THEORETICAL AND APPLIED MECHANICS
24, pp. 1-11, 1998, UDK 517.972

Brachystochronous movement on a manifold

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Submitted 10 June 1997

Abstract

The paper presents an analysis of brachystochronous movement of a conservative system during the action of holonomous relations. More precisely, it deals with solving the assignment of moving a system on a configurational manifold from one given position, where it remains at rest, to another given position, for the shortest possible time. The differential equations of this movement are formulated, and several special cases are analyzed. The geometrical characteristics of the regions of the possible brachystochronous movements are also indicated.

1 Introduction and basic assumptions

We consider a mechanical system (M, L); M - n-dimensional configurational manifold, L - Lagrangian, differentiable function defined at the fiber bundle TM

$$L : TM \to R, \quad L = L_2 + L_1 + L_o, \tag{1}$$

where the indices denote the homogeneity degree in relation to the generalized velocities. We shall solve the following

Assignment: Define curve $c : I \to M$ ($I \subset R$ time interval) along which the system should move so that from the state of rest in location a it comes to location b for the shortest possible time.

We call curve c, which represents the solution of this assignment, the brachystochrone of the system (M, L), while the movement along it will be called brachystochronous movement. For the movement of the system given by the Lagrangian L the first integral holds, i.e.,

$$L_2 - L_o = h,$$

which can be written in the form

$$T - U = 0, \quad (T = L_2,\ U = L_o + h). \tag{2}$$

According to the above, the assignment of defining the brachystochronous movement represents the problem of optimization in relation to transporting of system (1) from the location a to the location b, under the restriction given by integral (2) and the additional condition on the left-hand side. It can be formulated in the following way:

$$\mathcal{J}(c(\cdot), t) = \int_0^t d\tau \to \inf_c,$$

$$c \in \{\gamma : [0, t] \to M |\, T(\gamma, \dot{\gamma}) - U(\gamma) = 0\}, \dot{\gamma}(0) = 0.$$

Let us denote the local chart on M with $x = (x_1, ..., x_n)$, and the time coordinate with $t \in R$. The local chart at the fiber bundle are $(x, \dot{x})$, $\dot{x} = dx/dt$. Integral (2) in the local chart is

$$a_{ij}(x)\, \dot{x}^i \dot{x}^j - 2U(x) = 0, \qquad i, j = 1, ..., n.$$

If we perform the substitution of the parameter $t \longmapsto s(t)$ with the strictly increasing function $s : R \to R$, $s(0) = 0$, we shall obtain

$$a_{ij}(x)\, x'^i x'^j \left(\frac{ds}{dt}\right)^2 - 2U(x) = 0, \quad x' = dx/ds. \tag{3}$$

In that way, the assignment gets the form

$$\mathcal{J}\left(x\left(\cdot\right)\right)=\int_{0}^{1}\left(\frac{a_{ij}\left(x\right)x'^{i}x'^{j}}{U\left(x\right)}\right)^{\frac{1}{2}}ds\rightarrow\inf, \tag{4}$$

(the multiplier $2^{-1/2}$ has been omitted, since it does not affect the solution of the assignment). We can notice that the subintegral function is a linear form along x', wherefrom it follows that integral (4) does not depend upon parameterization. Therefore, the integral limits can be considered to be fixed. In order to formulate the assignment correctly, it is necessary to select the region of the functions in which the solution is searched for. We shall select it in the following way. Let us denote with Ω_{ab} the set in parts of smooth functions[1] which satisfy the conditions $s(0)=a$, $s(1)=b$. It is possible to complete this set into an infinitely dimensional manifold, which will also be denoted with Ω_{ab}. The curve on the manifold Ω_{ab} is a differentiable homotopy

$$c:[0,1]\times(-\epsilon,\epsilon)\rightarrow M;\quad c:(s,\alpha)\rightarrow c(s,\alpha), \tag{5}$$

for which the following holds

$$c\left(0,\alpha\right)=a,\quad c\left(1,\alpha\right)=b,\quad\forall\alpha\in\left(-\epsilon,\epsilon\right),$$

$$c\left(s,0\right)=c\left(s\right),\quad\forall s\in[0,1].$$

The tangential vector on Ω_{ab} in point $c(s)$ is a vector field along $c(s)$ which is annulled in points a and b:

$$X_{s}\in T_{c(s)}M,\quad X_{o}=X_{1}=0.$$

As a brachystochrone, in the sense of the definition given above, we shall consider the curve from Ω_{ab} upon which the functional (5) has the weak minimum.

[1] in sense of definition [1]

2 Necessary conditions

If the curve $c(s) = c(s,0)$ is the point of the weak local minimum of the functional $\mathcal{J}(c(\cdot,\alpha))$ upon Ω_{ab}, then [1]

$$\frac{d}{d\alpha}|_{\alpha=1}\mathcal{J}(c(\cdot,\alpha)) = 0.$$

From this condition and from Hilbert's lemma, it follows that $c(s)$ is a continuous differentiable, and that it satisfies Euler-Lagrangian equations. For the functional (4), these equations are

$$\frac{d}{ds}\frac{a_{ij}x'^j}{V} = \frac{1}{2UV}\left(U\frac{\partial a_{jk}}{\partial x^i}x'^j x'^k - \frac{\partial U}{\partial x^i}a_{jk}x'^j x'^k\right), \tag{6}$$

where, for the sake of brevity, we introduce the designation

$$V = \left(U a_{ij}x'^i x'^j\right)^{\frac{1}{2}}.$$

When T is a positively definite square form on T_pM for $\forall p \in M$, the coefficient of the form can be taken as the metric tensor field. The manifold M metricized by the tensor field $a_{ij}(x)$ represents Riemann's manifold upon which Levi-Civita's connection can be defined, the coefficients of which Γ^i_{jk} $(\Gamma_{ij,k})$ represent Christoffel's symbols of the I (II) kind. Then, Eqs. (6) obtain the form

$$x''^i + \Gamma^i_{jk}x'^j x'^k - \left[\frac{1}{2T}\left(\Gamma_{jk,l}x'^j x'^k x'^l + a_{jk}x'^j x''^k\right) + \frac{1}{2U}\frac{\partial U}{\partial x^j}x'^j\right]x'^i + \\ + \frac{T}{U}a^{ij}\frac{\partial U}{\partial x^j} = 0, \\ \left(2T = a_{ij}x'^i x'^j\right). \tag{7}$$

The above equations determine the brachystochronous movement trajectory, while the movement itself is obtained as the complex mapping $x(s(t))$, after $s(t)$ has been calculated by inverting the function $t(s)$.

Example. The equations of the brachystochrone in R^2. In Euclidean space the system of the coordinates can be selected so that all $\Gamma^i_{jk} = 0$. In that case, Eq. (7) is reduced to

$$x''_i - \left(\frac{\delta_{jk} x'^j x''^k}{2T} + \frac{1}{2U} \frac{\partial U}{\partial x^j} x'^j \right) x'^i + \frac{T}{U} \delta^{ij} \frac{\partial U}{\partial x^j} = 0;$$

$$\delta_{ij}, \delta^{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$$

If there exists a cyclical coordinate, for instance x_2, the previous system is equivalent to Eq.

$$\frac{x''}{1 + (x')^2} + \frac{1}{2U} \frac{\partial U}{\partial x} = 0 \quad \left(x = x^1, x' = dx^1/dx^2 \right)$$

which has the first integral $\left[1 + (x')^2\right] U(x) = const.$

3 The brachystochrone upon a two-dimensional manifold

Let us consider a system with two degrees of freedom. Let us designate the local chart upon a configurational manifold with x^1 and x^2, and presume that x^2 is a cyclical coordinate. We shall search the equation of the brachystochrone in the form $x_1 = f(x_2)$. From (3), taking into account that

$$\dot{x}_1 = \frac{dx_1}{dx_2} \dot{x}_2 = x'_1 \dot{x}_2,$$

we obtain

$$\left(a_{11} (x'_1)^2 + 2a_{12} x'_1 + a_{22} \right) \dot{x}_2^2 - 2U = 0.$$

The functional (4) is of the form

$$\int_\gamma \mathcal{L}\left(x_1, x_1'\right) dx_2,$$

where

$$\mathcal{L} = \left(\frac{a_{11}\left(x_1'\right)^2 + 2a_{12}x_1' + a_{22}}{2U}\right)^{1/2}.$$

Euler-Lagrangian equation in this case has the first integral

$$\frac{\partial \mathcal{L}}{\partial x_1'} x_1' - \mathcal{L} = h,$$

wherefrom the equation of the brachystochrone is obtained

$$\left(2h^2 U a_{11} - a_{12}^2\right)\left(x_1'\right)^2 + 2a_{12}\left(2h^2 U - a_{22}\right) x_1' + \left(2h^2 U - a_{22}\right) a_{22} = 0. \tag{8}$$

According to the above, the assignment is reduced to solving the differential equation of the form

$$\varphi(x)\left(x'\right)^2 + 2\psi(x)\,\theta(x)\,\nu(x) = 0$$

which is known in the literature (E. Kamke, DIFFERENTIALGLE-ICHUNGEN).

Example. A material point moves along a smooth rotational surface in the force field which is invariant in relation to the group of rotations around the axis of the surface symmetry. If the axis of rotation is designated with O_z and the rotation angle with φ, the equation of the curve the rotation of which forms the surface is of the form $\rho = \rho(z)$ (ρ - distance between the point and the axis of rotation). Then,

$$T - U = g_1(z)\,\dot{z}^2 + g_2(z)\,\dot{\varphi}^2 - U(z) = 0,$$

$$\left(g_1 = 1 + \left(\frac{d\rho}{dz}\right)^2, \quad g_2 = \rho^2\right).$$

Coordinate φ is cyclical. The solution in the form $z(\varphi)$ is obtained from Eq. (8), which, in this case, is reduced to

$$2h^2 U\left[\left(1+\rho'^2\right)z'^2+\rho^2\right]=\rho^4, \quad \left(z'=\frac{dz}{d\varphi}\right).$$

After the final equation of the brachystochrone has been determined, the brachystochronous movement is obtained in the following way:

$$\rho^2\dot{\varphi}=c \Rightarrow t=\frac{1}{c}\int_0^{\varphi}\rho^2\left[z\left(\theta\right)\right]d\theta=\frac{1}{c}\Phi\left(\varphi\right)\Rightarrow \varphi=\varphi\left(t\right).$$

In a special case, when $\rho = a = const.$ (the rotational surface is a cylinder), the equation of the brachystochrone reads

$$2c^2U\left(z\right)\left(z'^2+a^2\right)=a^4,$$

i.e., the same as in R^2.

4 The region of possible brachystochronous movements

From the condition that the brachystochronous movement is expressed by (2), whereby $T(x,\dot{x}) \geq 0$ we find that the region of possible movement is

$$D=\left\{x\in M\left|U\left(x\right)\geq 0\right.\right\}.$$

When $\min U(x) < 0$, the region D represents a manifold with an end. We shall consider some characteristics of the region D for this case.

Suppose that the region boundary is smooth

$$\operatorname{grad} U(x) \neq 0 \quad \text{for} \quad x \in \partial D \quad (\partial D \neq 0).$$

It follows therefrom that the mechanical system has no equilibrium positions upon ∂D. The problem of determining the brachystochrone is reduced to solving the assignment

$$\int_\gamma \left(g_{ij}x'^i x'^j\right)^{1/2} ds \to \min, \quad \gamma \in \Omega_{ab}, \quad g_{ij} = a_{ij}/U(x).$$

The field tensor $g_{ij}(x)$ defines Riemann's metrics on *int* D (the restriction of the functions $a_{ij}(x)$ on *int* D and $g_{ij}(x)$ belong to the same conforming structure). On ∂D field $g_{ij}(x)$ is not defined. When x passes sequence $(x_n) \in$ *int* D, which converges to a certain point on ∂D, $g_{ij}(x) \to \infty$ $(n \to \infty)$.

Let Ω_{ab} denote the set of all continuous curves on D which connect points p and q (the set is not empty with arbitrary selection of points p and q, since we consider D as a connected set). Let us define the mapping of d: $D \times D \to R$ by the formula

$$d(p,q) = \inf \int_\gamma \left(g_{ij}x'^i x'^j\right)^{1/2} ds, \quad \gamma \in \Omega_{ab}. \tag{9}$$

Lemma 1 *Mapping d is the metrics upon D.*

Proof. Among the conditions which d must satisfy in order to be the metrics, we take that it is necessary to prove only one: $d(p,q) = 0 \Rightarrow p = q$, since the others are obviously satisfied. If p and q from *int* D, this implication can be proved in the familiar manner [1]. Suppose that $p \in \partial D$ and $q \in D$. Let γ be a continuous curve which connects points p and q, and which does not wholly lie in ∂D (it is clear that such a curve always exists). Let U be the neighbourhood of point p in set D and let (U, ψ) be a chart on the manifold selected in such a way that $\psi(p) = 0$, $\psi(m) = (x^1, \ldots . x^n)$, $m \in U$. Let us parameterize curve γ so that

$$\gamma : s \longmapsto \gamma(s), \quad s \in [0, s_1], \quad \gamma(0) = p, \quad \gamma(s_1) = q.$$

Then

$$d(p,q) = \int_\gamma \left(g_{ij} x'^i x'^j\right)^{1/2} ds = \int_0^{s_1} \left[(g_{ij} \circ \gamma)\, x'^i x'^j\right]^{1/2} ds. \tag{10}$$

Since $g_{ij}(x)$ is a positive definite tensor for each $x \in intD$, there exists the neighbourhood $V \subset U$ of point p and $\varepsilon > 0$, so that the following inequality holds:

$$\left(g_{ij} x'^i x'^j\right)^{1/2} > \varepsilon \left[< \gamma', d\left(x^1 \circ \gamma\right) >^2 + ... + < \gamma', d\left(x^n \circ \gamma\right) >^2\right]^{1/2} \tag{11}$$

Let us use s_o to denote the upper limit of those values of parameter s for which $\gamma(s) \in V$. Taking into account (10) and (11), we have

$$\begin{aligned} d(p,q) &= \varepsilon \int_0^{s_o} \left[< \gamma', d\left(x^1 \circ \gamma\right) >^2 + ... + < \gamma', d\left(x^n \circ \gamma\right) >^2\right]^{1/2} ds \\ &> \varepsilon \int_0^{s_o} \left| < \frac{d\left(x^j \circ \gamma\right)}{ds} \frac{\partial}{\partial x^j}, d\left(x^i \circ \gamma\right) > \right| ds \\ &\Rightarrow \; d(p,q) > \varepsilon \left| x^i(s) \right|, \; \forall i. \end{aligned}$$

which proves the lemma.

We can notice that the point after which the brachystochronous movement commences must lie upon ∂D. If ∂D does not contain the equilibrium positions of the system, there exists the region $B = \{x \in D \,|0 \leq U(x) \leq \varepsilon\}$ (the boundary ∂D) in which point $x(t)$ cannot remain infinitely long [2], (the so-called rejection neighbourhood area).

Theorem 1. *Let points a and b be from int D, whereby $b \in B$. There exists a brachystochrone which passes through points a and b.*

Proof. According to Hopf-Rinow's theorem, there exists a unique geodesic line γ which connects points a and b. We are going to show that this geodesic line can be extended up to a point on ∂D, which means that it represents a brachystochrone through points a and b.

Function $d(\cdot, m)$, $m \in \partial D$ is continuous and limited from below. Therefore, the mapping $\partial(\cdot) : \partial D \to R^+$ (the distance of the point $(\cdot)$ from ∂D) can be defined by the formula

$$\partial(b) := \inf d(b, m), \quad m \in \partial D.$$

Since b can be selected such that $\partial(b) < \delta$ ($\delta > 0$ is arbitrarily small), basing upon the theorem on the possibility of the local lengthening of the geodesic line [3], it follows that γ can be lengthened up to its cutting point with ∂D.

Let us now consider the movement along the brachystochrone. The commencing conditions are $x(0) \in \partial D$, $\dot{x}(0) = 0$. From the assumption that there are no equilibrium positions on system ∂D, basing upon the existence of the rejection area, it follows that $x(t)$ must come out from some neighbourhood of point $x(0)$ for a definite time. The brachystochronous movement between points $x(0) \in \partial D$ and $a \in D$ will exists if there are no stopping points on the corresponding brachystochrone, which must be specially analyzed.

References

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Brahistohrono kretanje na mnogostrukostima

U radu se rešava zadatak odredjivanja glatke krive na konfiguracionoj mnogostrukosti konzervativnog sistema po kojoj bi reprezentativna tačka trebalo da se kreće da bi iz stanja mirovanja u zadanom položaju za najkraće vreme stigla u neki drugi, vezama dopušten položaj. Takvo kretanje se zove brahistohrono.

Odredjeni su potrebni uslovi brahistohronog kretanja na konfiguracionoj mnogostrukosti i posebno razmotren slučaj dvodimenzijske mnogostrukosti. Ispitane su neke geometrijske karakteristike oblasti mogućih brahistohronih kretanja.