THEORETICAL AND APPLIED MECHANICS
23, pp. 83-104, 1997, UDK 531.01

Double zero eigenvalues in regions of divergence in autonomous non-potential systems

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Submitted 27 March 1997

Abstract

Perfect bifurcational dissipative systems with trivial fundamental paths under follower loads in regions of divergence instability, are considered. Such autonomous non-potential systems under certain conditions may exhibit limit cycles in critical states of divergence. This is due to the coupling of divergence and flutter instability occurring at a double zero eigenvalue. The conditions for a double zero eigenvalue in critical states of divergence are properly established. A 2-DOF and 3-DOF systems are used as models to illustrate the new findings.

1 Introduction

Elastic instability of *non-potential dissipative* systems has been the subject of a large amount of work within the context of classical analysis, either static or dynamic [1-4]. Interesting examples range from the Takoma Narrows bridge wind-induced collapse, to offshore structures, to aerospace structures under follower forces produced by jet and rocket thrusts, to fluid-structure interaction, to shell-type "ovalling" [3], oscillations of pipes conveying fluid, etc.

In this work we study *perfect bifurcational* systems with trivial fundamental paths which are acted upon by *partial follower loads* with given *nonconservativeness* parameter η. Attentions is restricted to critical states in the region of *divergence* which are comprehensively discussed in the presence of *viscous damping*. Recent findings based on numerical analyses [5,6] of a 2-DOF (Degree Of Freedom) model have revealed interesting dynamic bifurcational phenomena. Conditions for which a *static* bifurcation (at a zero eigenvalue) is transformed to a *dynamic* bifurcation (at a double zero eigenvalue) are properly established. Recall that in such a case the systems under discussion exhibit a *coupled flutter-divergence* instability (although the critical states [5] are equilibria related to the 1^{st} branching point).

The new results are based on the *stability* analyses of a 2-DOF and 3-DOF models.

2 Basic considerations and problem description

The analysis that follows treats structural systems which are discrete or have been discretized by some approximation technique. Thus, one can consider a general N-DOF non-linear dissipative system under a partial follower load λ with nonconservativeness loading parameter η. Attention is focused on *perfect bifurcational systems with trivial fundamental paths* which due to the above type of loading are *autonomous non-potential systems* [3]. Their response at any time $t > 0$ can be described in terms of generalized *displacements*, $q_i = q_i(t)$ and generalized *velocities*, $\dot{q}_i = \dot{q}_i(t)$, $(i = 1, ..., n)$, via Lagrange's equations of motion. These 2^{nd} order differential equations, using the transformation of variables [5]

$$y_i = q_i, \qquad y_{n+i} = \dot{q}_i, \qquad (i = 1, ..., n), \tag{1}$$

can be written for a system initially ($t = 0$) at rest as follows [3]

$$\dot{\mathbf{y}} = \mathbf{Y}(\mathbf{y}; \mathbf{k}; \lambda; \eta), \quad \mathbf{y} \in \mathbf{E}^{2n}, \quad \mathbf{k} \in \mathbf{E}^{n}, \quad \lambda, \eta \in \mathbf{E}, \qquad (2)$$
$$\text{subject to } \mathbf{y}^{o}(t = 0) = 0.$$

where $\mathbf{y} = (y_1, ..., y_{2n})^T$ is the *state* vector in the Euclidean space $\mathbf{E}^{2n}$, being a continuous function of t and λ for fixed η, with T denoting transpose; $\mathbf{Y} = \mathbf{Y}(Y_1, ..., Y_{2n})^T$ is a nonlinear vector-function which by assumption satisfies the Lipschitz conditions for all t, λ and η, at least in the domain of interest; $\mathbf{k} = (k_1, ..., k_n)^T$ is a stiffness parameter vector, while λ and η are the main control parameters for *static* and *dynamic* bifurcations as well as for the *stability* of *equilibria* and *limit cycles.*

The existence of all possible equilibrium states $\mathbf{y}^E$ can be established by setting the L.H.S. of eq. (2) to zero, i.e.

$$\mathbf{Y}\left(\mathbf{y}^E; \lambda; \eta\right) = 0. \qquad (3)$$

Since we are discussing with perfect bifurcational systems with trivial fundamental paths eq. (3) is satisfied by the *trivial* (zero) solution $\mathbf{y}^E \equiv 0$, regardless of the values of λ and η; i.e.

$$\mathbf{Y}(0; \lambda; \eta) = 0. \qquad (4)$$

At the critical state $\lambda = \lambda^o$ (depending on η), the system exhibits also another solution different from zero $\mathbf{y}^E \neq 0$ (postbuckling path); namely it displays a bifurcation [6].

3 Local analysis

Using a local (linear) analysis one can discuss the nature of the *eigenvalues* of the Jacobian matrix evaluated at a known solution $\mathbf{y}^*$, being either an equilibrium (singular) point $\mathbf{y}^E$ or a non-equilibrium (regular) point $\mathbf{y}^R$. Setting $\mathbf{y} = \mathbf{y}^* + \xi$ and employing a Taylor's expansion around $\mathbf{y}^E = 0$, after linearization, we get the following linear variational equation

$$\dot{\xi} = \delta\mathbf{Y}^E = \mathbf{Y}^E_y \xi \qquad (5)$$

where $\mathbf{Y}_y^E = \mathbf{Y}_y^E(0;\lambda;\eta)$ is the Jacobian matrix [3] (evaluated at $y^E = 0$). The last matrix has the following characteristic equation [4,6]

$$\left|\rho^2[\alpha_{ij}] + \rho[c_{ij}] + [V_{ij}]\right| = 0, \qquad (6)$$

where $[\alpha_{ij}]$ is the *positive definite* function of the total *kinetic* energy; $[c_{ij}]$ is the *non-negative definite dissipation* matrix and $[V_{ij}]$ the *asymmetric stiffness* matrix.

Eq. (6) after expansion yields

$$f(\rho) = \rho^{2n} + a_1\rho^{2n-1} + a_2\rho^{2n-2} + \ldots + a_{2n-1}\rho + a_{2n} = 0, \qquad (7)$$

where

$$a_1 = \sum_{i=1}^{n} \tilde{c}_{ii} = \sum_{i=1}^{2n} \rho_i, \qquad a_{2n} = \det \mathbf{Y}_y = \det\left[\tilde{V}_{ij}\right] = \prod_{i=1}^{2n} \rho_i, \qquad (8)$$

with $[\tilde{c}_{ij}] = [\alpha_{ij}]^{-1}[c_{ij}]$, $\left[\tilde{V}_{ij}\right] = [\alpha_{ij}]^{-1}[V_{ij}]$ and ρ_i $(i = 1, \ldots, 2n)$ the Jacobian eigenvalues. All coefficients a_i, $(i = 1, \ldots, 2n)$ can be determined by means of Bocher's recurrence formulas [7]. It can be shown that a_1 is a function *only* of damping coefficients $\tilde{c}_{ii}$, while a_{2n} and a_{2n-1} of k_i, λ and η, and k_i, c_i, λ and η respectively, i.e.

$$\begin{aligned} a_{2n} &= a_{2n}(k_i, \lambda, \eta), \\ a_{2n-1} &= a_{2n-1}(k_i, c_{ij}, \lambda, \eta), \qquad i, j = 1, \ldots, n. \end{aligned} \qquad (9)$$

Note that a_{2n} leads to a n-degree algebraic polynomial in λ^c, while a_{2n-1} to an $(n-1)$-degree polynomial in λ^c. If there is no damping (unrealistic case) all odd terms in eq. (7) are zero, i.e. $a_1 = a_3 = \ldots = a_{2n-1} = 0$. Then, eq. (7) has automatically a double zero eigenvalue at the critical state (obtained via $a_{2n} = 0$).

The characteristic eq. (7) can also be written as follows

$$f(\rho) = \prod_{i=1}^{n} \left(\rho^2 + B_i\rho + C_i\right) = 0, \qquad (10)$$

where

$$\sum_{i=1}^{n} B_i = a_1 = \sum_{i=1}^{n} \tilde{c}_{ii}, \qquad \prod_{i=1}^{n} C_i = a_{2n}(k_i, \lambda, \eta),$$

and

$$a_{2n-1} = B_1 C_2 C_3 ... C_n + C_1 B_2 C_3 ... C_n + C_1 C_2 B_3 C_4 ... C_n + \\ ... + C_1 C_2 ... C_{n-1} B_n. \tag{11}$$

As is known the *precritical* state for a dissipative (viscous type) system is *asymptotically stable*, which implies [6]

$$B_i > 0, \quad C_i > 0, \quad (i = 1, ..., n). \tag{12}$$

4 Conditions for a double zero eigenvalue

The structural systems under discussion may lose their stability either through *static* (divergence) instability or through *dynamic* (flutter) instability depending on the value of the nonconservativeness parameter η varying from $\eta = 0$ (tangential load) to $\eta = 1$ (conservative load). The *boundary* between *existence* and *non-existence* of adjacent equilibria corresponds to a certain value of η, say $\eta = \eta_o$, which can be conveniently established according to a procedure presented by Kounadis [2]. This implies the solution of the following nonlinear algebraic equations

$$a_{2n}(\lambda^c, \eta; k_i) = \frac{\partial a_{2n}}{\partial \lambda}(\lambda^c, \eta; k_i) = 0, \tag{13}$$

with respect to λ^c (critical divergence load) and η for given values of the stiffness parameters k_i $(i = 1, ..., n)$. Clearly, since $a_{2n} = 0$ yields an algebraic polynomial of n-degree in λ^c, eqs. (13) constitute the *necessary conditions* for a *double* root in λ^c.

Let λ_o^c be the *smallest* positive value of λ^c (obtained from eqs. (13)) for which $\eta(\lambda_o^c; k_i) = \eta_o (> 0)$. It is clear that η_o (being a double or compound branching point) is either a *maximum* or a *minimum* of the function (curve) $\eta = \eta(\lambda^c; k_i)$, provided that $\partial a_{2n}(\lambda_o^c; k_i)/\partial\lambda^2 \neq 0$. The *boundary* between *existence* and *non-existence* of adjacent equilibria corresponds to the tangent at the *double critical point* $(\lambda_o^c; \eta_o)$ for given k_i of the curve η vs λ^c. A typical plot in which $(\lambda_o^c; \eta_o)$ may be a maximum or a minimum in the curve η vs $\lambda^c$is given in Fig. 1.

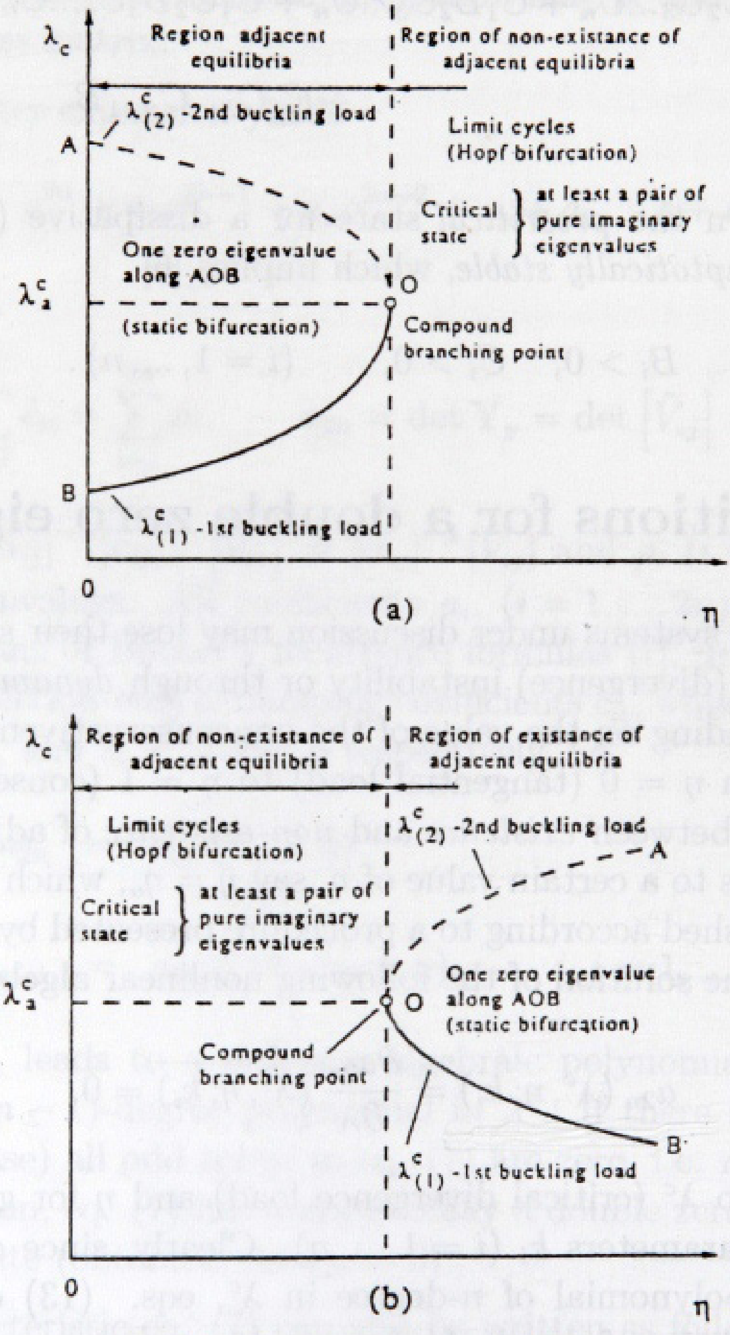

Fig. 1. The compound branching point 0 $(\lambda_o^c; \eta_o)$ boundary between the regions of existence and nonexistence of adjacent equilibria. Point 0 in the curve AOB may be either a maximum (a) or a minimum (b).

In what follows we discuss *critical states* only in regions of existence of adjacent equilibria (region of divergence). Without restricting the generality we may consider that the point $(\lambda_o^c; \eta_o)$ for given k_i corresponds to a *minimum* in the curve $\eta = \eta(\lambda^c, k_i)$ as shown in Fig. 2. We can also assume that this point is obtained by the coincidence of the 1^{st} and 2^{nd} buckling loads $\lambda_{(1)}^c$ and $\lambda_{(2)}^c$ since the coincidence of the 2^{nd} and 3^{rd} buckling loads $\lambda_{(2)}^c$ and $\lambda_{(3)}^c$ is rather unusual [1].

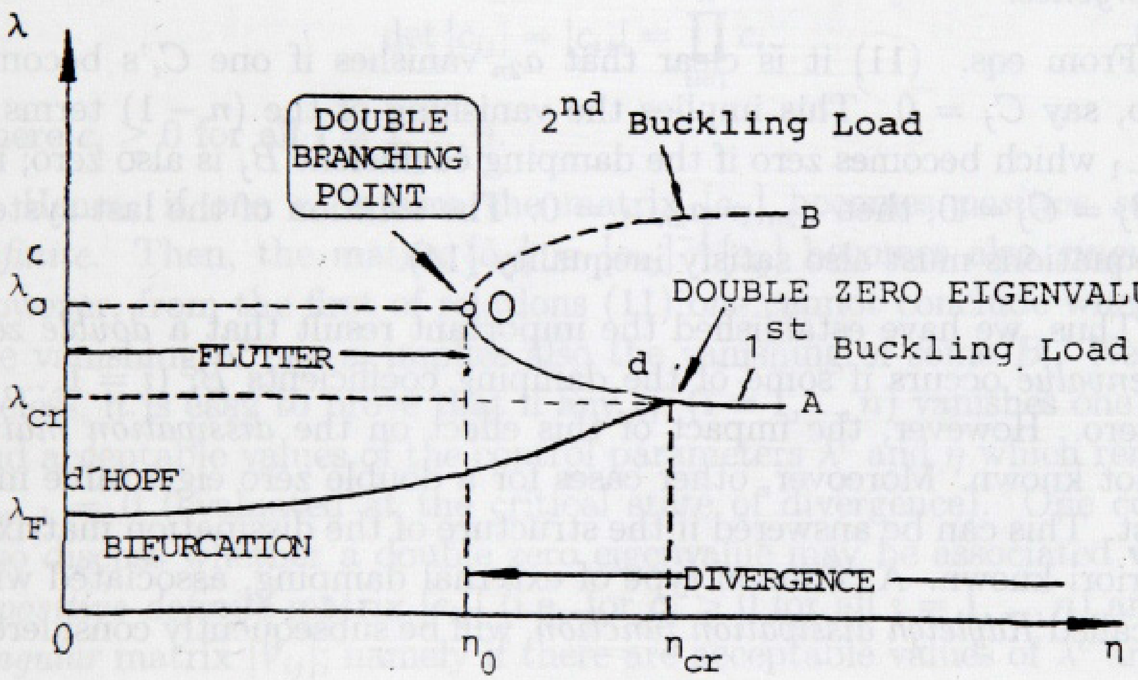

Fig. 2. Typical plot of λ^c vs η showing a double zero eigenvalue at point d.

In view of the above we are interesting in critical states associated with the first branching point in the region $\eta \geq \eta_o$. The (static) buckling loads $\lambda_{(1)}^c$ for which $a_{2n} = 0$ (curve AOB in Fig. 2) render the Jacobian matrix *singular* with one zero eigenvalue. Hence, a point $(\lambda_{cr}^c; \eta_{cr})$ on the curve AOB corresponds to a *double zero eigenvalue* if it satisfies both equations

$$a_{2n}\,(\lambda^c_{cr}, \eta_{cr}; k_i) = a_{2n-1}\,(\lambda^c_{cr}, \eta_{cr}; k_i; c_{ij}) = 0,$$

$$\text{such that} \quad \lambda^c_{cr} < \lambda^c_o \quad \text{or} \quad \eta_{cr} > \eta_o. \tag{14}$$

Recall that, as was shown [8], a *double zero eigenvalue* corresponds to the intersection of the locus of Hopf bifurcations with the curve of the first buckling load (i.e. of the curve η vs $\lambda^c_{(1)}$) as shown in Fig. 2. Thus, at a double zero eigenvalue we have an *interaction* of *flutter* and *divergence.*

From eqs. (11) it is clear that a_{2n} vanishes if one C_i's becomes zero, say $C_j = 0$. This implies the vanishing of the $(n-1)$ terms of a_{2n-1} which becomes zero if the damping coefficient B_j is also zero; i.e. if $B_j = C_j = 0$, then $a_{2n} = a_{2n-1} = 0$. The solution of the last system of equations must also satisfy inequality (14).

Thus, we have established the important result that a *double zero eigenvalue* occurs if some of the damping coefficients B_i $(i = 1, ..., n)$ is zero. However, the impact of this effect on the *dissipation matrix* is not known. Moreover, other cases for a double zero eigenvalue may exist. This can be answered if the structure of the dissipation matrix is a priori known. A common type of external damping, associated with so-called *Rayleigh dissipation function*, will be subsequently considered.

Rayleigh dissipation function

The Rayleigh dissipation function is a *non-negative* definite function of the form

$$F = (1/2)\, c_{ij}\dot{q}_i\dot{q}_j, \tag{15}$$

where the elements c_{ij} are functions of the damping coefficients of the members c_i, i.e. $c_{ij} = c_{ij}\,(c_i)$, where $i, j = 1, ..., n$. The Rayleigh *dissipation matrix* of an N-DOF cantilevered model [6,8-10] is of the following tri-diagonal form

$$[c_{ij}] = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 & 0 & 0 & \dots & 0 \\ -c_2 & c_2 + c_3 & -c_3 & 0 & 0 & \dots & 0 \\ 0 & -c_3 & c_3 + c_4 & -c_4 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & -c_{n-1} & c_{n-1} + c_n & -c_n \\ 0 & \dots & \dots & \dots & 0 & -c_n & c_n \end{bmatrix} \tag{16}$$

It can be shown that the determinant of the above dissipation matrix is given by

$$\det [c_{ij}] = |c_{ij}| = \prod_{i=1}^{n} c_i, \tag{17}$$

where $c_i \geq 0$ for all $i = 1, ..., n$.

Hence, if one c_i is zero the matrix $[c_{ij}]$ becomes *positive semi-definite.* Then, the matrix $[\tilde{c}_{ij}] = [\alpha_{ij}]^{-1} [c_{ij}]$ becomes also *singular.* However, from the first of relations (11) one cannot conclude whether the vanishing of one c_i implies also the vanishing of some B_i. Nevertheless, it is easy to prove that if any c_i, $(i = 1, ..., n)$ vanishes one can find acceptable values of the control parameters λ^c and η which render $a_{2n-1} = 0$ (evaluated at the critical state of divergence). One could also discuss whether a double zero eigenvalue may be associated with a *positive definite* matrix $[c_{ij}]$ (i.e. for $c_i > 0$ for all $i = 1, ..., n$) and a *singular* matrix $[V_{ij}]$; namely if there are acceptable values of λ^c and η which satisfy both these conditions.

The above will be established for cantilevered models with $n = 2$ and 3 degrees of freedom, as shown below. However, one could extend the present study to models with more than three degrees of freedom.

2-DOF model

This model shown in Fig. 3 is associated with the following matrices

$$[\alpha_{ij}] = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \quad [c_{ij}] = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix},$$

$$[V_{ij}] = \begin{bmatrix} 1 + \bar{k} - \lambda & -1 + \lambda(1-\eta) \\ -1 & 1 - \lambda\eta \end{bmatrix}, \tag{18}$$

where $\bar{k} = k_1/k_2$ (> 0) and c_i, $(i = 1, 2)$ *non-negative* damping coefficients.

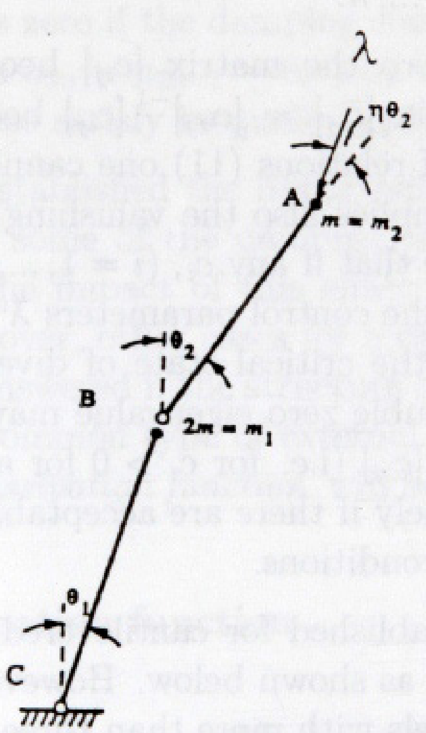

Fig. 3. Ziegler's two-degree-of-freedom model under partial follower load.

One can readily find that $a_3 = a_4 = 0$ imply

$$c_1 \left(1 - \lambda^c \eta\right) + c_2 \left(\bar{k} - 2\eta\lambda^c\right) = 0, \tag{19}$$

$$\eta\left(\lambda^{c}\right)^{2}-\eta\left(\bar{k}+2\right)\lambda^{c}+\bar{k}=0. \tag{20}$$

The smaller root of eq. (20) is

$$\lambda^{c}_{(1)}=\frac{1}{2}\left(\bar{k}+2-\sqrt{\left(\bar{k}+2\right)^{2}-\frac{4\bar{k}}{\eta}}\right). \tag{21}$$

Since the *double branching point* is defined by

$$\lambda^{c}_{o}\ =\ \frac{\bar{k}+2}{2},\qquad \eta_{o}=\frac{4\bar{k}}{\left(\bar{k}+2\right)^{2}}, \tag{22}$$

for a *double zero eigenvalue* according to inequality (14) we must also have

$$\lambda_{cr}=\lambda^{c}_{(1)}=\frac{1}{2}\left(\bar{k}+2-\sqrt{\left(\bar{k}+2\right)^{2}-\frac{4\bar{k}}{\eta}}\right)<\lambda^{c}_{o},\qquad \eta_{cr}>\eta_{o}. \tag{23}$$

One can readily show that if both damping coefficients c_1 and c_2 are *arbitrary* but *different from zero* (i.e. $c_1+c_2\neq 0$) a double zero eigenvalue is *excluded.* Indeed, in such a case eq. (19) yields

$$1=\eta\lambda^{o},\qquad\text{and}\qquad \bar{k}=2\eta\lambda^{o}, \tag{24}$$

which are not consistent unless $\bar{k}=2$. However, even in this case (i.e. $\bar{k}=2$), eq. (20) due to the 1^{st} of eqs. (24) leads to

$$\lambda^{c}_{(1)}=\lambda_{cr}=2, \tag{25}$$

which does not satisfy inequality (23) since the point corresponding to a *double zero* eigenvalue ($\eta_{cr}=0.5,\ \lambda_{cr}=2$) obviously *coincides* with the *double branching* point ($\eta_o=0.5,\ \lambda^c_o=2$) obtained from eqs. (22).

However, as was shown [4,5,8-10] a double zero eigenvalue may occur if either c_1 or c_2 is zero. In such a case the *dissipation* matrix becomes *singular* (i.e. positive semidefinite), since $\det\left[c_{ij}\right]=c_1c_2=0$. Namely, a double zero eigenvalue occurs when both matrices, i.e. the asymmetric *stiffness* matrix $[V_{ij}]$ and the symmetric *dissipation* matrix $[c_{ij}]$ become simultaneously *singular*. However, a double zero eigenvalue may also occur in case of a *singular* matrix $[V_{ij}]$ and *positive definite* matrix $[c_{ij}]$ (i.e. $c_1, c_2 \neq 0$). In this case one has to find positive values of $c_1$and c_2 which, when combined with acceptable value of $\bar{k}$, λ^c and η, satisfy eq. (19). Indeed, if $c_1/c_2 = 0.27511602$, $\eta = 0.33$, $\bar{k} = 0.5$ (implying $\eta_o = 0.32$, $\lambda_o^c = 1.25$) and $\lambda_{(1)}^c = 1.03240293$. then eq. (19) is obviously satisfied. In Fig. 4 one can see the corresponding phase-plane portrait $\left(\theta_1 \text{ vs } \dot{\theta}_1\right)$ associated with a limit cycle response.

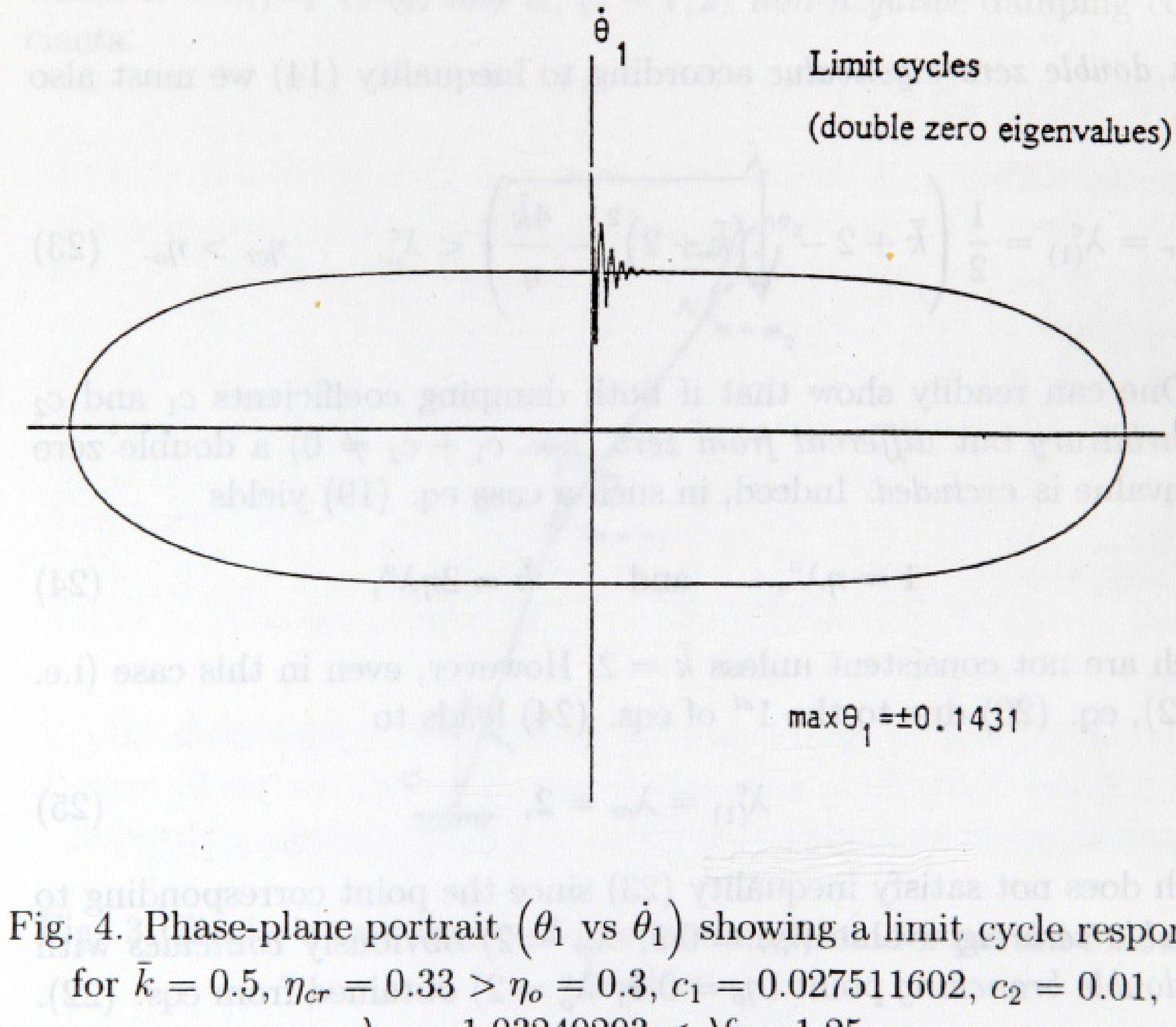

Fig. 4. Phase-plane portrait $\left(\theta_1 \text{ vs } \dot{\theta}_1\right)$ showing a limit cycle response for $\bar{k} = 0.5, \eta_{cr} = 0.33 > \eta_o = 0.3, c_1 = 0.027511602, c_2 = 0.01,$ $\lambda_{cr} = 1.03240293 < \lambda_o^c = 1.25.$

Finally, one can find the *limiting* value of η for the occurrence of a double zero eigenvalue. Solving eq. (20) for $\bar{k}$ and introducing its expression into eq. (19) we obtain

$$c_1 \left(1-\eta\lambda^c\right)^2 + c_2\eta\left(\lambda^c\right)^2\left(2\eta-1\right)=0. \tag{26}$$

Since, $1-\eta\lambda^c \neq 0$ and $c_i,\ (i=1,2),\ \eta,\ \lambda^c$ are *positive* quantities, eq. (26) is satisfied only if $\eta \leq 0.5$.

3-DOF model

The following matrices correspond to the 3-DOF model [8-10] shown in Fig. 5

$$[\alpha_{ij}] = \begin{bmatrix} 1+\bar{m}_1+\bar{m}_2 & 1+\bar{m}_2 & 1 \\ 1+\bar{m}_2 & 1+\bar{m}_2 & 1 \\ 1 & 1 & 1 \end{bmatrix},\ [c_{ij}] = \begin{bmatrix} \bar{c}_1+\bar{c}_2 & -\bar{c}_2 & 0 \\ -\bar{c}_2 & \bar{c}_2+\bar{c}_3 & -\bar{c}_3 \\ 0 & -\bar{c}_3 & \bar{c}_3 \end{bmatrix}, \tag{27}$$

$$[V_{ij}] = \begin{bmatrix} \bar{k}_1+\bar{k}_2-\lambda & -\bar{k}_2 & \lambda(1-\eta) \\ -\bar{k}_2 & 1+\bar{k}_2-\lambda & -1+\lambda(1-\eta) \\ 0 & -1 & 1-\eta\lambda \end{bmatrix}$$

where $\det[\alpha_{ij}] = \bar{m}_1\bar{m}_2$ and $\det[c_{ij}] = \bar{c}_1\bar{c}_2\bar{c}_3$ with $\bar{m}_1 = m_1/m_3$, $\bar{m}_2 = m_2/m_3$, $\bar{k}_1 = k_1/k_3$, $\bar{k}_2 = k_2/k_3$ and $c_i,\ (i=1,2,3)$ non-negative damping coefficients.

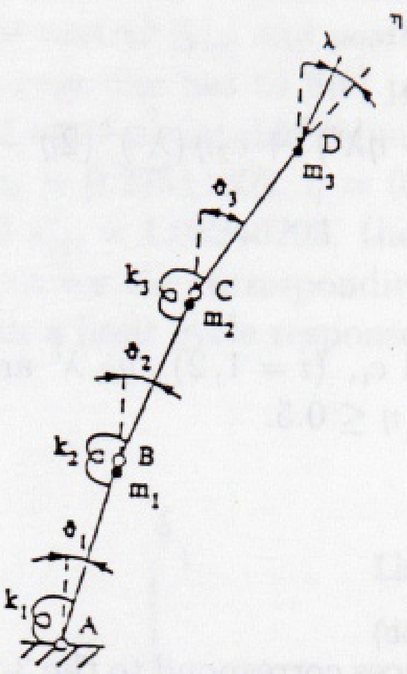

Fig. 5. 3-DOF cantilever model $ABCD$ under a partial follower compressive load at its tip.

One can show that $a_5 = a_6 = 0$ yield [8]

$$\bar{c}_1\left[\bar{k}_2+\eta\left(\lambda^c\right)^2-\lambda^c\eta\left(\bar{k}_2+2\right)\right]+\bar{c}_2\left[\bar{k}_1+2\eta\left(\lambda^c\right)^2-\lambda^c\eta\left(\bar{k}_1+3\right)\right]+$$

$$+\bar{c}_3\left[\bar{k}_1\bar{k}_2+2\eta\left(\lambda^c\right)^2-\lambda^c\eta\left(2\bar{k}_1+3\bar{k}_2\right)\right]=0, \tag{28}$$

$$\left(\lambda^c\right)^3-\left(\bar{k}_1+2\bar{k}_2+2\right)\left(\lambda^c\right)^2+\left(2\bar{k}_1+3\bar{k}_2+\bar{k}_1\bar{k}_2\right)\lambda^c-\frac{\bar{k}_1\bar{k}_2}{\eta}=0. \tag{29}$$

Since the signs of the terms are alternately positive and negative, eq. (29) has three *positive* roots.

Differentiation of the last equation yields

$$3\left(\lambda^c\right)^2-2\left(\bar{k}_1+2\bar{k}_2+2\right)\lambda^c+2\bar{k}_1+3\bar{k}_2+\bar{k}_1\bar{k}_2=0. \tag{30}$$

It can be proven [8] that the *smaller* root $\lambda^c_{(1)}$ of eq. (30)-being a double root of eq. (29)-corresponds to the *double branching point* (coincidence of the 1^{st} and 2^{nd} buckling loads $\lambda^c_{(1)} \equiv \lambda^c_{(2)}$). Hence

$$\lambda^c_o = \frac{1}{3}\left(A - \sqrt{A^2 - 3B}\right),$$

and due to eq. (28)

$$\eta_o = \frac{\bar{k}_1 \bar{k}_2}{\lambda^c_o \left[(\lambda^c_o)^2 - A\lambda^c_o + B\right]}, \qquad (31)$$

where

$$A = \bar{k}_1 + 2\bar{k}_2 + 2, \qquad B = 2\bar{k}_1 + 3\bar{k}_2 + \bar{k}_1\bar{k}_2.$$

For a double zero eigenvalue, in addition to eqs. (28) and (29), one must also have

$$\lambda_{cr} = \lambda^c_{(1)} < \lambda^c_o. \qquad (32)$$

One can show that if no one from c_1, c_2 or c_3 is zero a double zero eigenvalue cannot occur. Indeed, if $c_1 + c_2 + c_3 \neq 0$, then eq. (28) is satisfied if the following equations are satisfied

$$\eta\left(\lambda^c\right)^2 - \lambda^c\eta\left(\bar{k}_2 + 2\right) + \bar{k}_2 = 0,$$

$$2\eta\left(\lambda^c\right)^2 - \lambda^c\eta\left(\bar{k}_1 + 3\right) + \bar{k}_1 = 0, \qquad (33)$$

$$2\eta\left(\lambda^c\right)^2 - \lambda^c\eta\left(2\bar{k}_1 + 3\bar{k}_2\right) + \bar{k}_1\bar{k}_2 = 0,$$

where $\bar{k}_i$ $(i = 1, 2)$ and λ^c are taken *positive* values, while $0 \leq \eta \leq 1$. Under the circumstances, one can prove that there is no set of such values of k_i, λ^c and η satisfying the system of eqs. (29) and (33). Indeed, from these equations one can obtain

$$\eta = \frac{\bar{k}_1 \bar{k}_2}{\lambda^c \left[(\lambda^c)^2 - A\lambda^c + B\right]}, \qquad (34)$$

$$\eta = \frac{\bar{k}_2}{\lambda^c \left(\bar{k}_2 + 2 - \lambda^c\right)}, \qquad (35)$$

$$\eta = \frac{\bar{k}_1}{\lambda^c\left(\bar{k}_1 + 3 - 2\lambda^c\right)}, \tag{36}$$

$$\eta = \frac{\bar{k}_1\bar{k}_2}{\lambda^c\left(2\bar{k}_1 + 3\bar{k}_2 - 2\lambda^c\right)}. \tag{37}$$

From eqs. (34) and (36) it follows that

$$(\lambda^c)^2 - \left(\bar{k}_1 + 2\right)\lambda^c + 2\bar{k}_1 = 0, \tag{38}$$

from which we obtain

$$\lambda^c_{1,2} = \frac{1}{2}\left(\bar{k}_1 + 2 - \left|\bar{k}_1 - 2\right|\right). \tag{39}$$

Namely, either $\lambda^c_1 = \bar{k}_1$ or $\lambda^c_2 = 2$. Both these values are rejected. Indeed, if $\lambda^c_1 = \bar{k}_1 \neq 0$ from eqs. (36) and (37) we find respectively

$$\eta = \frac{1}{3 - \bar{k}_1}, \qquad \eta = \frac{\bar{k}_2}{3\bar{k}_2} = \frac{1}{3}, \quad (\text{for } \bar{k}_2 \neq 0). \tag{40}$$

However, the last relations are *not consistent* for $\bar{k}_1 \neq 0$. If $\lambda^c_2 = 2$ from eqs. (35) and (36) we find respectively

$$\eta = \frac{\bar{k}_2}{2\bar{k}_2} = \frac{1}{3}, \qquad (\text{for } \bar{k}_2 \neq 0), \qquad \eta = \frac{\bar{k}_1}{2\left(\bar{k}_1 - 1\right)}. \tag{41}$$

Clearly, the last relations are *not consistent* and thus the value $\lambda^c_2 = 2$ is also rejected.

The conclusion which is drawn from the above discussion is similar to that valid for the 2-DOF model. Namely, if $\det\left[c_{ij}\right] = \bar{c}_1\bar{c}_2\bar{c}_3 \neq 0$, then $a_5 \neq 0$ which *excludes* the case of a double zero eigenvalue.

However, a double zero eigenvalue may occur if either c_1 or c_2 or c_3 is zero. More specifically, if for given values of $\bar{k}_i$, $(i = 1, 2)$ one can find acceptable values for the control parameters λ^c and η for which the asymmetric *stiffness* matrix $[V_{ij}]$ becomes *singular* and the *dissipation* matrix $[c_{ij}]$ positive semi-definite, i.e. when

$$\det[V_{ij}] = 0, \qquad \det[c_{ij}] = \bar{c}_1\bar{c}_2\bar{c}_3 = 0, \tag{42}$$

then the *conditions* for a double zero eigenvalue are satisfied. The solution $\lambda = \lambda_{cr} > 0$, obtained from eqs. (42), is acceptable, if

$$\lambda_{cr} < \lambda_o^c, \tag{43}$$

where λ_o^c corresponds to the *double branching point* (coincidence of the 1^{st} and 2^{nd} buckling loads $\lambda_{(1)}^c$ and $\lambda_{(2)}^c$).

For instance, if $\det[c_{ij}] = 0$, due to $\bar{c}_2 = 0$ (while $\bar{c}_1\bar{c}_3 \neq 0$) and $\det[V_{ij}] = 0$, from eqs. (34), (35) and (37) one can find acceptable value of η, λ^c and $\bar{k}_1$, as functions of the parameter $\bar{k}_2$ (> 0). Indeed, from eqs. (34) and (35) we find that

$$(\lambda^c)^2 - 2\left(\bar{k}_2 + 1\right)\lambda^c + 3\bar{k}_2 = 0, \tag{44}$$

from which we obtain

$$\lambda_{1,2}^c = 1 + \bar{k}_2 \pm \sqrt{\bar{k}_2^2 - \bar{k}_2 + 1}. \tag{45}$$

On the other hand from eqs. (35) and (37) we get

$$\lambda^c = \frac{\bar{k}_2\left(3 - \bar{k}_1\right)}{2 - \bar{k}_1}. \tag{46}$$

If

$$\lambda_1^c = 1 + \bar{k}_2 - \sqrt{\bar{k}_2^2 - \bar{k}_2 + 1} > 0, \tag{47}$$

then eqs. (35) and (46) yield

$$\eta = \frac{1}{2 - \bar{k}_2 + \sqrt{\bar{k}_2^2 - \bar{k}_2 + 1}}, \qquad \bar{k}_1 = 2 - \frac{\bar{k}_2}{1 - \sqrt{\bar{k}_2^2 - \bar{k}_2 + 1}}. \tag{48}$$

Apparently, if $\bar{k}_2 > 1$ then $\bar{k}_1 > 1$. For instance if $\bar{k}_2 = 4$, then eqs. (47) and (48) give

$$\lambda_1^c = \lambda_{cr} = 1.394488725,$$

$$\eta_{cr} = 0.62283903, \tag{49}$$

$$\bar{k}_1 = 3.535183759,$$

$$\lambda_{cr} < \lambda_o^c, \tag{50}$$

then λ_{cr} is acceptable, since it implies a double zero eigenvalue at the 1^{st} branching point. Indeed, setting

$$A = \bar{k}_1 + 2\bar{k}_2 + 2 = 13.53518359,$$

$$B = 33.21110188, \tag{51}$$

we obtain

$$\lambda_o^c = \frac{1}{3}\left(A - \sqrt{A^2 - 3B}\right) = 1.464545428, \tag{52}$$

which due to eq. (49) satisfies inequality (50). Note also that

$$\eta_o = \frac{\bar{k}_1 \bar{k}_2}{\lambda_o^c\left[(\lambda_o^c)^2 - A\lambda_o^c + B\right]} = 0.621599832 < \eta_{cr}.$$

The curve η vs λ^c for the above values of $\bar{k}_1$ and $\bar{k}_2$ is given in Fig. 2.

A double zero eigenvalue may also occur in a *divergence* critical state (i.e. $\det[V_{ij}] = 0$) with a *positive definite* dissipation matrix $[c_{ij}]$(i.e. when $c_i \neq 0$ for $i = 1, 2, 3$). Indeed, for $\bar{k}_1 = 2$, $\bar{k}_2 = 0.5$, (implying $A = 5$ and $B = 6.5$) the curve η vsλ^c exhibits a *minimum* at $\lambda_o^c = 0.884930706$ and $\eta_o = 0.395330493$. For any divergence (critical) state in the vicinity of the point (λ_o^c, η_o) we can find suitable values of $c_i > 0$, (for $i = 1, 2, 3$) for which both eqs. (28) and (29) are satisfied. For instance, if $\lambda_{cr} = 0.80 < \lambda_o^c$ then $\eta_{cr} = 0.398089172 > \eta_o$. In this case one can choose $c_1 = c_3$ and $c_2/c_1 = 0.309027776$. Clearly, both eqs. (28) and (29) are satisfied for the above numerical values of $\bar{k}_1, \bar{k}_2, \lambda_{cr}, \eta_{cr}, c_1/c_3 = 1$ and c_2/c_1.

In view of the above, a double zero eigenvalue may be associated with a *singular* $[V_{ij}]$ and a dissipation matrix $[c_{ij}]$ which is either positive-*definite* or positive *semi-definite*.

It is worth noticing that the results in both models are also valid for *vanishing but non zero* damping; namely for *practically non-dissipative* systems. For instance, if $c_1 = 0$ and $c_2 \rightarrow 0$ the 2-DOF model exhibits a double zero eigenvalue under the conditions presented above. This is also true for the 3-DOF model, if $\bar{c}_2 = \bar{c}_3 = 0$ and $\bar{c}_1 = 0.01$. For instance, if $\bar{k}_1 = 6$, $\bar{k}_2 = 0.5$ (implying $A = 9, B = 16.5$) we find that a double zero eigenvalue occurs at $\lambda_{cr} = 0.63397$ $(< \lambda_o^c = 1.12917)$ and $\eta_{cr} = 0.42265$ $(> \eta_o = 0.3490)$. The corresponding phase-plane portrait is shown in Fig. 6.

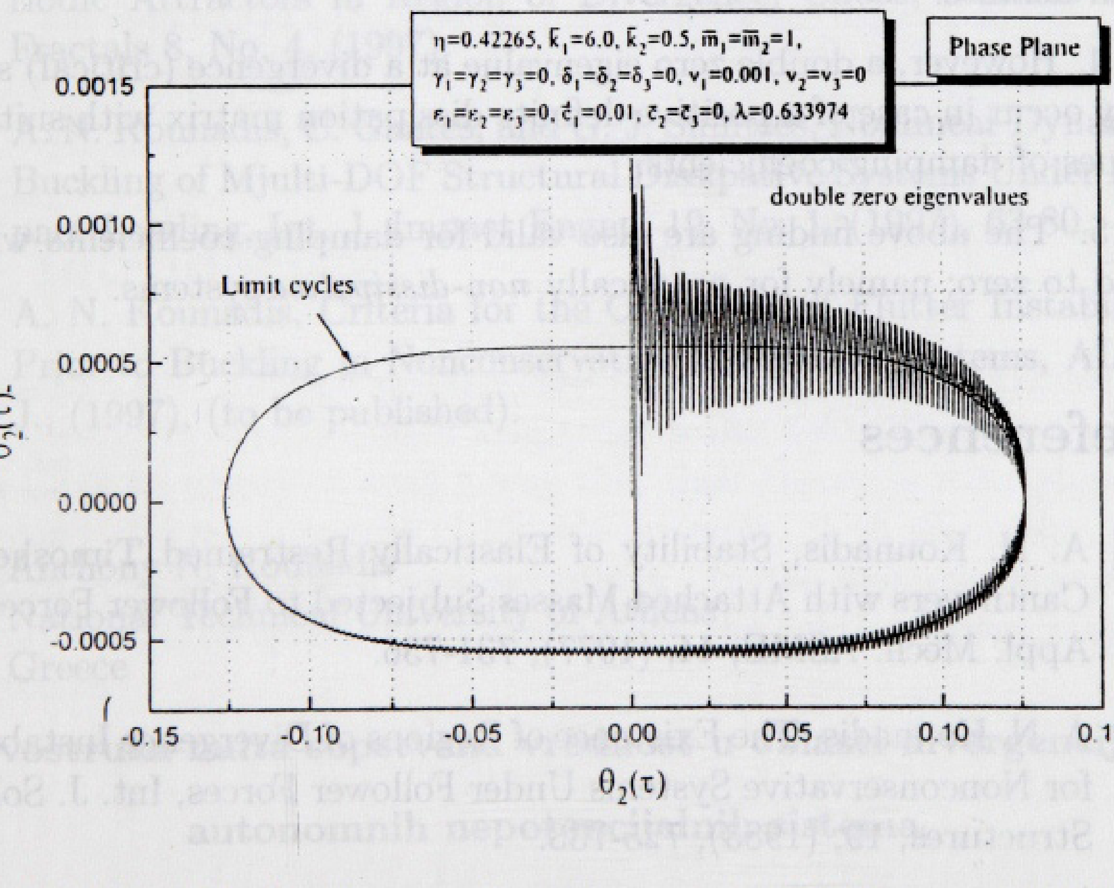

Fig. 6. The critical (divergence) state ($\eta_{cr} = 0.42265, \lambda_{cr} = 0.633974$) associated with a periodic attractor (due to a double zero eigenvalue).

5 Conclusions

Using a 2-DOF and a 3-DOF nonlinear dissipative models under partial follower load λ with non conservativeness parameter η, critical states of divergence associated with a *double zero* eigenvalue, are discussed in detail. The most important results for models with *Rayleigh dissipation* matrix are the following:

1. The conditions for the occurrence of a double zero eigenvalue (implying a coupled divergence-flutter instability) are properly established.

2. The case of a double zero eigenvalue is *excluded* for arbitrary *different from* zero damping coefficients (implying a positive definite dissipation matrix).

3. A double zero eigenvalue may also occur if one of the damping coefficients is zero which renders the dissipation matrix $[c_{ij}]$ positive semi-definite.

4. However, a double zero eigenvalue at a divergence (critical) state may occur in case of a positive definite dissipation matrix with suitable values of damping coefficients.

5. The above finding are also valid for damping coefficients which tend to zero; namely for practically *non-dissipative* systems.

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Dvostruka nulta sopstvena vrednost u oblasti divergencije autonomnih nepotencijalnih sistema

Posmatra se perfektna bifurkacija za disipativne sisteme sa trivijalnom fundamentalnom granom pod dsejstvom pratećeg opterećenja.

Ovakvi autonomni nepotencijalni sistemi, pod odredjenim uslovima, mogu pokazati oscilatorno kretanje u kritičnom stanju divergencije. Ovo je rezultat interakcije dva tipa nestabilnosti koji se javlja na dvostrukoj nultoj sopstvenoj vrednosti. Uslovi za pojavu ovog fenomena su detaljno ispitani. Za ilustraciju posmatrani su modeli kako sa dva i tako i sa tri stepena slobode.