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Contribution to the study of the universal equation of boundary layer of Saljnikov

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Abstract

The universal equation of Saljnikov as one universalization of Prandtl's equation of the boundary layer is examined. The corrections of the function $B(f_1)$ and the arbitrary constant a_0 are suggested. These corrections are necessary because they guarantee noticeable improvement of the behavior of the first and the second derivative of the nondimensional stream function.

1 Introduction

The universalization of Prandtl's equation of the boundary layer has been done by many authors among which Loitsianski and Saljnikov are the most famous. Some results obtained in the research of the universal equation of the boundary layer of Saljnikov are given in this paper. The corrections of the function $B(f_1)$ and the arbitrary constant a_0 are suggested here.

2 The universal equations of Loitsianski and Saljnikov

The universalization of the Prandtl's equation of the boundary layer

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=UU'+\nu\frac{\partial^2 u}{\partial y^2},$$

$$u=\frac{\partial\psi}{\partial y},\quad v=-\frac{\partial\psi}{\partial x},\tag{1}$$

$$u\left(x,y\right)=0,\quad v\left(x,y\right)=0,\quad \text{when}\quad y=0,$$

$$u\left(x,y\right)\to U\left(x\right),\quad \text{as}\quad y\to\infty,$$

has been done by Loitsianski [1] and Saljnikov [2], with the usual notations:

x - the coordinate measured along the contour from the stagnation point,

y - the coordinate perpendicular to the contour,

$u(x,y)$ - the velocity into the direction of x,

$v(x,y)$ - the velocity into the direction of y,

$U(x)$- free stream velocity,

$U\prime(x)$ - first derivative of the function $U(x)$,

ν - kinematic viscosity,

ψ - the stream function.

The difference in the method used by Loitsianski and Saljnikov is represented as follows. The physical function B, given by the following expression

$$B=\int_0^\infty\frac{\partial\Phi}{\partial\eta}\left(1-\frac{\partial\Phi}{\partial\eta}\right)d\eta.\tag{2}$$

Loitsianski treats as a constant

$$B=\int_0^\infty \frac{\partial\Phi}{\partial\eta}\left(1-\frac{\partial\Phi}{\partial\eta}\right)d\eta = B_0 = 0,4696. \tag{3}$$

Saljnikov treats the physical function B (2) as the function of $f_k(x)$

$$B=\int_0^\infty \frac{\partial\Phi}{\partial\eta}\left(1-\frac{\partial\Phi}{\partial\eta}\right)d\eta = B\,(f_k)\,, \tag{4}$$

where the nondimensional stream function $\Phi = \Phi(\eta, f_k)$ is the function of the independent variable η and the set of parameters $f_k = f_k(x)$; $(k = 1, 2, 3, ..., \infty)$.

For the independent variables Loitsianski uses

$$x \equiv x; \quad y = \frac{\delta^{**}}{B_0}\eta, \tag{5}$$

with $\delta^{**}\,(x)$ as the momentum thickness, and for the stream functions $\psi\,(x,y)$

$$\Psi\,(x,y) = \frac{U\delta^{**}}{B_0}\Phi\,(\eta, f_k)\,, \qquad (k = 1, 2, 3, ..., \infty)\,, \tag{6}$$

where B_0 is given by (3).

Substituting (5) and (6) into (1), Loitsianski obtains the universal equation of the boundary layer

$$\frac{\partial^3\Phi}{\partial\eta^3} + \frac{F_L + 2f_1}{2B_0^2}\Phi\frac{\partial^2\Phi}{\partial\eta^2} + \frac{f_1}{B_0^2}\left[1-\left(\frac{\partial\Phi}{\partial\eta}\right)^2\right] =$$

$$= \frac{1}{B_0^2}\sum_{k=1}^{\infty}\Theta_k\left(\frac{\partial^2\Phi}{\partial\eta\partial f_k}\frac{\partial\Phi}{\partial\eta} - \frac{\partial^2\Phi}{\partial\eta^2}\frac{\partial\Phi}{\partial f_k}\right),$$

(7)

$$\Phi\,(\eta, f_k) = 0, \qquad \frac{\partial\Phi}{\partial\eta} = 0, \quad \text{when} \quad y = 0;$$

$$\frac{\partial\Phi}{\partial\eta} \to 1, \quad \text{as} \quad \eta \to \infty,$$

where

$$F_L = 2\left[B_0\left(\frac{\partial^2\Phi}{\partial\eta^2}\right)_{\eta=0} - f_1\left(2+H\right)\right],$$

$$H\left(f_1\right) = \frac{\int_0^\infty\left(1-\dfrac{\partial\Phi}{\partial\eta}\right)d\eta}{B_0}, \quad \theta_k = \left[kF_L + (k-1)\right]f_k + f_{k+1}. \tag{8}$$

For the independent variables Saljnikov uses

$$x \equiv x; \quad y = U^{-\frac{b_0}{2}}\left(a_0 v\int_0^x U^{b_0-1}dx\right)^{\frac{1}{2}}\eta, \tag{9}$$

and for the stream function $\psi\left(x,y\right)$

$$\psi\left(x,y\right) = U^{-\frac{b_0}{2}}\left(a_0 v\int_0^x U^{b_0-1}dx\right)^{\frac{1}{2}}\Phi\left(x,\eta\right), \tag{10}$$

where a_0 and b_0 are the arbitrary constants.

Substituting (9) and (10) into (1) Saljnikov obtains the universal equation of the boundary layer

$$\frac{\partial^3\Phi}{\partial\eta^3} + \frac{1}{2B^2}\left[a_0B^2 + f_1\left(2-b_0\right)\right]\Phi\frac{\partial^2\Phi}{\partial\eta^2} + \frac{f_1}{B^2}\left[1-\left(\frac{\partial\Phi}{\partial\eta}\right)^2\right] =$$

$$= \frac{1}{B^2}\sum_{k=1}^{\infty}\Theta_k\left(\frac{\partial^2\Phi}{\partial\eta\partial f_k}\frac{\partial\Phi}{\partial\eta} - \frac{\partial^2\Phi}{\partial\eta^2}\frac{\partial\Phi}{\partial f_k}\right),$$

$$\Phi\left(x,\eta\right) = 0, \quad \frac{\partial\Phi}{\partial\eta} = 0, \quad \text{when} \quad \eta = 0;$$

$$\frac{\partial\Phi}{\partial\eta} \to 1, \quad \text{as} \quad \eta \to \infty, \tag{11}$$

where B is given by (4) and where

$$a_0 B^2 - b_0 f_1 = F_s - \frac{2}{B} \sum_{k=1}^{\infty} \Theta_k \frac{dB}{df_k},$$

that is

$$F_s = a_0 B^2 - b_0 f_1 + \frac{2}{B} \sum_{k=1}^{\infty} \Theta_k \frac{dB}{df_k}, \quad \theta_k = [kF_s + (k-1) f_1] f_k + f_{k+1} \tag{12}$$

To follow the basic idea of Saljnikov that B is given by (4), for the independent variables we can use

$$x \equiv x; \quad y = \frac{\delta^{**}}{B(f_k)} \eta, \tag{13}$$

and for the stream function $\Psi(x, y)$

$$\Psi(x, y) = \frac{U\delta^{**}}{B(f_k)} \Phi(\eta, f_k) \tag{14}$$

Substituting (13) and (14) into (1) we obtain the universal equation of boundary layer

$$\frac{\partial^3 \Phi}{\partial \eta^3} + \frac{F + 2f_k - \frac{2}{B} \sum_{k=1}^{\infty} \Theta_k \frac{dB}{df_k}}{2B^2} \Phi \frac{\partial^2 \Phi}{\partial \eta^2} + \frac{f_1}{B^2} \left[1 - \left(\frac{\partial \Phi}{\partial \eta} \right)^2 \right] =$$

$$= \frac{1}{B^2} \sum_{k=1}^{\infty} \Theta_k \left(\frac{\partial^2 \Phi}{\partial \eta \partial f_k} \frac{\partial \Phi}{\partial \eta} - \frac{\partial^2 \Phi}{\partial \eta^2} \frac{\partial \Phi}{\partial f_k} \right), \tag{15}$$

$$\Phi(\eta, f_k) = 0; \quad \frac{\partial \Phi}{\partial \eta} = 0 \quad \text{when} \quad \eta = 0,$$

$$\frac{\partial \Phi}{\partial \eta} \to 1 \quad \text{as} \quad \eta \to \infty,$$

where

$$F = 2\left[B\left(\frac{\partial^2\Phi}{\partial\eta^2}\right)_{\eta=0} - f_1\,(2+H)\right]. \tag{16}$$

In this paper the answers to the following questions are expected to be found:

2.1. Is the function F_s (12) a good choice for testing the physical quantity $z = \delta^{**2}/v$?

2.2. Which conditions should the function $B(f_k)$, that is $B(f_1)$ satisfy?

2.3. Which values should be arbitrary constants a_0 and b_0 have according to the function $B(f_1)$?

The next sections comprise the results of the research which are the starting points in finding the answers.

3 Is the function F_s (12) a good choice?

The original method of Loitsianski [1, page 463] is used here: using the asymptotic solution at $\eta \to \infty$ $(\sim)$ of the universal equation of boundary layer, the value of the characteristic function of boundary layer F, identical to the value F, which was starting point in the universalization of the equation of boundary layer (1), (11) and (15) for $\eta \to \infty$ $(\sim)$ we obtain

$$\int_0^\infty \frac{\partial^3\Phi}{\partial\eta^3}d\eta = -\left(\frac{\partial^2\Phi}{\partial\eta^2}\right)_{\eta=0}, \tag{17}$$

$$\int_0^\infty \Phi\frac{\partial^2\Phi}{\partial\eta^2}d\eta = B = \int_0^\infty \frac{\partial\Phi}{\partial\eta}\left(1-\frac{\partial\Phi}{\partial\eta}\right)d\eta, \tag{18}$$

$$\int_0^\infty \left[1-\left(\frac{\partial\Phi}{\partial\eta}\right)^2\right]d\eta = B\,(1+H), \tag{19}$$

$$\int_0^\infty \left(\frac{\partial^2 \Phi}{\partial \eta \partial f_k} \frac{\partial \Phi}{\partial \eta} - \frac{\partial^2 \Phi}{\partial \eta^2} \frac{\partial \Phi}{\partial f_k} \right) d\eta = -\frac{dB}{df_k}. \tag{20}$$

On the basis of equation (17), (18). (19) and (20), the equation (7) becomes

$$F_L = 2 \left[B_0 \left(\frac{\partial^2 \Phi}{\partial \eta^2} \right)_{\eta=0} - f_1 (2 + H) \right]. \tag{21}$$

As F_L (21) is identical to F_L (8), it means that F_L (8) is a good choice.

On the basis of equations (17), (18), (19) and (20), the equation (11) becomes

$$a_0 B^2 - b_0 f_1 + \frac{2}{B} \sum_{k=1}^{\infty} \Theta_k \frac{dB}{df_k} = 2 \left[B \left(\frac{\partial^2 \Phi}{\partial \eta^2} \right)_{\eta=0} - f_1 (2 + H) \right]. \tag{22}$$

The left side of equation (22) is identical to the equation (12), and the right side of equation (22) is identical to the equation (16). If (16) proves to be a good choice, it can be asserted that (12) is a good choice too. In that sense, on the basis of equations (17), (18), (19) and (20), the equation (15) becomes

$$F = 2 \left[B \left(\frac{\partial^2 \Phi}{\partial \eta^2} \right)_{\eta=0} - f_1 (2 + H) \right]. \tag{23}$$

As for F (23) is identical to F (16), it means that F (16) is a good choice, so the conclusion is that F_s (12) is a good choice too.

This analysis was necessary because of two reasons:

3.1. To assert that, before starting further research, F_s (12) is a good choice, and

3.2. The equation (11) should be further used in the following form

$$\frac{\partial^3\Phi}{\partial\eta^3}+\frac{F_s+2f_1-\frac{2}{B}\sum\limits_{k=1}^{\infty}\Theta_k\frac{dB}{df_k}}{2B^2}\Phi\frac{\partial^2\Phi}{\partial\eta^2}+\frac{f_1}{B^2}\left[1-\left(\frac{\partial\Phi}{\partial\eta}\right)^2\right]= \tag{24}$$

$$=\frac{1}{B^2}\sum_{k=1}^{\infty}\Theta_k\left(\frac{\partial^2\Phi}{\partial\eta\partial f_k}\frac{\partial\Phi}{\partial\eta}-\frac{\partial^2\Phi}{\partial\eta^2}\frac{\partial\Phi}{\partial f_k}\right),$$

where F_s is given by (12).

4 Which conditions should the function $B(f_1)$ satisfy?

Using the method of asymptotic solution of the nondimensional stream function $\Phi(\eta, f_k)$ at $\eta\to\infty$ ($\sim$), we obtain

$$\Phi(\eta, f_k)\sim\eta-B(f_k)\,H(f_k)\,. \tag{25}$$

In case of one-parameter solution ($f_1\neq 0$; $f_2=f_3=...=f_k=0$), from (25) we obtain

$$\Phi(\eta, f_1)\sim\eta-B(f_1)\,H(f_1)\,. \tag{26}$$

The first derivative (26) of the f_1, in the case of Loitsianski follows as

$$\frac{\partial\Phi_L}{\partial f_1}\sim -B_0\frac{\partial H_L}{\partial f_1}, \tag{27}$$

where

$$H_L(f_1)=\frac{\int\limits_0^{\infty}\left(1-\frac{\partial\Phi_L}{\partial\eta}\right)d\eta}{B_0}. \tag{28}$$

The first derivative (26) of the f_1, in the case of Saljnikov follows as

$$\frac{\partial \Phi_s}{\partial f_1} \sim -\frac{dB}{df_1} H_s(f_1) - B(f_1)\frac{dH_s}{df_1}, \tag{29}$$

where

$$H_s(f_1) = \frac{\int\limits_0^\infty \left(1 - \frac{\partial \Phi_s}{\partial \eta}\right) d\eta}{B(f_1)}. \tag{30}$$

In the specific case of the Blasius solution of a flat plate ($f_1 = 0$), used by both Loitsianski and Saljnikov, on the basis of (27), (28), (29) and (30), the function $B(f_1)$ should satisfy two conditions at the point $f_1 = 0$

$$B(0) = B_0 = 0,4696, \tag{31}$$

$$\left(\frac{dB}{df_1}\right)_{f_1=0} = 0. \tag{32}$$

The possible forms of the function $B(f_1)$, which satisfy the conditions (31) and (32), are given in Figure 1.

Based on the researches, not mentioned in this paper, which are based on the comparing the second derivative of the nondimensional stream function $\Phi''(0)$ of Loitsianski and Saljnikov, it is possible to conclude that for each value $B(f_1) < B_0$ the first derivative of the nondimensional stream function $\partial\Phi/\partial\eta$ will tend to one faster than with Loitsianski.

If we have in the mind the Hartree's conclusion [3], for the same differential equation, the better solution of the two is the one where $\partial\Phi/\partial\eta \to 1$ is faster (Figure 2), because then all the solutions where $B(f_1) < B_0$ are better than the solution where $B_0 = 0,4696$. And vice versa. Namely, for each value $B(f_1) > B_0$ the first derivative of the nondimensional stream function $\partial\Phi/\partial\eta$ will tend to one slower than with Loitsianski.

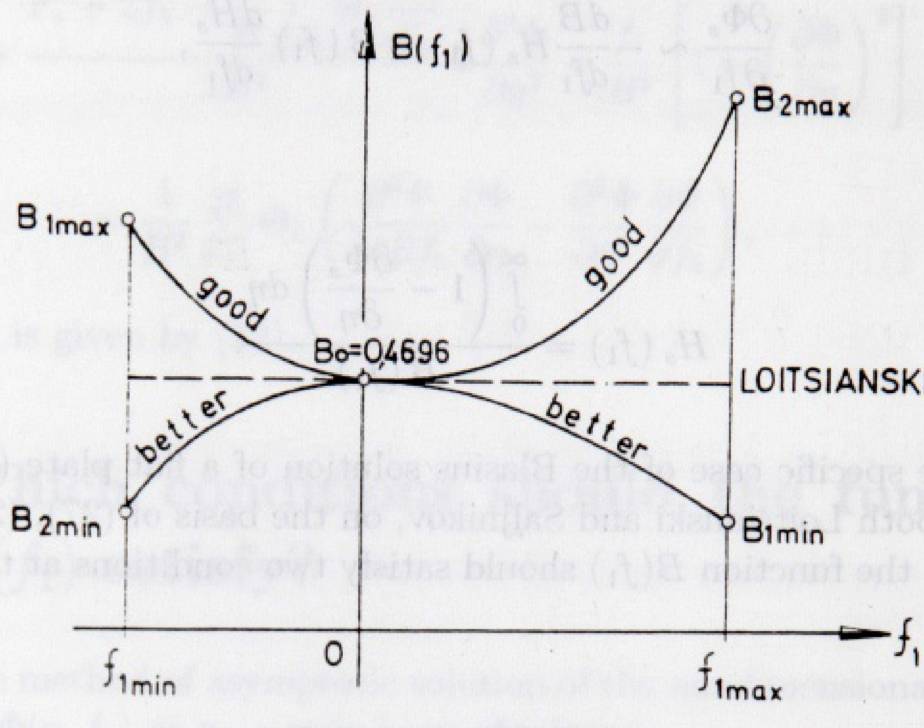

Fig. 1. The possible forms of $B(f_1)$.

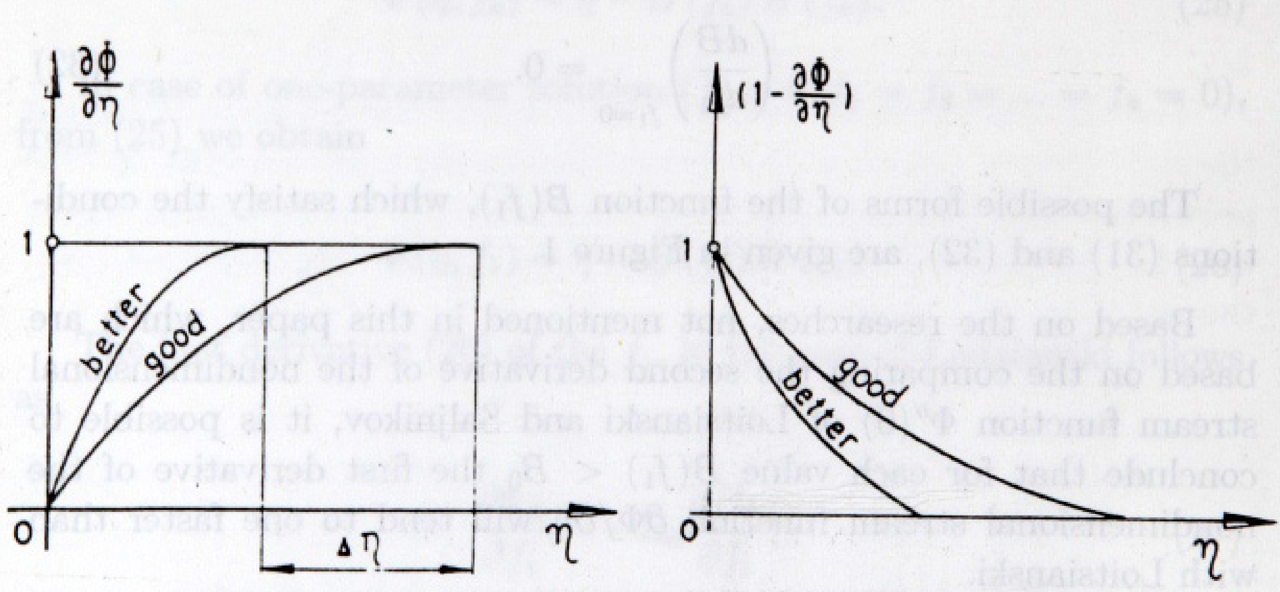

Fig. 2. The possible forms of $\partial\Phi/\partial\eta$.

Observing the numerical values of the function $B(f_1)$ of Saljnikov [2], it can be seen that $B(f_1)$ does not satisfy the condition (31), but it satisfies the condition (32). The numerical values of the second derivative of the nondimensional stream function $\Phi''(0)$ of Saljnikov [2] show that the first derivative of the nondimensional stream function $\partial\Phi/\partial\eta$ tends to one very slowly. The only reason is that with Saljnikov, $B(f_1) > B_0$ is significantly bigger.

5 Which values should the arbitrary constants a_0 and b_0 have?

The function F_s (12) should satisfy the following conditions at the point $f_1 = 0$

$$F_s(0) = 0,4408, \tag{33}$$

$$\left(\frac{dF_s}{df_1}\right)_{f_1=0} = -5,714. \tag{34}$$

In the case of one-parameter solution ($f_1 \neq 0$; $f_2 = f_3 ... f_k = 0$) from (12) we obtain

$$F(f_1) = \frac{a_0 B^2 - b_0 f_1}{1 - \dfrac{2}{B} f_1 \dfrac{dB}{df_1}}. \tag{35}$$

From (35) follows

$$F(0) = a_0 [B(0)]^2, \tag{36}$$

$$\left(\frac{dF}{df_1}\right)_{f_1=0} = 4a_0 B(0) \cdot \left(\frac{dB}{df_1}\right)_{f_1=0} - b_0. \tag{37}$$

Using the conditions (31), (32), (33) and (34), from (36) and (37) we obtain

$$a_0 = 2,0000, \tag{38}$$

$$b_0 = 5,7140. \tag{39}$$

For the arbitrary constant b_0 Saljnikov [2] obtained the value 5,7140, and for the arbitrary constant a_0 he obtained the value 0,4408.

From the expression for the function $H(f_1)$

$$H(f_1) = \frac{\delta^*}{\delta^{**}} = \frac{\int_0^\infty \left(1 - \frac{\partial \Phi}{\partial \eta}\right) d\eta}{\int_0^\infty \frac{\partial \Phi}{\partial \eta}\left(1 - \frac{\partial \Phi}{\partial \eta}\right) d\eta} = \frac{A(f_1)}{B(f_1)}, \tag{40}$$

the direct influence of the value of the arbitrary constant a_0 can not be observed. However, observing the numerical values of the function $A(f_1)$, got by Saljnikov, we can see that they are significantly bigger than the numerical values of Loitsianski. Having in mind, on the basic of the research in this paper, that the value of the arbitrary constant a_0 is directly related to the form of the function $B(f_1)$, that is, to the condition (31) and if, for example the values $A(f_1)$ of Loitsianski (Figure 3) and $A(f_1)$ of Saljnikov (Figure 4) are graphically shown for $f_1 = 0,0854$, the conclusion stated in the Section 4 of this paper is confirmed. The behavior for any value of parameter f_1 is similar.

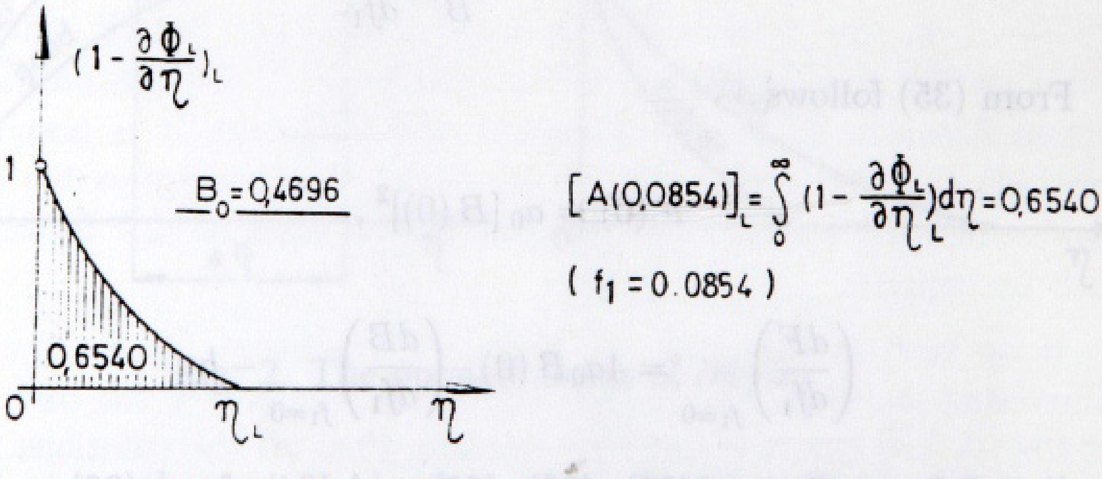

Fig. 3. The values $A(f_1)$ of Loitsianski.

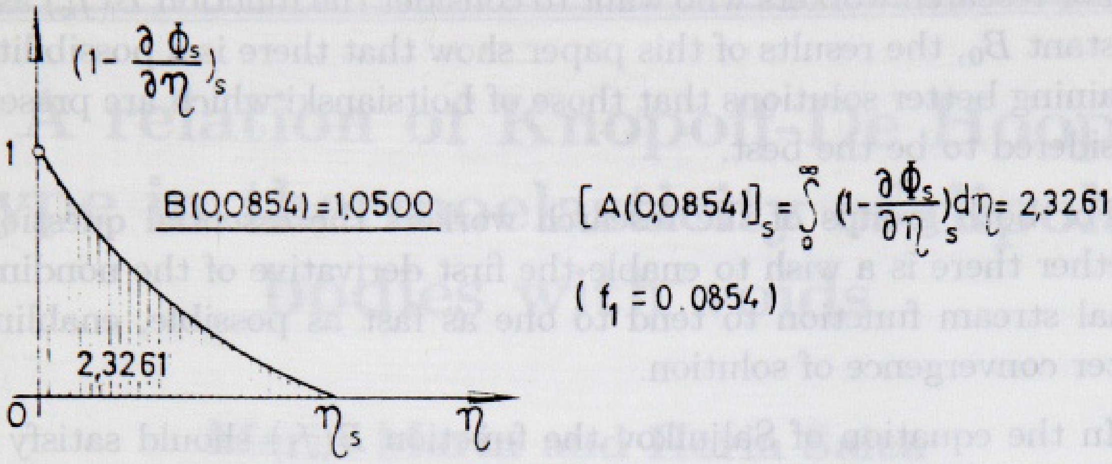

Fig. 4. The values $A\,(f_1)$ of Saljnikov.

From the expressions of Saljnikov [2] for the skin friction

$$\tau_w = \frac{\mu U^{1+\frac{b_0}{2}}}{\left(a_0\nu \int\limits_0^\infty U^{b-1}dx\right)^{\frac{1}{2}}}\Phi''(0)\,, \tag{41}$$

both the direct influence of the arbitrary constant a_0, and the value $\Phi''(0)$ can be observed. For the suggested new value for a_0 (2,000) from (41) it results that $\Phi''(0)$ should increase significantly. It means that the first derivative of the nondimensional stream function of Saljnikov will be noticeably improved.

6 Conclusion

For research workers who do not want to consider the function $B(f_k)$ constant, the results of this paper do not change the basic idea, because the characteristic function of the boundary layer F_s (12) is a good choice, but they change the form of function $B(f_1)$ and the arbitrary constant a_0

These corrections are necessary because they guarantee noticeable improvement of the behavior of the first and the second derivative of the nondimensional stream function.

For research workers who want to consider the function $B(f_k)$ as the constant B_0, the results of this paper show that there is a possibility of obtaining better solutions that those of Loitsianski which are presently considered to be the best.

For both groups of the research workers the essential question is whether there is a wish to enable the first derivative of the nondimensional stream function to tend to one as fast as possible, enabling a better convergence of solution.

In the equation of Saljnikov the function $B(f_1)$ should satisfy the conditions (31) and (32), and the arbitrary constant a_0 should have the value of 2,0000.

References

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Doprinos studiji Saljnikove univerzalne jednačine graničnog sloja

U radu je ispitivana univerzalna jednačina Saljnikova kao univerzalizacija Prantlove jednačine graničnog sloja. Predložene su korekcije za funkcije $B\,(f_1)$ i proizvoljnu konstantu a_0.