THEORETICAL AND APPLIED MECHANICS
24, pp. 125-138, 1998, UDK 539.194

Sonic waves in thermally conducting and dissociating gases

Arisudan Rai

Submitted 3 April 1996

Abstract

The propagation of sonic waves has been studied through thermally conducting and dissociating gases. The law of propagation is determined and it is found that the velocity of propagation relative to the normal gas velocity is the effective isothermal velocity of sound. The differential equations governing the growth and decay of sonic discontinuities are obtained and solved. The effects of dissociation, thermal conduction and that of the wave geometry on the global behaviour of the wave amplitude have been studied. It is found that there is an interesting competition between dissociation to resist the steepening tendency of a compressive weak wave to stabilize it and the thermal conduction effects to destabilize the wave. It is concluded that under dissociation effects, shock wave formation is either disallowed or delayed while thermal conduction accelerates the process of termination of a weak wave into a shock wave. Three cases, diverging waves, converging waves and plane waves have been studied separately with reference to the growth and decay behaviour of their amplitudes.

1 Introduction

During the last decades several researchers in the field of gas dynamics have taken keen interest in the problem of formation of shock waves. The present communication provides an analytical solution of the problem of growth of sonic discontinuities into shock waves after a finite critical time. Following the singular surface theory of Thomas [1], the problem of growth and decay of weak discontinuities has been investigated by several workers [2-8] using different material media. In a recent paper, Shankar and Prasad [9] have extended the paper [2] to include the unsteady motion ahead of the wave surface in which the discussion of growth equation is incomplete. Also the results obtained in [11] may create confusion in the minds of future researchers in the field. We find that not all compressive waves will grow into shock waves, i.e. there exists a critical value λ_c of the initial discontinuity $\lambda(0)$, such that waves with $\lambda(0) < \lambda_c$ damp out, waves with $\lambda(0) = \lambda_c$ take a stable wave form; and waves with $\lambda(0) > \lambda_c$ grow into a shock wave in finite time. Shyam and Sharma [5] and Pandey and Ram [6] studied the behaviour of weak discontinuities in radiating gas flows by neglecting thermal conduction effects, but in a gaseous flow under high temperature conditions it is more realistic to consider the thermal conduction effects along with dissociation effects. Using the ray theory Upadhyay [7] obtained the growth equation for a so9nic discontinuity propagating through thermally conducting gases but he did not discuss the growing and decaying tendencies. Rai and Sisodia [8] and Rai and Vishwakarma [9] studied the global behaviour of weak discontinuities in radiation gas dynamics and in magneto gas dynamics analytically. Ram and Gaur [10] discussed the sonic wave propagation through thermally conducting and dissociating gases but their discussions of growth equation is incomplete so that exact behaviour of waves is not fully understood. The main object of the present paper is to investigate the essential features of the effects of the dissociation, thermal conduction and that of the wave geometry on the global behaviour of sonic waves. In the following analysis, we shall neglect the molecular transport effects leading to viscosity and diffusion. The temperature range is assumed from 1000 K to 7000 K so that the only chemical reaction involved is that of dissociation and hence the contributions of energy from electronic excitation and ionization are neglected [12].

A simple dissociating gas is defined as a mixture resulting from a dissociation reaction in a symmetrical diatomic gas A_2 each A_2 molecule being made up from $2A_1$ atoms. The reaction is

$$A_2 + X \underset{K_r}{\overset{K_f}{\rightleftarrows}} 2A_1 + X,$$

where the species X can be either A_2 or A_1 and K_f and K_r are the reaction rate constants for the forward and reverse reactions. The rate of production of an atom mass per unit volume is given by Lightthill [13] in the form

$$\frac{\partial \alpha}{\partial t} + u\frac{\partial \alpha}{\partial x} = \frac{4\rho D^2 K_r (1+\alpha)}{R^2 T_d^2}\left(\rho_d (1-\alpha)\exp\left(-T_d/T\right) - \rho\alpha^2\right), \quad (1)$$

where ρ_d, $T_d D$, and R are respectively the characteristic density for dissociation, the characteristic temperature for dissociation, the dissociation energy per unit mass, the atom mass fraction or degree of dissociation and the gas constants for A_2. Although ρ_d is a function of temperature T, it has been seen that the variation of ρ_d over the temperature range for dissociation i.e. from 1000 K to 7000 K is very slight. Hence for practical purposes the useful simplification of ρ_d as a constant should lead to negligible errors.

Let us assume the existence of a moving surface $S(t)$ of a weak discontinuity called a "Sonic Wave" across which the flow parameters are continuous but there first and higher derivatives are discontinuous. The boundary conditions are

$$[p] = 0, \quad [\rho] = 0, \quad [u] = 0, \quad [\alpha] = 0, \quad [T] = 0,$$

$$\left[\frac{\partial p}{\partial x}\right] = 0, \quad \left[\frac{\partial u}{\partial x}\right] = 0, \quad \left[\frac{\partial \rho}{\partial x}\right] = 0, \quad (2)$$

where $[Z]$ denotes the discontinuity in the quantity enclosed. The geometrical and kinematical compatibility conditions of first and second orders due to [14] for a singular surface can be expressed in the form

$$\left[\frac{\partial z}{\partial x}\right]=B,\quad \left[\frac{\partial z}{\partial t}\right]=-BG,\quad \left[\frac{\partial^2 z}{\partial x^2}\right]=\bar{B},\quad \left[\frac{\partial^2 z}{\partial t^2}\right]=-\bar{B}G+\frac{\delta B}{\delta t},\tag{3}$$

where Z refers for any of the flow variables, B is a scalar function defined over $S(t)$, G is the velocity of the surface $S(t)$ into a uniform medium at rest and $\delta/\delta t$ is the operator of time derivatives as observed from the wave front itself.

2 Law of propagation

The basic equations governing the coupling of the thermally conducting and simple dissociating gas [13] are

$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+\rho\frac{\partial u}{\partial x}=0,\tag{4}$$

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\frac{\partial p}{\partial x}=0,\tag{5}$$

$$\rho\frac{\partial h}{\partial t}-\frac{\partial p}{\partial t}+\rho u\frac{\partial h}{\partial x}-u\frac{\partial p}{\partial x}-K\frac{\partial^2 T}{\partial x^2}=0\tag{6}$$

where u, p, ρ, and h, denote respectively the gas velocity, pressure, density and enthalpy of the gas mixture. K is the coefficient of thermal conduction.

The caloric and thermal equations of states of an ideal dissociating gas are given by [12]

$$h=(4+\alpha)\,RT+\alpha D,\tag{7}$$

and

$$p=(1+\alpha)\,\rho RT.\tag{8}$$

By virtue of equations (1), (7) and (8), the energy equation (6) can be written in the form:

$$\frac{\partial p}{\partial t}-\rho u\frac{\partial u}{\partial t}-\rho u^2\frac{\partial u}{\partial x}+\gamma_e p\frac{\partial u}{\partial x}+F(p,\rho,\alpha)-\frac{(1+\alpha)}{3}K\frac{\partial^2 T}{\partial x^2}=0,\tag{9}$$

where

$$F(p,\rho,\alpha)=\frac{4\rho D^2 K_r}{3R^2T_d^2}\left\{\left(\rho_d\left(1-\alpha\right)\right)e^{-T_d/T}-\rho\alpha^2\right)\left(-\rho\left(1+\alpha\right)\right)^2 D+3p\right\},$$

and

$$\gamma_e=\frac{4+\alpha}{3},$$

is the effective exponent of heat for the gas mixture.

Taking jumps in the equations (4), (5) and (1) with the help of (2) and (3), we obtain

$$(u-G)\,\zeta+\rho\lambda=0, \tag{10}$$

$$\rho\,(u-G)\,\lambda+\xi=0, \tag{11}$$

$$\mu\,(u-G)=0, \tag{12}$$

where

$$\left[\frac{\partial\rho}{\partial x}\right]=\zeta,\quad \lambda=\left[\frac{\partial u}{\partial x}\right],\quad \xi=\left[\frac{\partial p}{\partial x}\right]\quad\text{and}\quad \mu=\left[\frac{\partial\alpha}{\partial x}\right].$$

From the law of conservation of energy across $S(t)$, we have [14],

$$\left[\frac{\partial T}{\partial x}\right]=0. \tag{13}$$

Differentiating Eq. (8) with respect to x and then taking jump across the wave front, we get the relation

$$\xi=\frac{p}{\rho}\zeta=a_e^2\zeta, \tag{14}$$

where a_e is the effective isothermal velocity of sound.

From Eqs. (10), (11) and (14) we get

$$\zeta\left\{(u-G)^2-a_e^2\right\}=0. \tag{15}$$

The assumption that $S(t)$ is a regular singular surface implies that $\zeta \neq 0$, we have

$$(u-G)^2 = a_e^2. \tag{16}$$

The relation shows that the velocity of propagation relative to the normal gas velocity is the effective isothermal velocity of sound. From Eqs. (10), (11) and (14), we get

$$\zeta = \frac{\rho\lambda}{(G-u)} = \frac{\xi}{(G-u)^2}. \tag{17}$$

If the medium ahead of a sonic discontinuity is uniform and at rest, the sonic discontinuity will propagate with a constant isothermal speed of sound. In this case the relations (17) and (16) will reduce to the following forms

$$\zeta = \frac{\rho\lambda}{G} = \frac{\xi}{G^2}; \quad G^2 = a_e^2. \tag{18}$$

3 The growth and decay of sonic discontinuity

Differentiating Eqs. (4) and (5) with respect to x and making use of the second order compatibility conditions (3), we obtain

$$\frac{\delta\zeta}{\delta t} = G\bar{\zeta} - 2\zeta\lambda - \rho_o\bar{\lambda} + 2\lambda\Omega\rho_o, \tag{19}$$

$$\rho_o\frac{\delta\lambda}{\delta t} = G\zeta\lambda - \bar{\xi} - \rho_o\lambda^2 + \rho_o G\bar{\lambda}, \tag{20}$$

where

$$\bar{\zeta} = \left[\frac{\partial^2\rho}{\partial x^2}\right], \quad \bar{\xi} = \left[\frac{\partial^2 p}{\partial x^2}\right], \quad \bar{\lambda} = \left[\frac{\partial^2 u}{\partial x^2}\right],$$

and Ω is the mean curvature of the wave surface $S(t)$. Taking the jump of the equation (9), we get

$$\left(\frac{1+\alpha}{3}\right)K\left[\frac{\partial^2 T}{\partial x^2}\right] = \left(C_e^2 a_e - a_e^3\right)\zeta, \tag{21}$$

where

$$C_e = \gamma_e p/\rho.$$

Now differentiating equation (1) with respect to x and making use of geometrical and kinematical compatibility conditions of first and second order due to Thomas [14], we get,

$$-a_e\bar{\mu} = A\zeta,$$

where

$$\bar{\mu} = \left[\frac{\partial^2\alpha}{\partial x^2}\right],$$

$$A = \lfloor \frac{4D^2(1+\alpha)K_r}{R^2T_d^2}\rceil \left((1-\alpha)\rho_d \exp\left(-T_d/T\right) - \rho\alpha^2\right). \qquad (22)$$

Differentiating equation (8) twice with respect to x and then taking jumps, we get

$$\bar{\xi} = G^2\bar{\zeta} + \frac{3\rho K}{K}\left(C_e^2 a_e - a_e^3\right)\zeta - \frac{Ap}{(1+\alpha)G}\zeta. \qquad (23)$$

Making application of (23) in equations (19) and (20), we obtain

$$\frac{\delta\lambda}{\delta t} + \lambda\left(\left(\frac{3\rho_o R}{2K}\left(C_e^2 - a_e^2\right) - \frac{Ap}{(1+\alpha)G^2}\right) - a_e\Omega(t)\right) + \lambda^2 = 0. \qquad (24)$$

This equation governs the growth and decay of the sonic discontinuities propagating in an ideal dissociating and thermally conducting gas at rest. We define the jump discontinuity $\lambda(t)$ as a time dependent amplitude of the weak wave under study. Let $S(t_o)$ represents the sonic wave surface at time t_o and let σ represents the distance measured from $S(t_o)$ the normal trajectories to the family of surfaces $S(t)$ in the direction of propagation. Then $\sigma = G(t - t_o)$. Hence, we can write

$$\frac{\delta\lambda}{\delta t} = G\frac{d\lambda}{d\sigma}. \qquad (25)$$

Using equation (25) and $\sigma = Gt$ at time t in equation (24), we obtain

$$\frac{d\lambda}{dt} + \lambda\left(\left(\frac{3\rho_o R}{3K}\left(C_e^2 - a_e^2\right) - \frac{Ap}{(1+\alpha)G^2}\right) - a_e\Omega(t)\right) + \lambda^2 = 0. \quad (26)$$

The solution of the equation (26) is of the form

$$\lambda(t) = F(t)\left(\frac{1}{\lambda(0)} + \int_0^t F(\tau)\,d\tau\right)^{-1}, \quad (27)$$

where,

$$F(t) = \exp(-c_1 t)\left((1 - K_1 Gt)(1 - K_2 Gt)\right)^{-1/2},$$

$$c_1 = \left(3\rho_o R/2K\left(c_e^2 - a_e^2\right) - Ap/2(1+\alpha)G^2\right) > 0,$$

$$\Omega(t) = (\Omega_o - K_o Gt)/\left(1 - 2\Omega_o Gt + K_o G^2 t^2\right),$$

$$\Omega_o = (K_1 + K_2)/2, \quad \text{and} \quad K_o = K_1 K_2,$$

K_1 and K_2 are principal curvatures of $S(t_o)$, $\lambda(0)$ is the initial amplitude of the wave at time t_o.

Now we shall discuss the growth and decay behaviour of $\lambda(t)$ for diverging, converging and plane waves.

4 Case (1): Diverging waves

For diverging waves of curved surfaces, we have $K_1 < 0$ and $K_2 < 0$, so that the solution (27) is

$$\lambda(t) = I(t)\left(1/\lambda(0) + \int_0^t I(t')\,d(t')\right)^{-1}, \quad (28)$$

where

$$I(t) = \exp(-C_1 t)\left((1 + |K_1|\, a_e t)(1 + |K_2|\, a_e t)\right)^{-1/2}.$$

If $\lambda(0) > 0$, then the denominator of (28) is positive, finite and non-zero for every t over $[0, \infty]$. Since

$$\lim_{t\to\infty} I(t) = 0.$$

It follows that $\lambda(t)$ is a monotonic decreasing function of t and tends to zero as $t \to \infty$. This implies that the wave will decay and will be damped out ultimately.

If $\lambda(0) < 0$, then there exists a critical value λ_c of $|\lambda(0)|$ given by,

$$\lambda_c = \left(\int_0^\infty I(t)\, dt\right)^{-1} > 0. \tag{29}$$

If $|\lambda(0)| < \lambda_c$, then the denominator of (28) is still finite and non-zero for every t over $[0, \infty]$, and we have

$$\lim_{t\to\infty} |\lambda(t)| = 0,$$

which shows that under this case, even a compressive wave ($\lambda(0) < 0$) will not grow in time and will be damped out ultimately.

When $|\lambda(0)| = \lambda_c$, then both the numerator and the denominator of $\lambda(t)$ tends to zero as $t \to \infty$, hence applying de l' Hospital's rule, we get

$$\lim_{t\to\infty} |\lambda(t)| = C_1,$$

which implies that under this case, a compressive wave will neither grow into a shock wave nor decay out, and for a large time period it will assume a stable wave form of constant amplitude.

When $|\lambda(0)| > \lambda_o$, then there exists a finite and non-zero time t_c given by

$$\int_0^{t_c} I(t)\, dt = \frac{1}{|\lambda(0)|}, \tag{30}$$

such that

$$\lim_{t\to t_c} |\lambda(t)| = \infty.$$

This implies that in this case compressive weak wave will terminate in a shock wave after a finite critical time t_c.

From (29) and (30), we get

$$\frac{d\lambda_c}{dC_1} = \lambda_c^2 \int_0^\infty tI\left(t\right)dt > 0, \quad \frac{dt_c}{dc_1} = \frac{1}{I\left(t_c\right)} \int_0^{t_c} tI\left(t\right)dt > 0.$$

These inequalities imply that both λ_c and t_c increase with increasing C_1. But C_1 increases under dissociation effects and decreases under thermal conduction effects. This means that dissociation either disallows shock formation of delays it, while thermal conduction destabilizes because it accelerates the process of termination of a weak wave in a shock wave.

Differentiating (29) and (30) with respect to $|K_1|$, we get the following inequalities,

$$\frac{d\lambda_c}{d\left|K_1\right|} = \frac{1}{2}\lambda_c^2 a_e \int_0^\infty t\left(1+\left|K_1\right|a_e t\right)^{-1} I\left(t\right)dt > 0, \tag{31}$$

$$\frac{dt_c}{d\left|K_1\right|} = \frac{a_e}{2I\left(t_c\right)} \int_0^{t_c} t\left(1+\left|K_1\right|a_e t\right)^{-1} I\left(t\right)dt > 0. \tag{32}$$

Inequalities (31) and (32) show that the stability of a weak compressive wave will increase under curvature effects i.e. weak compressive waves with curved surfaces are more stable than plane waves.

5 Case (2): Converging waves

For converging waves, we have both $K_1 > 0$ and $K_2 > 0$. In this case $\lambda(t)$ is given by (27).

Let t^* be the smallest root of the equation

$$\left(1 - K_1 a_e t\right)\left(1 - K_2 a_e t\right) = 0.$$

If $\lambda(0) > 0$ and $K_1 \neq K_2$, then

$$\int_0^{t^*} F\left(t\right)dt < \infty,$$

and the denominator of (28) is positive and finite for every t over $(0, \infty]$. Hence it follows from (28) that

$$\lim_{t \to t^*} \lambda(t) = \infty,$$

because of curvature effects. This implies that in this case a converging wave forms a caustic within a finite time $t^* > 0$.

If $K_1 = K_2 > 0$, then

$$\int_0^t F(t)\, dt = \infty,$$

but

$$\lim_{t \to t^*} \lambda(t) = \lim_{t \to t^*} \frac{F'(t)}{F(t)} = \infty.$$

Hence in this case also a converging wave will form a caustic within a finite time t^*.

If $\lambda(0) < 0$ and $K_1 \neq K_2$, then there exists a critical value λ_c^* of $|\lambda(0)|$ given by

$$\lambda_c^* = \left\{ \int_0^{t^*} F(t)\, dt \right\}^{-1},$$

such that for $|\lambda(0)| > \lambda_c^*$, there exists a non-zero finite time $t_c < t^*$ so that

$$\int_0^{t_c} F(t)\, dt = \frac{1}{|\lambda(0)|} \quad \text{and} \quad \lim_{t \to t_c} |\lambda(t)| = \infty.$$

In this case a converging wave grows into a shock wave at $t = t_c$ because of nonlinear steepening effects, and the formation of a caustic under curvature effects will not occur. However, when $|\lambda(0)| = \lambda_c^*$, then there occurs simultaneous formation of a shock wave due to steepening and a caustic due to curvature effects. If $|\lambda(0)| < \lambda_c^*$, then a caustic will be formed under curvature effects, and the formation of a shock wave is ruled out. If $K_1 = K_2 > 0$, then $\lambda_c^* = 0$ and hence all compressive wave will terminate in shock waves before a caustic can be formed, no matter how small $|\lambda(0)|$ may be.

6 Case (3): Plane waves

For plane waves, we have $\Omega = 0$, so the solution of (26) has the form

$$\lambda(t) = \lambda(0)\exp(-C_1 t)\{1 + \lambda(0)/C_1(1 - \exp(-c_1 t))\}^{-1}. \quad (33)$$

From (33) it is obvious that for $\lambda(0) > 0$, we have

$$\lim_{t\to\infty} \lambda(t) = 0,$$

which implies that all expansion waves will decay out.

If $\lambda(0) < 0$, then there exists a critical time t_c given by

$$t_c = \frac{1}{c_1}\ln\left\{\frac{|\lambda(0)|}{|\lambda(0)| - c_1}\right\},$$

such that

(i) if $|\lambda(0)| < c_1$, then $\lim_{t\to\infty}|\lambda(t)| = 0$,

(ii) if $|\lambda(0)| < c_1$, then $|\lambda(t)| = |\lambda(0)|$, and

(iii) if $|\lambda(0)| > c_1$, then $\lim_{t\to t_c}|\lambda(t)| = \infty$.

Thus it follows that a shock wave will be formed under condition (iii), condition (ii) implies that the wave amplitude $\lambda(t)$ does not undergo distortion, which is a very special class of non-linear waves. Condition (1) ensures that even a weak compressive wave is not likely to grow into a shock wave and it will be damped out.

Acknowledgment: The author is thankful to Professor Rishi Ram for his valuable suggestions during the preparation of this work.

References

[1] T.Y. Thomas, J. Math. Mech., 6, 455 (1957).

[2] R. Ram, Appl. Sci. Res., 34, 93 (1978).

[3] A. Rai and M. Gaur, Acta Physica Polonica, A57, 5, 653 (1980).

[4] R. Ram and S. Srinivasan, ZAMP, 26, 307 (1975).

[5] R. Shyam and V.D. Sharma, Acta Astron, 8, 31 (1981).

[6] B.D. Panday and R. Ram, AIAAJ, 18, 855 (1980).

[7] K.S. Upadhyaya, tensor (N.S.), 21, 296 (1967).

[8] A.S. Rai & P.S. Sisodia, Acta Physica Polonica A, 82(3), 461 (1992).

[9] A.S. Rai & J.P. Vishwakarma, Int. J. of Engg. Sc., 32(4), 635 (1994).

[10] R. Ram and M. Gaur, Ind. Nat. Sc. Acad., 42(96), 473 (1976).

[11] R. Shankar and M. Prasad, ZAMP, 30, 937 (1979).

[12] J.F. Clarke, J. Fluid Mech., 7, 577 (1960).

[13] M.J. Lighthill, J. Fluid Mech., 2, 1 (1957).

[14] T.Y. Thomas, J. Math. Mech. 6, 311 (1957).

Arisudan Rai
Dept. of Mathematics
NERIST, Nirjuli - 791 109
Arunachal Pradesh, India

Zvučni talasi u disocirajućem termički provodljivom gasu

U radu se proučava prostiranje zvučnih talasa kroz termički provodljiv disocirajući gas. Zakon prostiranja je odredjen a pronadjeno je i da je brzina prostiranja u odnosu na brzinu za normalan gas efektivna izotermalna brzina zvuka. Diferencijalne jednačine uspona i pada

zvučnih diskontinuiteta su izvedene i rešene. Efekti disocijajacije, provodjenja toplote i geometrije talasa na globalno ponašanje amplitude talasa su posebno analizirani. Tri slučaja divergirajući talasi, konvergirajući talasi i ravni talasi su posebno proučavani i to sa posebnim osvrtom na ponašanje njihovih amplituda.