf{i}$, $u_{12} = \mathbf{e}_1 \cdot \mathbf{j}$, ..., $u_{33} = \mathbf{e}_3 \cdot \mathbf{k}$. Therefore, we have

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix} \begin{bmatrix} \bar{c}_1 \\ \bar{c}_2 \\ \bar{c}_3 \end{bmatrix}. \tag{3}$$

From Fig. 2 it follows that

$$\mathbf{U}(\vartheta_1, \vartheta_2, \vartheta_3) = \begin{pmatrix} \cos\vartheta_2\cos\vartheta_3 & \begin{matrix}\cos\vartheta_1\cos\vartheta_2\sin\vartheta_3 \\ +\sin\vartheta_2\sin\vartheta_1\end{matrix} & \begin{matrix}\cos\vartheta_2\sin\vartheta_1\sin\vartheta_3 \\ -\cos\vartheta_1\sin\vartheta_2\end{matrix} \\ -\sin\vartheta_3 & \cos\vartheta_1\cos\vartheta_3 & \sin\vartheta_1\cos\vartheta_3 \\ \cos\vartheta_3\sin\vartheta_2 & \begin{matrix}\cos\vartheta_1\sin\vartheta_3\sin\vartheta_2 \\ -\sin\vartheta_1\cos\vartheta_2\end{matrix} & \begin{matrix}\sin\vartheta_1\sin\vartheta_2\sin\vartheta_3 \\ +\cos\vartheta_1\cos\vartheta_2\end{matrix} \end{pmatrix}. \tag{4}$$

Note the $\mathbf{U}$ is *regular* around the identity i.e., for small ϑ_1, ϑ_2 and ϑ_3. For the case when the system $\bar{x}$, $\bar{y}$, $\bar{z}$ is transformed into $\bar{x}_{10}$, $\bar{x}_{20}$ and $\bar{x}_{30}$ we use $\boldsymbol{\omega}_0$ to denote the "angular velocity" determined by the same system as (2) with ϑ_i replaced with ϑ_{0i}, where ϑ_{0i} denote the ship angles that bring $\bar{x}$, $\bar{y}$ and $\bar{z}$ to $\bar{x}_{10}$, $\bar{x}_{20}$ and $\bar{x}_{30}$. The unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}_{10}$, $\mathbf{e}_{20}$, $\mathbf{e}_{30}$ are connected through a one-parameter family (the parameter is S) of orthogonal transformations Λ as

$$\mathbf{e}_i = \mathbf{\Lambda}\mathbf{e}_{i0}, \quad \mathbf{e}_{i0} = \mathbf{\Lambda}^T\mathbf{e}_i. \tag{5}$$

The transformation $\mathbf{\Lambda}$ could be written in the direct notation as

$$\mathbf{\Lambda} = \mathbf{e}_k \otimes \mathbf{e}_{k0}. \tag{6}$$

where $\otimes$ denotes the tensor product. The elements λ_{ij} (when $\mathbf{\Lambda}$ is expressed in a specific coordinate system) of the matrix $\mathbf{\Lambda}$ could be expressed in terms of Euler angles connecting $\bar{x}_{10}$, $\bar{x}_{20}$, $\bar{x}_{30}$ and $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$. The deformed configuration of the rod is known if the radius vector $\mathbf{r} = \mathbf{r}(S)$ and the orthogonal transformation matrix $\mathbf{\Lambda}(S)$ are known. The vectors $\boldsymbol{\omega}_0$ and $\boldsymbol{\omega}$ are taken to define the first measure of deformation of the rod in the initial and deformed configurations. With the transformation matrix $\mathbf{\Lambda}$ introduced by (5) the first strain measure is, see [4], [6]

$$\mathbf{\Omega} = \boldsymbol{\omega} - \hat{\boldsymbol{\omega}}_0 = \boldsymbol{\omega} - \mathbf{\Lambda}\boldsymbol{\omega}_0 = \Omega_i\mathbf{e}_i. \tag{7}$$

The second (vector) strain measure $\mathbf{\Gamma}$ is defined as

$$\mathbf{\Gamma} = \frac{d\mathbf{r}(S)}{dS} - \mathbf{\Lambda}\frac{d\mathbf{r}_0}{dS} = \mathbf{r}' - \mathbf{\Lambda}\mathbf{e}_{10} = \mathbf{r}' - \mathbf{e}_1 = \Gamma_i\mathbf{e}_i. \tag{8}$$

If $\mathbf{\Gamma} = 0$, the equation (8) gives

$$\mathbf{r}' = \mathbf{\Lambda e}_{10} = \mathbf{e}_1, \tag{9}$$

i.e., the rod axis is inextensible, since $|\mathbf{r}'| = |\mathbf{\Lambda e}_{10}| = |ds/dS| = 1$ and the tangent to the rod axis in the deformed state is orthogonal to the cross-section in the deformed state (classical Kirchhoff theory). Since the two vectors $\mathbf{\Omega}$ and $\mathbf{\Gamma}$ are taken as the strain measures those two vectors are the arguments in the constitutive equations. We take the simplest possible form of constitutive equations that are linear in $\mathbf{\Omega}$ and $\mathbf{\Gamma}$. This theory was proposed by Haringx [8] and used by Elyseyev [7] for dynamics of rods; Atanackovic and Spasic [9] for heavy column; Atanackovic and Glavardanov [6] for perfect Greenhill problem and Guran and Atanackovic for fluid conveying pipe, [16]. Thus, we assume that the constitutive equations have the form

$$\begin{aligned} \mathbf{R} &= B_{11}\Gamma_1\mathbf{e}_1 + B_{22}\Gamma_2\mathbf{e}_2 + B_{33}\Gamma_3\mathbf{e}_3; \\ \mathbf{M} &= A_{11}\Omega_1\mathbf{e}_1 + A_{22}\Omega_2\mathbf{e}_2 + A_{33}\Omega_3\mathbf{e}_3, \end{aligned} \tag{10}$$

where $B_{11}, ...A_{33}$ are constants. Sometimes (see Rogers [10]) the vector $\hat{\mathbf{\Gamma}} = \mathbf{\Gamma} + \mathbf{e}_1$instead of $\mathbf{\Gamma}$ is taken as a strain measure.

Finally we write equilibrium equations. In vector form they are

$$\mathbf{R}' = -\mathbf{q}; \qquad \mathbf{M}' = -(\mathbf{\Gamma} + \mathbf{e}_1) \times \mathbf{R} - \mathbf{m}, \tag{11}$$

where $\mathbf{R}$ is the contact force, $\mathbf{M}$ is the contact couple (representing the influence of the part $[0, S)$ of the rod on the part $[S, L]$) and $\mathbf{q}$ and $\mathbf{m}$ are the intensities of the distributed forces and couples (per unit length of the rod axis in the undeformed state), respectively. The complete set of equations describing an elastic rod with shear and compressibility follows from equilibrium geometrical and constitutive equations. We write the complete set of equations in the local coordinate system $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$. Then we have, see [4]:

1. Equilibrium equations (11)

$$
\begin{aligned}
&R_1' + R_3\omega_2 - R_2\omega_3 = -q_1,\\
&R_2' + R_1\omega_3 - R_3\omega_1 = -q_2,\\
&R_3' + R_2\omega_1 - R_1\omega_2 = -q_3,\\
&M_1' + M_3\omega_2 - M_2\omega_3 = R_2\Gamma_3 - R_3\Gamma_2 - m_1,\\
&M_2' + M_1\omega_3 - M_3\omega_1 = R_3\Gamma_1 - R_1\Gamma_3 + R_3 - m_2,\\
&M_3' + M_2\omega_1 - M_1\omega_2 = R_1\Gamma_2 - R_2\Gamma_1 - R_2 - m_3.
\end{aligned}
\tag{12}
$$

2. "Strain-displacement" relations ((2) and (8))

$$
\begin{aligned}
&\omega_1 = \vartheta_1' \cos\vartheta_2 \cos\vartheta_3 - \vartheta_3' \sin\vartheta_2,\\
&\omega_2 = \vartheta_2' - \vartheta_1' \sin\vartheta_3,\\
&\omega_3 = \vartheta_1' \cos\vartheta_3 \sin\vartheta_2 + \vartheta_3' \cos\vartheta_2,\\
&x_1' + (\omega_2 x_3 - \omega_3 x_2) - 1 = \Gamma_1,\\
&x_2' + (\omega_3 x_1 - \omega_1 x_3) = \Gamma_2,\\
&x_3' + (\omega_1 x_2 - \omega_2 x_1) = \Gamma_3.
\end{aligned}
\tag{13}
$$

3. Constitutive equations (10)

$$
\begin{aligned}
&R_1 = B_{11}\Gamma_1, \qquad R_2 = B_{22}\Gamma_2, \qquad R_3 = B_{33}\Gamma_3,\\
&M_1 = A_{11}(\omega_1 - \omega_{10}), \qquad M_2 = A_{22}(\omega_2 - \omega_{20}),\\
&M_3 = A_{33}(\omega_3 - \omega_{30}).
\end{aligned}
\tag{14}
$$

The components of ω_0 are known from the equations describing the natural state of the rod. Then, there are the following eighteen unknowns: six couple force variables ($M_i, i = 1,2,3$ and $R_i, i = 1,2,3$), six geometrical variables (three components of the position vector $\mathbf{r}$ and three Euler type angles) and six strains (three components of $\mathbf{\Omega}$ and three components of $\mathbf{\Gamma}$). These eighteen quantities must satisfy the following eighteen equations: six scalar equations of equilibrium (12), six equations defining strain measures in terms of geometrical quantities (13) and six constitutive equations (14).

When compared with the classical Kirchhoff theory the difference is in the fact that now both $\mathbf{R}$ and $\mathbf{M}$ are constitutive quantities. The Kirchhoff theory follows from (14) if we assume that the extensional B_{11} and shear $B_{22}$and B_{33} rigidities tend to infinity so that $\mathbf{\Gamma} = 0$.

The equations (12)-(14) describe the general spatial deformation of an elastic rod with the constitutive equations of the form (14). We turn now to the special case corresponding to a generalised Greenhill problem with the boundary conditions shown in Fig. 1.

3 A rod with force at the boundary as an imperfection

Consider the rod shown in Fig. 1. Let $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ be the unit vectors along the $\bar{x}$, $\bar{y}$ and $\bar{z}$ axis respectively. The distributed forces, the vector $\boldsymbol{\omega}_0$ and the distributed couples are assumed in the form

$$\mathbf{q} = 0; \qquad \boldsymbol{\omega}_0 = 0; \qquad \mathbf{m} = 0. \tag{15}$$

The equilibrium equations (12) now become

$$R_1' + R_3\omega_2 - R_2\omega_3 = 0,$$

$$R_2' + R_1\omega_3 - R_3\omega_1 = 0,$$

$$R_3' + R_2\omega_1 - R_1\omega_2 = 0,$$

$$
\begin{aligned}
A_{11}\omega_1' + (A_{33} - A_{22})\,\omega_3\omega_2 &= \left(\frac{1}{B_{33}} - \frac{1}{B_{22}}\right) R_2R_3,\\
A_{22}\omega_2' + (A_{11} - A_{33})\,\omega_1\omega_3 &= \left(\frac{1}{B_{11}} - \frac{1}{B_{33}}\right) R_1R_3 + R_3,\\
A_{33}\omega_3' + (A_{22} - A_{11})\,\omega_1\omega_2 &= \left(\frac{1}{B_{22}} - \frac{1}{B_{11}}\right) R_1R_2 - R_2.
\end{aligned}
\tag{16}
$$

The "stress-strain" relations could be written as

$$
\begin{aligned}
\vartheta_1' &= \frac{c_2}{c_3}\omega_1 + \frac{s_2}{c_3}\omega_3,\\
\vartheta_2' &= \frac{c_2s_3}{c_3}\omega_1 + \omega_2 + \frac{s_2s_3}{c_3}\omega_3,\\
\vartheta_3' &= -s_2\omega_1 + c_2\omega_3,\\
x' &= c_2c_3\left(1 + \frac{R_1}{B_{11}}\right) - s_3\frac{R_2}{B_{22}} + c_3s_2\frac{R_3}{B_{33}},\\
y' &= (c_1c_2s_3 + s_1s_2)\left(1 + \frac{R_1}{B_{11}}\right) + c_1c_3\frac{R_2}{B_{22}} +\\
&\quad + (c_1s_2s_3 - c_2s_1)\frac{R_3}{B_{33}},\\
z' &= (c_2s_1s_3 - c_1s_2)\left(1 + \frac{R_1}{B_{11}}\right) + s_1c_3\frac{R_2}{B_{22}} +\\
&\quad + (s_1s_2s_3 + c_1c_2)\frac{R_3}{B_{33}},
\end{aligned}
\tag{17}
$$

where $\mathbf{r} = \mathbf{r}(S) = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ is the position vector of a generic point in the deformed configuration and $c_1 = \cos\vartheta_1,\ s_1 = \sin\vartheta_1, ...,\ s_3 = \sin\vartheta_3$

The boundary conditions corresponding to (16), (17) are

$$\begin{gathered} R_1(L) = -F; \quad R_2(L) = \alpha_2, \quad R_3(L) = \alpha_3, \\ A_{11}\omega_1(L) = M_u, \quad \vartheta_2(L) = 0, \quad \vartheta_3(L) = 0, \\ \vartheta_1(0) = 0, \quad \vartheta_2(0) = 0, \quad \vartheta_3(0) = 0, \\ x(0) = 0, \quad y(0) = 0, \quad z(0) = 0. \end{gathered} \tag{18}$$

In $(18)_{2,3}$ we used α_2 and α_3 to denote components of the concentrated force representing load imperfection. The forces α_2 and α_3 act on the rigid plate at the point **K** so that as the plate rotates during the deformation the forces change their action lines to keep the same orientation with respect to plate. Note also that for the case of circular cross-section we have $A_{22} = A_{33}$ while A_{11} could be obtained from St. Venant's solution of the torsion problem, [10]. The system (16), (17) we write in non-dimensional form. Thus, we introduce

$$\begin{gathered} R_1^* = \frac{R_1 L^2}{A_{22}}, \quad \alpha_2^* = \frac{\alpha_2 L^2}{A_{22}}, \quad \alpha_3^* = \frac{\alpha_3 L^2}{A_{22}}, \\ R_2^* = \frac{R_2 L^2}{A_{22}} - \alpha_2^*, \quad R_3^* = \frac{R_3 L^2}{A_{22}} - \alpha_3^*, \\ \omega_1^* = \omega_1 L, \quad \omega_2^* = \omega_2 L, \quad \omega_3^* = \omega_3 L, \\ \lambda_1 = \frac{F L^2}{A_{22}}, \quad \lambda_2 = \frac{M_u L}{A_{22}}, \\ x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad z^* = \frac{z}{L}, \quad t = \frac{S}{L}. \end{gathered} \tag{19}$$

For the bifurcation analysis we need the first nine equations of the system (16), (17) since the last three variables (x, y and z) could be determined after the first nine equations are solved. We write next the system (16), $(17)_{1,2,3}$ in compact form. Thus we introduce the vector

$\mathbf{w} = (R_1^*, R_2^*, ..., \vartheta_3)$. Suppose further that $\mathbf{w} \in Y$, where

$$Y = \left\{ \mathbf{w} : \textstyle\int_0^1 \mathbf{w}\cdot\mathbf{w}^T dt < \infty, \quad \int_0^1 \dot{\mathbf{w}}\cdot\dot{\mathbf{w}}^T dt < \infty, \right.$$

$$R_1^*(1) = -\lambda_1, \quad R_2^*(1) = 0, \quad R_3^*(1) = 0, \tag{20}$$

$$\omega_1^*(1) = \lambda_2, \quad \vartheta_2(1) = 0, \quad \vartheta_3(1) = 0,$$

$$\left. \vartheta_1(0) = 0, \quad \vartheta_2(0) = 0, \quad \vartheta_3(0) = 0 \right\},$$

where $\dot{(\cdot)} = d(\cdot)/dt$. The system (16), $(17)_{1,2,3}$ could now be written as

$$\dot{\mathbf{w}} = \mathbf{F}(\mathbf{w}, \lambda_1, \lambda_2, \alpha_2^*, \alpha_3^*). \tag{21}$$

In the component form (21) reads ($\mathbf{F}$ is defined by (16), $(17)_{1,2,3}$)

$$\dot{R}_1^* = -\left[(R_3^* + \alpha_3^*)\,\omega_2^* - (R_2^* + \alpha_2^*)\,\omega_3^*\right],$$

$$\dot{R}_2^* = -\left[R_1^*\omega_3^* - (R_3^* + \alpha_3^*)\,\omega_1^*\right],$$

$$\dot{R}_3^* = -\left[(R_2^* + \alpha_2^*)\,\omega_1^* - R_1^*\omega_2^*\right],$$

$$\dot{\omega}_1^* = \frac{A_{22}}{A_{11}}\left[\left(1 - \frac{A_{33}}{A_{22}}\right)\omega_3^*\omega_2^* + \right.$$
$$\left. + \left(\frac{1}{B_{33}} - \frac{1}{B_{22}}\right)\frac{A_{22}}{L^2}(R_2^* + \alpha_2^*)(R_3^* + \alpha_3^*)\right],$$

$$\dot{\omega}_2^* = \left(\frac{A_{33}}{A_{22}} - \frac{A_{11}}{A_{22}}\right)\omega_1^*\omega_3^* + \tag{22}$$
$$+ \left(\frac{1}{B_{11}} - \frac{1}{B_{33}}\right)\frac{A_{22}}{L^2}R_1^*(R_3^* + \alpha_3^*) + (R_3^* + \alpha_3^*),$$

$$\dot{\omega}_3^* = \frac{A_{22}}{A_{33}}\left[\left(\frac{A_{11}}{A_{22}} - 1\right)\omega_1^*\omega_2^* - \right.$$
$$\left. - \left(\frac{1}{B_{11}} - \frac{1}{B_{22}}\right)\frac{A_{22}}{L^2}R_1^*(R_2^* + \alpha_2^*) - (R_2^* + \alpha_2^*)\right],$$

$$\dot{\vartheta}_1 = \frac{c_2}{c_3}\omega_1^* + \frac{s_2}{c_3}\omega_3^*,$$

$$\dot{\vartheta}_2 = \frac{c_2 s_3}{c_3}\omega_1^* + \omega_2^* + \frac{s_2 s_3}{c_3}\omega_3^*,$$

$$\dot{\vartheta}_3 = -s_2\omega_1^* + c_2\omega_3^*.$$

Suppose that the rod has circular cross-section. Then $A_{22} = A_{33}$ and $B_{22} = B_{33}$. We shall study first the perfect system corresponding to the case when the load imperfections are equal to zero

$$\dot{\mathbf{w}} = \mathbf{F}(\mathbf{w}, \lambda_1, \lambda_2, 0, 0) = \mathbf{F_0}(\mathbf{w}, \lambda_1, \lambda_2). \tag{23}$$

Note that

$$\mathbf{F_0}(-\mathbf{w}, \lambda_1, \lambda_2) = -\mathbf{F_0}(\mathbf{w}, \lambda_1, \lambda_2). \tag{24}$$

The perfect system has been analysed [6]. It was shown there that by introducing the trivial solution of (23) in the form

$$\begin{gathered} R_1^{*0} = -\lambda_1, \quad R_2^{*0} = 0, \quad R_3^{*0} = 0, \\ \omega_1^{*0} = \lambda_2, \quad \omega_2^{*0} = 0, \quad \omega_3^{*0} = 0, \\ \vartheta_1^0 = \lambda_2 t, \quad \vartheta_2^0 = 0, \quad \vartheta_3^0 = 0. \end{gathered} \tag{25}$$

and the perturbations ΔR_1, ..., $\Delta\vartheta_3$ by

$$\begin{gathered} R_1^* = R_1^{*0} + \Delta R_1^*, \quad R_2^* = R_2^{*0} + \Delta R_2^*, \quad R_3^* = R_3^{*0} + \Delta R_3^*, \\ \omega_1^* = \omega_1^{*0} + \Delta\omega_1^*, \quad \omega_2^* = \omega_2^{*0} + \Delta\omega_2^*, \quad \omega_3^* = \omega_3^{*0} + \Delta\omega_3^*, \\ \vartheta_1 = \vartheta_1^0 + \Delta\vartheta_1, \quad \vartheta_2 = \vartheta_2^0 + \Delta\vartheta_2, \quad \vartheta_3 = \vartheta_3^0 + \Delta\vartheta_3, \end{gathered} \tag{26}$$

we obtain the following system

$$\Delta \ddot{R}_2^* + \lambda_2 \left(\frac{A_{11}}{A_{22}} - 2 \right) \Delta \dot{R}_3^* + \frac{\left(\Delta \dot{R}_2^* - \lambda_2 \Delta R_3^* \right) \left(\Delta R_2^* \Delta \dot{R}_2^* + \Delta R_3^* \Delta \dot{R}_3^* \right)}{\lambda_1^2 - \Delta R_2^{*2} - \Delta R_3^{*2}} +$$
$$+ \left\{ \lambda_2^2 \left(\frac{A_{11}}{A_{22}} - 1 \right) + \sqrt{\lambda_1^2 - \Delta R_2^{*2} - \Delta R_3^{*2}} + \right.$$
$$\left. + \frac{A_{22}}{L^2} \left(\frac{1}{B_{22}} - \frac{1}{B_{11}} \right) \left(\lambda_1^2 - \Delta R_2^{*2} - \Delta R_3^{*2} \right) \right\} \Delta R_2^* = 0,$$

$$\Delta \ddot{R}_3^* - \lambda_2 \left(\frac{A_{11}}{A_{22}} - 2 \right) \Delta \dot{R}_2^* + \frac{\left(\Delta \dot{R}_3^* + \lambda_2 \Delta R_2^* \right) \left(\Delta R_2^* \Delta \dot{R}_2^* + \Delta R_3^* \Delta \dot{R}_3^* \right)}{\lambda_1^2 - \Delta R_2^{*2} - \Delta R_3^{*2}} +$$
$$+ \left\{ \lambda_2^2 \left(\frac{A_{11}}{A_{22}} - 1 \right) + \sqrt{\lambda_1^2 - \Delta R_2^{*2} - \Delta R_3^{*2}} + \right.$$
$$\left. + \frac{A_{22}}{L^2} \left(\frac{1}{B_{22}} - \frac{1}{B_{11}} \right) \left(\lambda_1^2 - \Delta R_2^{*2} - \Delta R_3^{*2} \right) \right\} \Delta R_3^* = 0, \tag{27}$$

under the condition that $\lambda_1^2 - \Delta R_2^{*2} - \Delta R_3^{*2} \neq 0$. The non-trivial solution of (23) exists if there is non-trivial solution of (27) subject to

$$\Delta R_2^*(0) = 0, \quad \Delta R_3^*(0) = 0, \quad \Delta R_2^*(1) = 0, \quad \Delta R_3^*(1) = 0. \tag{28}$$

By using Liapunov-Schmidt method [12] it was shown by Atanackovic and Glavardanov [6] that the non-trivial solutions of (27) exist for λ_1 and λ_2 satisfying

$$k_{1,2} = \frac{\lambda_2}{2} \left(\frac{A_{11}}{A_{22}} \text{-} 1 \right) \pm \sqrt{\left(\frac{A_{11}}{A_{22}} \frac{\lambda_2}{2} \right)^2 + \lambda_1 \left[1 + \lambda_1 \frac{A_{22}}{L^2} \left(\frac{1}{B_{22}} \text{-} \frac{1}{B_{11}} \right) \right]},$$
$$k_1 - k_2 = 2n\pi, \quad n = 1, 2, 3, ... \tag{29}$$

The non-trivial solutions for $n = 1$ are of the form

$$\Delta R_2^* = a \left(\sin k_1 t - \sin k_2 t \right) + \hat{R}_2,$$
$$\Delta R_3^* = a \left(-\cos k_1 t + \cos k_2 t \right) + \hat{R}_3, \tag{30}$$

where $\hat{R}_2$ and $\hat{R}_3$ are unknown functions and a is the real constant satisfying the bifurcation equation (see [6])

$$f\left(\Delta \lambda_1, a \right) = c_1 a \Delta \lambda_1 + c_3 a^3 + O\left(\left(a^2 \Delta \lambda_1 \right), a \left(\Delta \lambda_1 \right)^2, a^4 \right), \tag{31}$$

where

$$c_1 = 2\left[1 + 2\lambda_{1cr}\frac{A_{22}}{L^2}\left(\frac{1}{B_{22}} - \frac{1}{B_{11}}\right)\right],$$
$$c_3 = \frac{1}{\lambda_{1cr}^2}\left[2\pi^2 - 3\lambda_{1cr} - 6\lambda_{1cr}^2\frac{A_{22}}{L^2}\left(\frac{1}{B_{22}} - \frac{1}{B_{11}}\right)\right], \tag{32}$$

The type of bifurcation depends on the sign of coefficients c_1 and c_3. For example suppose that we consider the special case of Bernoulli-Euler rod. Then

$$B_{11} \to \infty, \qquad B_{22} \to \infty. \tag{33}$$

It follows from (32) that $c_1 > 0$ and that

$$c_3 < 0 \quad \text{if} \quad \lambda_{1cr} > 2\frac{\pi^2}{3},$$
$$c_3 > 0 \quad \text{if} \quad \lambda_{1cr} < 2\frac{\pi^2}{3}. \tag{34}$$

In the first case the bifurcation at λ_{1cr} is super-critical while in the second case is sub-critical. The point $\lambda_{1cr} = 2\pi^2/3$ that separates sub from super-critical bifurcation is here determined explicitly. This is a consequence of relatively simple form of the equilibrium equations (as compared with equilibrium equations when standard Euler angles are used) because we used ship angles and R_2^* and R_3^* as dependent variables.

We turn now to the problem with imperfections. For this case the parameters $\alpha_2^* \neq 0$ and $\alpha_3^* \neq 0$. From $(22)_4$ follows $\omega_1^*(t) = 0$. From the fact that $\mathbf{R} = const.$ we obtain

$$\mathbf{R}(t) = -\lambda_1\mathbf{i} + \alpha_2^*\mathbf{e}_2(1) + \alpha_3^*\mathbf{e}_3(1)$$
$$= R_1^*\mathbf{e}_1 + R_2^*\mathbf{e}_2 + R_3^*\mathbf{e}_3 \tag{35}$$
$$= (-\lambda_1 + \Delta R_1^*)\,\mathbf{e}_1 + \Delta R_2^*\mathbf{e}_2 + \Delta R_3^*\mathbf{e}_3.$$

Note that (35) could be solved for $R_1^* = -\lambda_1 + \Delta R_1^*$ so that

$$\Delta R_1^* - \lambda_1 = \sqrt{\lambda_1^2 + \alpha_2^{*2} + \alpha_3^{*2} - \Delta R_2^{*2} - \Delta R_3^{*3}}. \tag{36}$$

From $(22)_{2,3}$ we obtain ω_2^* and ω_3^*. When we substitute this result in $(22)_{5,6}$ and use decomposition (26), we obtain the following system

$$
\begin{aligned}
&\Delta \ddot{R}_2^* + \frac{\left[\Delta \dot{R}_2^* - \lambda_2 \left(\Delta R_3^* + \alpha_3^*\right)\right]\left[\left(\Delta R_2^* + \alpha_2\right)\Delta \dot{R}_2^* + \left(\Delta R_3^* + \alpha_3^*\right)\Delta \dot{R}_3^*\right]}{\lambda_1^2 + \alpha_2^{*2} + \alpha_3^{*2} - \Delta R_2^{*2} - \Delta R_3^{*2}} + \\
&\quad + \lambda_2 \left(\frac{A_{11}}{A_{22}} - 2\right)\Delta \dot{R}_3^* + \left(\Delta R_2^* + \lambda_2\right)\Big\{\lambda_2^2 \left(\frac{A_{11}}{A_{22}} - 2\right) + \\
&\quad + \sqrt{\lambda_1^2 + \alpha_2^{*2} + \alpha_3^{*2} - \Delta R_2^{*2} - \Delta R_3^{*2}} + \\
&\quad + \frac{A_{22}}{L^2}\left(\frac{1}{B_{22}} - \frac{1}{B_{11}}\right)\left(\lambda_1^2 + \alpha_2^{*2} + \alpha_3^{*2} - \Delta R_2^{*2} - \Delta R_3^{*2}\right)\Big\} = 0, \\
&\Delta \ddot{R}_3^* + \frac{\left[\Delta \dot{R}_3^* - \lambda_2 \left(\Delta R_2^* + \alpha_2^*\right)\right]\left[\left(\Delta R_2^* + \alpha_2\right)\Delta \dot{R}_2^* + \left(\Delta R_3^* + \alpha_3^*\right)\Delta \dot{R}_3^*\right]}{\lambda_1^2 + \alpha_2^{*2} + \alpha_3^{*2} - \Delta R_2^{*2} - \Delta R_3^{*2}} + \\
&\quad + \lambda_2 \left(\frac{A_{11}}{A_{22}} - 2\right)\Delta \dot{R}_3^* + \left(\Delta R_3^* + \alpha_3^*\right)\Big\{\lambda_2^2 \left(\frac{A_{11}}{A_{22}} - 1\right) + \\
&\quad + \sqrt{\lambda_1^2 + \alpha_2^{*2} + \alpha_3^{*2} - \Delta R_2^{*2} - \Delta R_3^{*2}} + \\
&\quad + \frac{A_{22}}{L^2}\left(\frac{1}{B_{22}} - \frac{1}{B_{11}}\right)\left(\lambda_1^2 + \alpha_2^{*2} + \alpha_3^{*2} - \Delta R_2^{*2} - \Delta R_3^{*2}\right)\Big\} = 0,
\end{aligned}
\tag{37}
$$

subject to (28). The system (37), (28) is the central result of this note. We analyse the number of solution of (37), (28) for α_2^*, α_3^* small and for $\lambda_1 = \lambda_{1cr} + \Delta\lambda_1$ and λ_{2cr} with $\Delta\lambda_1$ small. Thus we treat (37) as one parameter bifurcation problem by fixing $\lambda_2 = \lambda_{2cr}$ and by allowing λ_1 to vary in the interval $[\lambda_{1cr} - \Delta\lambda_1 < \lambda_1 < \lambda_{1cr} + \Delta\lambda_1]$. The parameters λ_{1cr} and λ_{2cr} (critical load parameters) satisfy (29) for $n = 1$, i.e.,

$$
\pi^2 = \left(\frac{A_{11}\lambda_{2cr}}{2A_{22}}\right)^2 + \lambda_{1cr}\left[1 + \lambda_{1cr}\frac{A_{22}}{L^2}\left(\frac{1}{B_{22}} - \frac{1}{B_{11}}\right)\right]. \tag{38}
$$

The solution of (37), (28) is assumed in the form

$$
\begin{aligned}
\Delta R_2^* &= a\left(\sin k_1 t - \sin k_2 t\right) + \hat{R}_2, \\
\Delta R_3^* &= a\left(-\cos k_1 t + \cos k_2 t\right) + \hat{R}_3,
\end{aligned}
\tag{39}
$$

where a is determined from the bifurcation equation and $\hat{R}_2$ and $\hat{R}_3$ represent "the extra little part" of the solution, see [13]. For the method of analysis of the system (37) with assumed solution (39) see for example [14] and [15]. By substituting (39) into (37), multiplying $(37)_1$ by $(39)_1$ and $(37)_2$ by $(39)_2$ adding the result and integrating between 0 and 1, we obtain the bifurcation equation (after lengthy calculations) for the problem with imperfections in the form

$$\begin{aligned} G\left(\Delta\lambda_1, a\right) = {} & c_1 a \Delta\lambda_1 + c_3 a^3 + c_4 \alpha_2 + c_5 \alpha_2 a^2 + c_6 \alpha_3 + \\ & + c_7 \alpha_3 a^2 + c_8 \Delta\lambda_1 \alpha_2 + c_9 \Delta\lambda_1 \alpha_3 + h.o.t, \end{aligned} \tag{40}$$

where c_1 and c_3 are given by (32) and

$$\begin{aligned} c_4 = {} & \left[\lambda_{2cr}^2 \left(\frac{A_{11}}{A_{22}} - 1\right) + \lambda_{1cr} + \right. \\ & \left. + \frac{A_{22}}{L^2}\left(\frac{1}{B_{22}} - \frac{1}{B_{11}}\right)\lambda_{1cr}^2\right] \frac{2\pi}{\pi^2 - A^2}\left(1 + \cos A\right), \\ c_5 = {} & \frac{8\pi^3\left(1 + \cos A\right)}{A^4 - 10A^2\pi^2 + 9\pi^4}\left\{\frac{A^2 + 2A\lambda_{2cr} + 3\pi^2}{\lambda_{1cr}^2} - \right. \\ & \left. - 6\left[\frac{1}{2\lambda_{1cr}^2} + \frac{A_{22}}{L^2}\left(\frac{1}{B_{22}} - \frac{1}{B_{11}}\right)\right]\right\}, \\ c_6 = {} & \left[\lambda_{2cr}^2 \left(\frac{A_{11}}{A_{22}} - 1\right) + \lambda_{1cr} + \right. \\ & \left. + \frac{A_{22}}{L^2}\left(\frac{1}{B_{22}} - \frac{1}{B_{11}}\right)\lambda_{1cr}^2\right] \frac{2\pi}{\pi^2 - A^2}\sin A, \\ c_7 = {} & \frac{8\pi^3 \sin A}{A^4 - 10A^2\pi^2 + 9\pi^4}\left\{\frac{A^2 + 2A\lambda_{2cr} + 3\pi^2}{\lambda_{1cr}^2} - \right. \\ & \left. - 6\left[\frac{1}{2\lambda_{1cr}} + \frac{A_{22}}{L^2}\left(\frac{1}{B_{22}} - \frac{1}{B_{11}}\right)\right]\right\}, \\ c_8 = {} & \left[1 + 2\lambda_{1cr}\frac{A_{22}}{L^2}\left(\frac{1}{B_{22}} - \frac{1}{B_{11}}\right)\right]\frac{2\pi}{\pi^2 - A^2}\left(1 + \cos A\right), \\ c_9 = {} & \left[1 + 2\lambda_{1cr}\frac{A_{22}}{L^2}\left(\frac{1}{B_{22}} - \frac{1}{B_{11}}\right)\right]\frac{2\pi}{\pi^2 - A^2}\sin A, \\ A = {} & \frac{\lambda_{2cr}}{2}\left(\frac{A_{11}}{A_{22}} - 2\right). \end{aligned} \tag{41}$$

From (41) it follows that the values (and therefore signs) of constants c_1, ..., c_9 depend on the values of $(\lambda_{1cr}, \lambda_{2cr})$ that is on the point on the eigenvalue curve determined by (38). The imperfections α_2^* and α_3^* form universal unfolding to the perfect bifurcation problem, if (see [14], [15]).

$$J = \det \begin{vmatrix} 0 & 0 & f_{a\,\Delta\lambda} & f_{aaa} \\ 0 & f_{\Delta\lambda\,a} & f_{\Delta\lambda\,\Delta\lambda} & f_{\Delta\lambda\,aa} \\ G_{\alpha_2} & G_{\alpha_2\,a} & G_{\alpha_2\,\Delta\lambda} & G_{\alpha_2\,aa} \\ G_{\alpha_3} & G_{\alpha_3\,a} & G_{\alpha_3\,\Delta\lambda} & G_{\alpha_3\,aa} \end{vmatrix} \neq 0, \qquad (42)$$

where $f_a = (\partial f/\partial a)_{a=0,\,\Delta\lambda_1=0}$, etc. By using (31) and (40) in (42) we obtain

$$J = c_1\,[2c_5c_6 - c_4c_7] - 6c_3\,[c_6c_8 - c_4c_9] \neq 0. \qquad (43)$$

From (43) we could determine when α_2^* and α_3^* form a universal unfolding. In the special case when $A = 0$ that is when $\lambda_{2cr} = 0$, then $c_6 = c_7 = c_9 = 0$ and $J = 0$. Thus in the case of rod loaded by a force only the imperfection taken in the form of concentrated force at the top of the rod does not generate a universal unfolding to the perfect bifurcation problem.

4 Conclusions

In this paper we formulated a generalised version of the classical Greenhill problem, i.e., a problem of determining the critical load for a rod loaded by force and a couple. We used generalised form of constitutive equations allowing compressibility of the axis and the influence of shear stresses. We also introduced the load imperfections in the form of a small concentrated force acting at the point **K**.

1. For the perfect case, described by (28) we reduced the problem of determining critical load parameters to (29). It is a generalization of the classical result (see [1], [4]).

2. For the case when the imperfections are taken in the form of small concentrated force at the end **K** we show that the equilib-

rium equations reduce to the system (37) subject to the boundary conditions (28).

3. The bifurcation equation corresponding to (37) is given in the form (40). From this equation it is concluded that the imperfections analyzed here form universal unfolding to the perfect bifurcation problem if (43) is satisfied. Note that coefficients in (40) depend on the point $(\lambda_{1cr}, \lambda_{2cr})$ on eigenvalue curve (38). In the case when the couple is equal to zero, the imperfection analysed here *do not* constitute the universal unfolding.

4. The load imperfection that we use here is force of follower type. Its direction is fixed with respect to the plate at the end **K** of the rod. It would be interesting to examine other types of load imperfections, like vertical distributed load (weight of the rod).

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Generalizacija Grinhilovog problema sa smicanjem, kompresibilnošću i imperfekcijama

Klasičan problem Grinhila o stabilnosti stuba opterećenog silom i spregom je tretiran u generalisanoj verziji. Naime , pretpostavlja se da pored koncentrisane sile i sprega na gornjem kraju stuba deluje i mala koncentrisana sila. Takodje, pretpostavlja se da stub opisuju konstitutivne jednačine Haringsovog tipa koje u obzir uzimaju i uticaj smicanja i kompresibilnosti.