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On the equilibrium stability of mechanical systems with dissipation

Miroslav Vesković

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Abstract

The paper considers the problem of the equilibrium stability of the first and second kinds of the nonholonomic dissipative systems with a bilateral and unilateral constraints. We have proved the theorems on the instability under the following the assumptions that: the constraint equations, the kinetic and potential energies and the Rayleigh's dissipation function are infinitely differentiable functions; in the equilibrium position the potential energy has not the minimum. The suggested theorems obtained represent an extension to nonholonomic systems of some results [1], [6], [9], referring to the equilibrium stability of holonomic systems. The proving technique will be similar to that used in the paper [1].

1 Introduction

Let us consider a scleronomic dissipative mechanical system in an n–dimensional configuration space, the movement of which is limited by

$m, m < n-1$ of the bilateral differential constraints and by one unilateral holonomic constraint. We shall designate the described system by M. Let $q = q^1, \ldots, q^{nT}$ be the vector of the generalized coordinate;

$$T = \frac{1}{2}\dot{q}^T A(q)\dot{q},$$

- the kinetic energy of the system ($A(q)$ is the symmetrical positively definite $n \times n$ matrix), $\Pi(q)$ potential energy;

$$\Phi = \frac{1}{2}\dot{q}^T D(q)\dot{q},$$

- Rayleigh's dissipative function ($D(q)$ - is the symmetrical positive semi definite $n \times n$ matrix). We shall take that the differential constraints are of the following form

$$B^T(q)\dot{q} = 0, \tag{1}$$

($B(q)$ is the matrix of $n \times m$ and $\text{rank}B = m$). We assume that the unilateral holonomic constraint in the form

$$b^T q \geq 0, \tag{2}$$

where b–constant n vector column. Generality of consideration is not reduced by this assumption because the unilateral constraint

$$f(q) \geq 0, \tag{3}$$

can, by a special choice of generalized coordinates, always be reduced to the form (2).

Whereby the function $f(q)$ is required to have the corresponding degree of smoothness. The system under consideration is characterized by impact interactions. However, we shall consider exclusively motions in the time intervals when the impact interactions do not exist. Those motions represent the solutions to Lagrange's equations with the constraint multipliers:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = -\Pi'(q) - D(q)\dot{q} + B\lambda + R, \quad B^T(q)\dot{q}, \quad b^T q \geq 0, \tag{4}$$

where: $\lambda = \lambda_1, \ldots, \lambda_m{}^T$ – is the column vector of arbitrary coefficients; $\Pi'(q) = \partial\Pi/\partial q$; R is the reaction originating from the constraints (2). From the standpoint of the classic model of dynamics of system with uniteral constraints, in the time interval when the trajectory belong's to the plane $\sigma = \left\{q \in R^n,\ b^T q = 0\right\}$, the reaction

$$R = \lambda_{m+1} b, \tag{5}$$

whereby

$$\lambda_{m+1}(t) \geq 0. \tag{6}$$

If the system does not more along the plane σ, that is $b^T q > 0$,

$$R = 0. \tag{7}$$

Alongside with system M, let us also consider the "auxiliary" system M^* in $(n-1)$–dimensional configuration space, which is obtained by substituting the unilateral constraint (2) by the bilateral one having the form

$$b^T q = 0.$$

The movement of system M^* is at the same time also the movement of system M, if the condition (6) is fulfilled.

Let $\Pi'(q_0) = B(q_0)\lambda(q_0, 0) + \lambda_{m+1}(q_0, 0)b$, then $q = q_0$ is the equilibrium position of system M. The point $q = q_0$ can be either the equilibrium position of the first kind (in which case the reactions of constraints are different from zero, and $\Pi'(q_0) \neq 0$) or of the second kind (then the reactions of constraint are equal to zero, and $\Pi'(q_0) = 0$). It is clear that the point $q = q_0 \in \sigma$ is also the equilibrium position of system M and system M^*. If $q = q_0 \in \sigma$ is the equilibrium position of the second kind for system M, then, $q = q_0 \in \sigma$ is the equilibrium position of the second kind for system M^*. The converse statement is not however, true. For instance, if

$$\Pi'(q_0) = \psi b, \quad \psi \neq 0 \tag{8}$$

(that is $B(q_0)\lambda(q_0, 0) = 0$ and $\lambda_{m+1} = \psi$) then $q = q_0 \in \sigma$ is the equilibrium position of the second kind for system M^*, whereas it is of the first kind for system M.

In the case of holonomic dissipative systems with bilateral constraints, V.V. Kozlov proved that, when the time $t \to -\infty$, the centre of attraction of which is in the equilibrium position, the trajectories with asymptotic behaviour exist. The proof is derived on the following conditions [1]

(1) *$A(q)$, $\Pi(q)$, $D(q)$ are infinitely differentiable functions for each q,*

(2) *$D(q)$ is the positively definite matrix for each q.*

(3) *The first nontrivial form of Π_p of Maclaurin's potential energy development has not the minimum in the point $q = 0$.*

(4) *$p \geq 2$.*

Corollary: Let the following conditions (1), (2), (3) and (4) hold. Then

$$q = 0, \quad \dot{q} = 0,$$

is an unstable equilibrium.

For $p = 2$ this is Lyapunov's result [1]. Then the asymptotic trajectories are of the following form

$$q(t) = ce_1 e^{\chi t} + o(e^{\chi t}), \tag{9}$$

when $t \to -\infty$, where $c = const > 0$; $\chi = const. > 0$; $o(.)$ - designates a vector function that is infinitely small when its argument tends to zero; e_1 - unit vector which begins at $q = 0$. For $p \geq 3$ the asymptotic trajectories could be expressed in the following way:

$$q(t) = c \cdot (-t)^{-1/(p-2)} \cdot e_* + o((-t)^{-1/(p-2)}), \tag{10}$$

when $t \to -\infty$. [1] Here e_* - is the vector that defines the point in the unity sphere $\|q\| = 1$, in which Π_p has the minimum; $\|u\| = u^T D(0) u^{1/2}$, u - is n vector column.

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[1]In the original version V. V. Kozlov has given asymptotic solutions in the form $q(t) = \sum_{k=1}^{\infty} a_k(t) e^{\rho k t}$, $\rho = const. > 0$, for $p = 2$, i.e. $q(t) = \sum_{k=1}^{\infty} a_k(-t) e^{-k/(p-2)}$. for $p \geq 3$. In the first case $a_k(\cdot)$ are polinomial vector function depending on t, and in the other case the polynomial vector function depending on $\ln(-t)$. Also the coefficient by the first therm is ce_*, $c =$

In our considerations, we shall first extend the result given in [1] to the nonholonomic dissipative systems with the bilateral constraints (system M^* belongs to this class of mechanical systems too), and then also to system M. As in [1] we assume that $\Pi(q)$, $A(q)$, $D(q)$ and $B(q)$ are infinitely differentiable functions. In the absence of dissipative forces, our results also hold in the case of mechanical systems in which $A(q)$, $\Pi(q)$ and $B(q)$ have the degree of smoothness which is lower than the infinite differentiability. Assuming that our system is "infinitely differentiable" the proofs for our statement are based upon the method applied in [1]. When the "smoothness degree of the system" is of a lower order than the infinite differentiability, we shall employ the method used in [5], [7], which is basically a modification of the method used in [2].

The following notation will be used:

$$P^0 = \left\{ q \in R^n : b^T q = 0 \right\}, \quad P^+ = \left\{ q \in R^n : b^T q > 0 \right\}. \qquad (11)$$

Since the constraints (1) make it possible for the mechanical system to occupy any position in the space $P^0 \cup P^+$, we say that it is the space of possible motions. Let the potential energy of the dissipative system with the bilateral holonomic and nonholonomic constraints in the equilibrium system $q = 0$ have a strict local minimum. Then, the equilibrium $q = 0$, $\dot{q} = 0$ is stable. The inversion of this statement is the result given in [1]. The statement (in the absence of dissipative forces known as Lagrange - Dirichlet theorem) can also be extended to system M: if in the area of possible motions $P^0 \cup P^+$ in the equilibrium position $q = 0$ the potential energy has a strict local minimum, then the equilibrium $q = 0$, $\dot{q} = 0$ is stable. Our considerations of the equilibrium instability of system M also represent, the inversion of this statement.

$\left(-1/\left(p\Pi_p\left(c_*\right)(p-2)\right)\right)^{1/(p-2)}$. If A, Π and D are only infinitely differentiable function, the series diverges. But, based on A. N. Kuznecov's theory (see [1]), each series are preceded by two forms, which formaly satisfy equations of motion (4), the solution of which is $q(t)$ so that

(1) $q(t) \to 0$ when $t \to \infty$ and

(2) the two series shown above are the asymptotic series $q(t)$ when $t \to \infty$.

2 The instability theorem in case of systems with a bilateral constrainsts

Let us observe a nonholonomic dissipative system the motion of which is limited by the bilateral constraints. The motion of such a system is represented by the solutions of the equations:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = -\Pi'(q) - D(q)\dot{q} + B(q)\lambda, \quad B^T(q)\dot{q} = 0. \tag{12}$$

If $\Pi'(0) = 0$, then $q = 0$ the position of equilibrium. Assume that the functions $\Pi(q), A(q), B(q)$ and $D(q)$ are infinitely differentiable, and $\Pi(0) = 0$. Let

$$\Pi = \Pi_p + \Pi_{p+1} + \cdots, \tag{13}$$

be the Maclaurin series of Π. Here Π_s is a homogeneus form of degree s, so that $p \geq 2$. Note the subspace which is orthogonal to the nonholonomic constraint vectors in point $q = 0$,

$$\pi = \left\{ q \in R^n : B_0^T q = 0 \right\}, \tag{14}$$

where $B_0 = B(0)$.

If W is a function, we let $\hat{W}$ denote its restriction to π. It is clear that $\hat{\Pi}_s : \pi \to R$ is a homogeneous form of the degree of s.

Theorem 1. Let the following conditions be satisfied:

(a) Matrix $D(q)$ is positively definite for each q.

(b) Form $\hat{\Pi}_p$ in point $q = 0 \in \pi$ has not a local minimum.

Then $q = 0$, $\dot{q} = 0$ is an unstable equilibrium.

Theorem 1 represents an extension of the result from [1] to nonholonomic systems and is equal to it if $m = 0$. This theorem holds in the case system M^*. Let us note that on the conditions of Theorem 1, the constraints (1) can be integrable and non-integrable. If $D \equiv 0$, the following statement holds true [4]: *If $\hat{\Pi}_p$, $p \geq 2$ in point $q = 0$ has no minimum, then $q = 0$, $\dot{q} = 0$ is an unstable equilibrium.* Consequently, Theorem 1 can also be formulated in the following way: The added forces of the complete dissipation do not affect the instability under the conditions of the theorem given in [4].

For $p = 2$ Theorem 1 has been proved in [10]. Let us assume that $p \geq 3$. Before proving Theorem 1, let us show that a result of general which refers to the differential equations of the form

$$\dot{q} = f_{p-1}(q) + O(\|q\|) + O(\|\dot{q}\|^2) + O(\|q\|^p), \tag{15}$$

holds. We shall take that $O(\|\cdot\|)$ is the infinitely differentiable n-vector function in the neigbourhood of the point $q = 0$, and $f_{p-1}(q)$ - n vector form. Let $f_{p-1}(0) = 0$, $O(\|0\|) = 0$ and $p - 1 \geq 2$. According to this $q(t) = 0$, $\dot{q}(t) = 0$ is the solution of the equations (15). Note the equation

$$\dot{q} = f_{p-1}(q). \tag{16}$$

Let us find out the solution of this equation the form

$$q(t) = a_*, \tag{17}$$

where a_* - is a n vector different from zero; μ - is a scalar. For such a solution to exist, it is necessary that vector a_* is a nontrivial solution of the algebraic equation

$$\mu a_* = f_{p-1}(a_*), \tag{18}$$

whereby $\mu = 1/(p-2)$.

Theorem 2. If there exists the point e_* upon the unit sphere $\|q\| = 1$, $\|u\| = u^T I u^{1/2}$ (I -unit $n \times n$ matrix) in which

$$f_{p-1}(e_*) = \chi e_*,$$

then, the equations (15) have the solution which asymptotically approaches, so that

$$\|q(t) \oplus \dot{q}(t)\| \to 0,$$

when $t \to -\infty$.

Corollary: Let the equations (15) describe the dynamics of a mechanical system. Under the conditions of Theorem 2, $q = 0$, $\dot{q} = 0$ is an unstable equilibrium.

The asymptotic solution is of the form

$$q(t) = \left(\frac{1}{\chi(p-2)}\right)^{1/(p-2)} e_*(-t)^{-1/(p-2)} + o\left((-t)^{-1/(p-2)}\right), \quad (19)$$

when $t \to -\infty$. So, the solution (19) consists of the sum of solutions, the "simplified" equation (16) and the member $o(\cdot)$ which is infinitely small in comparison with its argument when it approaches zero. The proof of Theorem 2 is contained in a hidden form in [1].

The proof of Theorem 1. We shall firstly show that the differential equations of motion (12) can be expressed in the form (15). For that purpose, let us express (12) in the following way:

$$A(q)\ddot{q} + \Gamma(q,\dot{q}) = -\Pi'(q) - D(q)\dot{q} + B(q)\lambda, \quad B(q)^T\dot{q} = 0, \quad (20)$$

$\Gamma(q,\dot{q})$ denotes the square n - vector function, with respect to $\dot{q}$. Let us eliminate the coefficients $\lambda = \lambda_1, \ldots, \lambda_n{}^T$ in the following way: let us find $\dot{q}$ from (20) (it is possible, since $\det D \neq 0$) and then substitute the equations thus obtained in the remaining m equations (20). In that way, we obtain a system of n equations of the form

$$\dot{q} = D^{-1}(q)\left(I - C(q)\right)\Pi'(q) - G(q,\dot{q}) - O\left(\|\ddot{q}\|\right), \quad (21)$$

where $C(q) = B(q)\left(B(q)^T D(q)^{-1} B(q)\right)^{-1} B(q)^T D(q)^{-1}$; $G(q,\dot{q})$ is a square n - vector function with respect to $\dot{q}$. Without reducing the generality, we can assume that the generalized coordinates are chosen so that

$$D(q) = I + \overline{D}(q), \quad \overline{D}(0) = 0. \quad (22)$$

Taking also into account (21) as well as the fact that $\Pi(q), A(q), D(q), B(q)$ are infinitely differentiable functions, equations of motions (21) can be represented in the form of the equations (15),

$$\dot{q} = -\left(I - C_o\right)\Pi_p'(q) + O\left(\|\dot{q}\|^2\right) + O\left(\|\ddot{q}\|\right) + O\left(\|q\|^p\right). \quad (23)$$

Notice that $I - C_0$ is the projector upon the subspace π, $C_0 = C(0)$. The right hand side of the equations (23) is infinitely differentiable.

Let $e_* \in \pi$ be a unit sphere point at which $\hat{\Pi}_p$ has the minimum. According to the theorem of the conditional extreme, it further follows that

$$\Pi_p'(e_*) = -\chi e_* + B_0 \nu. \tag{24}$$

Thereby,

$$\chi = -p\hat{\Pi}_p(e_*), \tag{25}$$

and, as $\det B_0^T B_0 \neq 0$,

$$\nu = \left(B_o^T B_0\right)^{-1} B_o^T \Pi_p'(e_*), \tag{26}$$

Since $\hat{\Pi}_p$, basing upon condition (b) of Theorem 1, has negative values, $\chi > 0$; ν is m - vector column. By substituting (26) in (24), we obtain

$$(I - C_o)\,\Pi_p'(e_*) = -\chi e_*. \tag{27}$$

Thus the existence of the vector e_* represents a necessary condition of the Theorem 2, and hence the asymptotic solution of the equations (12) in the form (19). Finally, from the fact that the asymptotic solution does exist, and that the equations (12) are autonomous, it follows that $q = 0, \dot{q} = 0$ is an unstable equilibrium. Theorem 1 holds.

Note. If $D = 0$ and the constraints (2) are not present, in [4] it has also been shown that there exist asymptotic solutions, which are of the form

$$q(t) = e_o e^{\chi t} + o\left(e^{\chi t}\right), \tag{28}$$

when $t \to -\infty$, for $p = 2$, and

$$q(t) = \left(\sqrt{\frac{2\chi}{p}\frac{p-2}{2}}\right)^{2/(2-p)} e_0(-t)^{-2/(p-2)} + o\left((-t)^{-2/(p-2)}\right), \tag{29}$$

when $t \to -\infty$, for $p \geq 3$. Therefore, $e_0 \in \pi$ is a vector which determines the point upon the sphere $\|q\|_0 = 1$, where $\hat{\Pi}_p$ has the minimum (here $\|\cdot\|_0$ - denotes the norm determined by the kinetic energy calculated in point $q = 0$). In the general case our e_* and e_0 in [4] do not overlap.

3 The instability theorem in case of systems with a unilateral constraint

In this part of the paper, we shall consider the problem of the equilibrium instability of system M. We shall only study the equilibrium positions belonging to the subspace

$$\pi = \left\{ q \in R^n : B_0^T q = 0, b^T q = 0 \right\}. \tag{30}$$

Let $\Pi(q), A(q), B(q)$ and $D(q)$ in the neighbourhood of the point $q = 0$ be infinitely differentiable functions ($q = 0$ the equilibrium position and $\Pi(0) = 0$). In the general case, Maclaurin's development of the potential energy is of the form

$$\Pi(q) = c^T q + \Pi_p(q) + \ldots, \qquad p \geq 2, \tag{31}$$

where $c = (c_1, ..., c_n)^T = const$. In the case of the subsidiary system M^* (see Section 1), we shall eliminate the constraint multipliers $\lambda = (\lambda_1, ..., \lambda_m)^T$ and λ_{m+1} (the elimination procedure is exactly the same as the one used in the proof of Theorem 1),

$$\left\{ \begin{array}{c} \lambda \\ \lambda_{m+1} \end{array} \right\} = \left(\left[\begin{array}{c} B^T \\ b^T \end{array} \right] D^{-1} \left[\begin{array}{cc} B & b \end{array} \right] \right)^{-1} \left[\begin{array}{c} B^T \\ b^T \end{array} \right] \Pi' + O(\|\ddot{q}\|) + O\left(\|\dot{q}\|^2\right). \tag{32}$$

As the vectors of nonholonomic constraints and the vector b are linearly independent for every q, the inverse matrix in (32) exists for every q.

Without reducing generality, we can take that $D(q) = I + \overline{D}(q)$, $\overline{D}(0) = 0$, where is the unit matrix $n \times n$. Basing upon this and the fact that $D(q)$ and $B(q)$ are infinitely differentiable functions, we obtain that

$$\left\{ \begin{array}{c} \lambda \\ \lambda_{m+1} \end{array} \right\} =$$

$$\left[\begin{array}{cc} B_o^T B_o & B_o^T b \\ b^T B_o & b^T b \end{array} \right]^{-1} \left[\begin{array}{c} B_o^T \\ b^T \end{array} \right] \Pi' + O(\|\ddot{q}\|) + O\left(\|\dot{q}\|^2\right) + O(\|q\|)\,\Pi', \tag{33}$$

The proofs of the instability theorems will consist of two stages: we shall first show that system M^* has an asymptotic solution, and, subsequently, that this also represents the solutions of the motion equations of system M. The condition that the motion $q(t)$ of system M^* is at the same time the motion of system M can be written in the following form:

$$\lambda_{m+1}\left(q(t), \dot{q}(t), \ddot{q}(t)\right) \geq 0. \tag{34}$$

Let us first consider the case when $q = 0$ is the equilibrium position of the first kind, with the form $\Pi'(0) = \psi b$, $\psi > 0$. Let $\overline{\Pi}_p$ designate the restriction of the function Π_p upon the subspace π'.

Theorem 3. The equilibrium $q = 0, \dot{q} = 0$ is unstable if the following conditions are fulfilled:

(a) Matrix $D(q)$ is positively definite for each q.

(b) The form $\overline{\Pi}_p$ in point $q = 0$ has no minimum.

Proof of Theorem 3. Point $q = 0 \in \pi'$ is the equilibrium position of the second kind for system M^*. Function $\overline{\Pi}_p$, at some point $e_* \in \pi'$ of the unit sphere has a minimum as well as $\overline{\Pi}_p(e_*) < 0$. Keeping in mind Theorem 1, we can conclude that the subsidiary system M^* has the asymptotic motion $q(t)$ so that $\|q(t) \oplus \dot{q}(t)\| \to 0$ when $t \to -\infty$. It still remains to prove that $q(t)$ is at the same time the motion of system M. Indeed,

$$\Pi'(0) = B_o \lambda(0) + \lambda_{m+1}(0)\, b = \psi b, \tag{35}$$

and therefrom, since the vectors of the nonholonomic vectors and vector b- are linearly independent,

$$\lambda_{m+1}(0) = \psi > 0,$$

and

$$\lambda_{m+1}\left(q(t), \dot{q}(t), \ddot{q}(t)\right) = \psi + O\left(q(t), \dot{q}(t), \ddot{q}(t)\right) \geq 0.$$

From the fact that system M has asymptotic motion and that the motion equations describing the dynamics of system M are autonomous, it follows that the equilibrium $q = 0, \dot{q} = 0$ is unstable. □

The case when $q = 0 \in \pi'$ is the equilibrium position of the second kind for system M (and, consequently, for system M* too) is of a more subtle problem. Now

$$\Pi(q) = \Pi_p(q) + \Pi_{p+1}(q) + \ldots, \quad p \geq 2,$$

and

$$\left\{ \begin{array}{c} \lambda \\ \lambda_{m+1} \end{array} \right\} =$$

$$\begin{bmatrix} B_o^T B_o & B_o^T b \\ b^T B_o & b^T b \end{bmatrix}^{-1} \begin{bmatrix} B_o^T \\ b^T \end{bmatrix} \Pi_p' + O(\|\ddot{q}\|) + O\left(\|\dot{q}\|^2\right) + O(\|q\|^p).$$

Since the matrix

$$\begin{bmatrix} B_o^T B_o & B_o^T b \\ b^T B_o & b^T b \end{bmatrix},$$

is symmetrical, it follows that there exist A_m, v and k so that

$$\begin{bmatrix} B_o^T B_o & B_o^T b \\ b^T B_o & b^T b \end{bmatrix} \begin{bmatrix} A_m & v \\ v^T & k \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & 1 \end{bmatrix},$$

where A_m is the symmetrical $m \times m$ matrix; $v - m$ column vector, I_m- unit matrix $m \times m$; k - scalar. From this matrix equation we obtain that

$$v^T = -\frac{1}{k} b^T B_0 \left(B_o^T B_0\right)^{-1},$$

and

$$k = 1/\left(b^T b - b^T C_0 b\right).$$

Taking into consideration the structure of the matrix C_0 (see paragraph 2), it is shown that $b^T b - b^T C_0 b > 0$ it follows that $k > 0$. Finally, we obtain that

$$\lambda_{m+1}(q, \dot{q}, \ddot{q}) = \frac{1}{k} b^T (I - C_0) \Pi_p' + O(\|\ddot{q}\|) + O\left(\|\dot{q}\|^2\right) + O(\|q\|^p),$$

where $k > 0$.

Let S designate the unit sphere with the centre in point $q = 0$. We shall also use the following designation

$$S^0 = \left\{ q \in R^n : B_0^T q = 0, b^T q = 0, \|q\| = 1 \right\},$$

$$S^+ = \left\{ q \in R^n : B_0^T q = 0, b^T q > 0, \|q\| = 1 \right\},$$

Theorem 4. Let matrix D be positively definite. The equilibrium $q = 0, \dot{q} = 0$ is unstable if one of the following conditions is fulfilled:

(a) At the set S^+ the restriction $\hat{\Pi}_p$ onto the subspace π has at least one point $e_* \in \pi$ at which $\hat{\Pi}_p$ has the local minimum and $\hat{\Pi}_p(e_*) < 0$.

(b) In the set $S^+ \cup S_0$, the form Π_p has the minimum point $e_* \in \pi'$, whereby $\Pi_p(e_*) < 0$, and e_*, and is not the minimum of the function upon $S \cap \pi$.

Proof of Theorem 4. Since $\hat{\Pi}_p$ has the minimum at point $e_* \in \pi \cap S^+$, then, in case (a), there exists the asymptotic motion which, during the whole time, lies in Γ^+, i.e., it is the motion of system M, but not of M^*. Basing upon Theorem 1, it follows that the equilibrium $q = 0, \dot{q} = 0$ is unstable.

We still have to study the case (b). Basing upon the conditions of the theorem, it follows that $e_* \in \pi'$, and therefrom also that e_* is the local minimum point of the form $\overline{\Pi}_p$. This means that system M^* has the asymptotic motion $q(t)$. Let us prove that $q(t)$ is also the motion of system M. If $p = 2$, we have that

$$\Pi_2'(q(t)) = ce^{\chi t}\Pi_2'(e_*) + o\left(e^{\chi t}\right),$$

when $t \to -\infty$, and for $p \geq 3$, respectively,

$$\Pi_p'(q(t)) = c^{p-1}(-t)^{-(p-1)/(p-2)}\,\Pi_p'(e_*) + o\left((-t)^{-(p-1)/(p-2)}\right),$$

when $t \to -\infty$. Now, as a contrast to the case when the equilibrium position is of the first kind, the sign λ_{m+1}, at sufficiently high t, is defined by the expression

$$b^T(I - C_0)\,\Pi_p'(e_*).$$

By using the conditions of Theorem 4, it can be shown for case 2), analogous to the procedure in [6] that

$$b^T (I - C_0) \Pi_p' (e_*) > 0.$$

This proves Theorem 4.

Theorems 4 and 5 represents an extension to the nonholonomic systems with the dissipation of the result from [6], [9], which refers to holonomic systems without dissipation.

4 On instability of equilibrium of systems in case of absence of dissipative forces

Let the following conditions be fulfilled:

(1) unilateral constraint and dissipative forces are not present ($D \equiv 0, b = 0$),

(2) the matrices $A(q)$ and $B(q)$ are twice continuously differentiable functions in R^n,

(3) $0 < \epsilon \leq 1, p \geq 2$,

(4) $\Pi(q), \Pi_p(q) : R^n / \{0\} \rightarrow R$ are, respectively, twice and three times continuously differentiable functions in $R^n / \{0\}, \Pi_p$ is the homogeneus function of degree p, and, for $i = 0, 1, 2$

$$\Pi^{(i)}(q) = \Pi_p^{(i)}(q) + O\left(\|q\|^{p+\epsilon-i}\right), \quad \text{when} \quad \|q\| \rightarrow 0.$$

Theorem 5. The equilibrium $q = 0, \dot{q} = 0$ is unstable if the for following conditions are fulfilled:

(a) the potential $\Pi(q)$ and the matrices fulfil the conditions (2), (3) and (4),

(b) the function $\hat{\Pi}_p$ at point $q = 0 \in \pi$ has no minimum.

Corollary 1. If the potential Π at point $q = 0$ has the maximum (it need not be strict) and $\hat{\Pi}_p \neq 0$ then, under the conditions (2), (3) and (4) $q = 0, \dot{q} = 0$ is the unstable equilibrium.

Corollary 2. If the function $\Pi_p(q)$ has no minimum at point $q = 0$ then, under the conditions (2), (3) and (4), the equilibrium $q = 0, \dot{q} = 0$ of the non-holonomic Chaplygin's systems is unstable.

The proof of Theorem 5 is included in [7] the proving technique will be similar to that used in Ref. [5], which is: the existence of asymptotic solutions is proved in the form of a sum which consists of a "basic" member when $t \to -\infty$ (see 28 and 29). In contrast to Chapter 2, the "basic" member is the solution the "simplified" equation, the form of which is:

$$\ddot{q} = -(I - C_o)\,\Pi_p'(q)$$

where $C(q) = B(q)\left(B^T(q)\,A^{-1}(q)\,B(q)\right)^{-1} B^T(q)\,A^{-1}(q)$;
$C_0 = C(0)$; I - unit $n \times n$ matrix.

Let us now consider the case when $D = 0$ and $b \neq 0$. On the basis of the result of Theorem 5, the same conclusions as in Theorem 3 and Theorem 4 may be derived. Proving is then completely analogous with that in those theorems.

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Miroslav Vesković
Faculty of Mechanical Engineering
University of Kragujevac
Dositejeva 19
36 000 Kraljevo
Yugoslavia

O stabilnosti ravnoteže disipativnog mehaničkog sistema

U radu se razmatra problem stabilnosti ravnoteže neholonomnog disipativnog sistema sa bilaterarnim i unilaterarnim vezama. Teoreme o nestabilnosti su dokazane pod pretpostavkom da su jednačne veze, kinetička i potencijalna energija i Rejlijeva disipativna funkcija beskonačno diferencijabilne funkcije, kao i da u položaju ravnoteže potencijalna energija ne ostvaruje minimum. Teoreme su generalizacija rezultata koji važe za holonomne sisteme, [1], [6], [9].