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The conservation laws in theory of rods

M. Vukobrat and D. Kuzmanović

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Abstract

This paper used a modified method suggested by J.W. Eischen and G. Herrmann [4] to obtain the balance laws in the theory of rods. In this way we were able to show that the J-type balance laws can be deduced simply by this method, applying the analogue operators like gradient, curl and divergence, by subjection to the Lagrangian function and "Lagrangian moment".

1 Introduction

The conservation laws for linear elastostatic without defects, based on Noether's theorem, were discussed in a more general form by Günter [1]. Independently of Günter, three types of conservation (or balance) laws, both for linearized and finite elastostatics, were established by Knowles and Sternberg [2].

Fletcher [4] extended the invariant integrals presented by Knowles and Sternberg [2] to linear elastodynamics, again relying on manipulations dictated by Noether's [5] theorem.

The paper by Eischen and Herrmann [5] was motivated primarily by the interest generated by Rice's J-integral, and a quest for its theoretical undergoing. Two additional laws were found which were subsequently associated with the L_k and M integrals of defect mechanics by Budianski and Rice [3].

The purpose of this paper is to apply the ideas of the papers mentioned above to the theory of rods [6]. It will be shown that balance laws of the type described above can be deduced simply by subjecting the Lagrangian density of familiar operations from vector calculus such as gradient, divergence and curl.

The dynamics of a rod is equivalent to the two conservation laws obtained below, that express the balance of linear and angular momentum. If the rod is hyperelastic it implies a third conservation law that expresses conservation of energy. The additional conservation laws that are of primary concern here arise in the cases of a uniform, hyperelastic rod, where the symmetry is translation in arc-length, and in a class of transversely isotropic rods, where the symmetry is rotation about the centerline of the rod. The last conservation law is obtained upon considering the scale changes.

2 Conservation laws

The Lagrangian density L may be viewed as a function of the particle velocity, total strain tensor, and coordinates. Accordingly, it is defined by

$$L = W - T = L\left(t, s, \Phi, \dot{\Phi}, \Phi'\right) \tag{1}$$

where W and T represent the elastic strain and kinetic energy densities, respectively. The $\Phi = \Phi(\Phi_i) \in R_m, i = 1, 2, ..., m$, are functions, which describe the behavior of the material system under consideration. We suppose that these fields are twice continuously differentiable. Here Φ_i' denotes the partial derivative with respect to arc-length s, while $\dot{\Phi}_i$ denotes the partial derivative with respect to t.

Given the particular form of equation (1), the usual variation of the Lagrangian will lead to the equations of motion, provided that the terms according for the work of the body forces and surface tractions are included.

The desired results are obtained next by simple vector and tensor calculus operations involving the Lagrangian.

I. According to Eischen, Herrmann [4], the first balance law is obtained by considering the partial derivative of the Lagrangian, i.e.

$$\frac{dL}{ds} = \frac{\partial L}{\partial \Phi} \cdot \frac{\partial \Phi}{\partial s} + \frac{\partial L}{\partial \dot{\Phi}} \cdot \frac{\partial \dot{\Phi}}{\partial s} + \frac{\partial L}{\partial \Phi'} \cdot \frac{\partial \Phi'}{\partial s} + \left.\frac{\partial L}{\partial s}\right|_{\text{expl}}, \tag{2}$$

where dot denotes the inner product.

Relation (2) can be expressed in the form

$$\frac{dL}{ds} = \left.\frac{\partial L}{\partial s}\right|_{\text{expl}} + \left(\frac{\partial L}{\partial \Phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\Phi}} - \frac{d}{ds}\frac{\partial L}{\partial \Phi'}\right)\Phi' + \\ + \frac{d}{ds}\left(\frac{\partial L}{\partial \Phi'} \cdot \Phi'\right) + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\Phi}} \cdot \Phi'\right). \tag{3}$$

From (3) we obtain

$$\left.\frac{\partial L}{\partial s}\right|_{\text{expll}} = \frac{d}{ds}\left(L - \frac{\partial L}{\partial \Phi'} \cdot \Phi'\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\Phi}} \cdot \Phi'\right) + \rho f \Phi', \tag{4}$$

where we assumed that the Lagrange equations have been satisfied, i.e.

$$\frac{\partial L}{\partial \Phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\Phi}} - \frac{d}{\mathrm{d}s}\frac{\partial L}{\partial \Phi'} = \rho f. \tag{5}$$

Relation (3) can also be expressed in the form

$$\frac{d}{ds}\left(L - \frac{\partial L}{\partial \Phi'} \cdot \Phi'\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\Phi}} \cdot \Phi'\right) - \rho f \Phi' + \left.\frac{\partial L}{\partial s}\right|_{\text{expl}}. \tag{6}$$

II. The second balance law is obtained by differentiation with respect to parameter t (time)

$$\frac{dL}{dt} = \left.\frac{\partial L}{\partial t}\right|_{\text{expl}} + \frac{\partial L}{\partial \Phi} \cdot \dot{\Phi} + \frac{\partial L}{\partial \Phi'} \cdot \left(\dot{\Phi}'\right) + \frac{\partial L}{\partial \dot{\Phi}} \cdot \ddot{\Phi} =$$

$$= \left.\frac{\partial L}{\partial t}\right|_{\text{expl}} + \left(\frac{\partial L}{\partial \Phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\Phi}} - \frac{d}{ds}\frac{\partial L}{\partial \Phi'}\right)\dot{\Phi} + \tag{7}$$

$$+ \frac{d}{ds}\left(\frac{\partial L}{\partial \Phi'}\dot{\Phi}\right) + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\Phi}} \cdot \dot{\Phi}\right).$$

Similarly, using the Lagrange equation, we obtain

$$\frac{d}{dt}\left(L - \frac{\partial L}{\partial \dot{\Phi}} \cdot \dot{\Phi}\right) = \frac{d}{ds}\frac{\partial L}{\partial \Phi'} \cdot \dot{\Phi} - \rho f \dot{\Phi} + \left.\frac{\partial L}{\partial t}\right|_{\text{expl}} \tag{8}$$

III. The third balance law is obtained, by analogy in [5] (divergence of "Lagrangian moment"), as a derivative with respect to parameter s, of the function (Ls).

$$\frac{d}{ds}(Ls) = \left.\frac{\partial (Ls)}{\partial s}\right|_{\text{expl}} + \frac{\partial (Ls)}{\partial \Phi}\Phi' + \frac{\partial (Ls)}{\partial \Phi'}\Phi'' + \frac{\partial (Ls)}{\partial \dot{\Phi}}\left(\dot{\Phi}\right)'. \tag{9}$$

Relation (9) can be transformed in the form of

$$\frac{d}{ds}(Ls) = \left.\frac{\partial L}{\partial s}\right|_{\text{expl}} s + L + \left[\frac{\partial L}{\partial \Phi} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\Phi}} - \frac{\partial}{\partial s}\frac{\partial L}{\partial \Phi'}\right]\Phi' s + \tag{10}$$

$$+ \frac{d}{ds}\left[\frac{\partial L}{\partial \Phi'}\Phi' s\right] - \frac{\partial L}{\partial \Phi'}\Phi' + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\Phi}}\Phi' s\right].$$

Using the Lagrange equations, the above relation becomes

$$\left.\frac{\partial L}{\partial s}\right|_{\text{expl}} s = \frac{d}{ds}\left[\frac{\partial L}{\partial \Phi'}\Phi' s\right] - L + 2W - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\Phi}}\Phi' s\right] + \rho f \Phi' s. \tag{11}$$

Further, if we assume that L is the homogenous function of the second degree[1], than

$$\frac{\partial L}{\partial \Phi'}\Phi' = 2W, \tag{12}$$

and from

$$-L + 2W = \frac{d}{dt}\left[Lt + t\dot{\Phi}\ddot{\Phi}\right] - t\left[\frac{\partial L}{\partial t} + 2\dot{\Phi}\ddot{\Phi}\right], \tag{13}$$

we finally obtain

$$\frac{d}{ds}\left[Ls - \frac{\partial L}{\partial \Phi'}\cdot\left(\Phi' s + \dot{\Phi} t\right)\right] = $$
$$= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\Phi}}\cdot\left(\Phi' s + \dot{\Phi} t\right)\right] + \left.\frac{\partial L}{\partial s}\right|_{\text{expl}} s - \rho f\left(\Phi' s + \dot{\Phi} t\right). \tag{14}$$

Now, we will apply this relations in the theory of the rods.

3 Rod dynamics

The independent variables are two scalars, namely undeformed arc-length s, and time t. The dependent variables are a vector function $\mathbf{r}(s,t) \in R^3$ and an orthonormal frame *directors* $\mathbf{d}_i(s,t) \in R^3$, $i = 1, 2, 3$. The derivation of the equations of motion will only be outlined here, as the rod model that will be adopted is a comparatively standard one.

The kinematics of the rod are encapsulated in the relations [6]

$$\mathbf{r}' = \mathbf{v}, \tag{15}$$

$$\mathbf{d}_i' = \mathbf{u} \times \mathbf{d}_i, \tag{16}$$

[1] which holds in the case of linear clastisity

and

$$\dot{\mathbf{d}}_i = \boldsymbol{\omega} \times \mathbf{d}_i. \tag{17}$$

The densities of linear and angular momentum are, respectively,

$$\mathbf{p}(s,t) \equiv \rho(s)\,\dot{\mathbf{r}}, \tag{18}$$

and

$$\boldsymbol{\pi}(s,t) \equiv I_{ij}(s)\,\omega_j \rho(s)\,\dot{\mathbf{d}}_i. \tag{19}$$

Here $\rho(s)$ is the mass per unit arc-length associated with the material cross-section at point s, which is determined entirely by the reference configuration. The quantities $I_{ij}(s)$ are components of the inertia tensor of the material cross-section at s expressed with respect to the triad $\{\mathbf{d}_i\}$. It will be assumed that these inertia coefficients are independent of the strain u_i and v_i.

In a rod model the stresses acting across each material cross-section are reduced to a net force $\mathbf{n}(s,t)$ and moment $\mathbf{m}(s,t)$. Then, the balance of linear and angular momentum yield the equations

$$\dot{\mathbf{p}} = \mathbf{n}', \tag{20}$$

and

$$\dot{\boldsymbol{\pi}} = \mathbf{m}' + \mathbf{r}' \times \mathbf{n}. \tag{21}$$

The system is closed by specification of a constitutive law relating the stresses $\mathbf{n}$ and $\mathbf{m}$ to the strains $\mathbf{u}$ and $\mathbf{v}$. However, it may immediately be observed that, trivially and independent of the constitutive relations, (20) is in the form with density $\mathbf{p}$ and flux $\mathbf{n}$. Moreover, using (18) and (20), equation (21) can be rewritten in the conservation form

$$(\boldsymbol{\pi} + \mathbf{r} \times \mathbf{p})_t = (\mathbf{m} + \mathbf{r} \times \mathbf{n})_s\,. \tag{22}$$

We next demonstrated that, for a hyperelastic rod, the equations of motion (20) and (21) imply a third (well-known) conservative law, namely conservation of the total energy.

A rod is said to be hyperelastic if there exists a scalar valued strain energy density function $W(v_i, u_i, s)$, dependent upon the six strains and arc-length, with the property that the components of stress n_i and m_i identically satisfy the constitutive relations

$$n_i = W_{v_i} = \frac{\partial W}{\partial v_i}, \tag{23}$$

and

$$m_i = W_{u_i} = \frac{\partial W}{\partial u_i}. \tag{24}$$

Here W_{v_i} and W_{u_i} denotes the partial derivative of W with respect to the argument v_i and u_i, respectively.

The conservation laws for rods

I. If the material is homogeneous and is free of any defects so that the Lagrangian density does not depend explicitly on the parameter s, then $(\partial L/\partial s)_{\text{expl}} = 0$, and in absence of body force, the equation (6) becomes a statement of the conservation law

$$(\mathbf{m}\cdot\mathbf{u}+\mathbf{n}\cdot\mathbf{v}-L')=\overline{(\boldsymbol{\pi}\cdot\mathbf{u}+\mathbf{p}\cdot\mathbf{v})} \tag{25}$$

where

$$\frac{\partial L}{\partial \underset{1}{\Phi}'}=\frac{\partial W}{\partial \mathbf{v}}=\mathbf{n},\quad \frac{\partial L}{\partial \underset{2}{\Phi}'}=\frac{\partial W}{\partial \mathbf{u}}=\mathbf{m},$$
$$\frac{\partial L}{\partial \underset{1}{\dot{\Phi}}}=\frac{\partial W}{\partial \dot{\mathbf{v}}}=\mathbf{p},\quad \frac{\partial L}{\partial \underset{2}{\dot{\Phi}}}=I_{ij}\omega\mathbf{d}_i. \tag{26}$$

Note that the above relations are valid for uniform hyperelastic (if the coefficients ρ and I_{ij} are constants) rods.

This relation represents Eshelby energy - momentum tensor type, which arises in a conservation law for elastostatics of a homogeneous elastic material in multispace dimensions.

II. In a special case, when Lagrangian is not a function of parameter t, and in absence of body force, from (8) we obtain

$$\overline{\left(W+\frac{1}{2}\mathbf{p}\cdot\dot{\mathbf{r}}+\frac{1}{2}\boldsymbol{\pi}\cdot\boldsymbol{\omega}\right)}^{\,\cdot}=(\mathbf{n}\cdot\dot{\mathbf{r}}+\mathbf{m}\cdot\boldsymbol{\omega})'. \tag{27}$$

This relation represents the conservation of energy.

III. Similarly, by substituting (26) into (14) we obtained

$$[Ls - \mathbf{n}\cdot(\mathbf{v}s + \dot{\mathbf{r}}t) - \mathbf{m}\cdot(\mathbf{u}s + \boldsymbol{\omega}t)]' = \overline{[Lt - \dot{\mathbf{r}}\cdot(\mathbf{v}s + \dot{\mathbf{r}}t) - \boldsymbol{\omega}\cdot(\mathbf{u}s + \boldsymbol{\omega}t)]}^{\cdot}\,. \tag{28}$$

This relation represents the conservation law obtained upon consideration of the family of scale changes.

4 Isotropic rods

We assume that the rod is transversely isotropic. Full isotropics is not physically justified for rods.

In the paper [6] (see also [Antman]), demonstrating that the hypotesis of isotropic elastic response is equivalent to the specialized constitutive relations

$$\mathbf{n} = N_1\mathbf{v} + N_2\mathbf{u} + N_3\mathbf{d}_3, \tag{29}$$

$$\mathbf{m} = M_1\mathbf{v} + M_2\mathbf{u} + M_3\mathbf{d}_3, \tag{30}$$

where the N_i and M_i, $i = 1, 2, 3$, are scalar function each. Isotropy of the dynamics properties of the rod follows for the angular momentum, have the special form

$$\boldsymbol{\pi} = I_1\boldsymbol{\omega} + (I_3 - I_1)\,\omega_3\mathbf{d}_3, \tag{31}$$

where $\mathbf{d}_3$ is a principal axis of inertia with associated principal inertia I_3. Also, we suppose that is $I_1 = I_2$.

With constitutive relations (23)-(24), and additional restriction $M_1 - N_2$, it is easily shown that the conservation law (25) reduces to the form [6]

$$\dot{\overline{(\boldsymbol{\pi}\mathbf{d}_3)}} = \frac{d}{ds}(\mathbf{m}\cdot\mathbf{d}_3). \tag{32}$$

5 Strings, inextensible rods, and planar dynamics

In this section we shall consider the conservation laws that arise in a certain degenerate case of the rod model described in Section 2.

A model of a string is obtained if the strain v is declared to satisfy $v_1 = v_2 = 0$, so that (6) reduces to

$$\mathbf{v} = v_3 \mathbf{d}_3. \tag{33}$$

The balance laws in this case have the forms

$$\dot{\boldsymbol{\pi}} = \mathbf{m}', \tag{34}$$

$$\dot{\overline{\mathbf{r} \times \mathbf{p}}} = \mathbf{r} \times \mathbf{n}', \tag{35}$$

because the constitutive laws for the force and moment are taken to be

$$n_1 = n_2 = 0, \quad n_3 = N_3, \tag{36}$$

and

$$m_1 = m_2 = 0, \quad m_3 = M_3. \tag{37}$$

Here N_3 and M_3 are scalar functions.

In this special case the conservative law (20), (22) is reduced to

$$\dot{\overline{\mathbf{r} \times \mathbf{p}}} = \mathbf{r} \times \mathbf{n}'. \tag{38}$$

This string is uniform and hyperelastic if there is a strain energy density function in the form of $W(v_3, u_3)$, such that $N_3 = W_{v_3}$ and $M_3 = W_{u_3}$ (W must be independent of u_i, $i = 1, 2$, to be consistent with the requirements $n_i = 0$, $i = 1, 2$,). For such a string the conservation law (25) takes the form

$$\dot{\overline{(\boldsymbol{\pi} \cdot \mathbf{u} + \mathbf{p} \cdot \mathbf{v})}} = (m_3 u_3 + n_3 v_3 - L)'. \tag{39}$$

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M. Vukobrat
Faculty of Transport and Traffic
V. Stepe 305, 11000 Beograd
Yugoslavia

D. Kuzmanović
Faculty of Mining and Geology
Djusina 7, 11000 Beograd
Yugoslavia.

Zakoni održanja u teoriji štapova

U radu se prikazuje metod, koji su predložili J.W. Eischen i G. Herrmann [4] kao metod za dobijanje zakona balanasa u teoriji štapova. Pokazuje se da se zakoni balansa J-tipa mogu da dobiju koristeći ovaj metod primenom operatora: gradijent, divergencija i rotor na takozvani "Lagranžev moment". Na kraju se razmatraju neki od specijalnih slučajeva.