THEORETICAL AND APPLIED MECHANICS
23, pp. 35-54, 1997, UDK 517.93

# The operator stiffness matrix of the fixed-end composite member

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Submitted 20 March 1996

**Abstract**

In the present paper the stiffness matrix of the fixed-end composite member, which is usually called a member type of "k", is derived. The elements of the stiffness matrix are determined without mathematical negligences. The solutions contains only the inevitable approximation concerning the desriptions of the rheological properties of materials. Concrete is considered as an aging linear viscoelastic material. The relaxation of the prestressing steel is taken into account. The expressions are developed for any given concrete creep function and for a member of variable cross section. Following these expressions, as a special case, the expressions for the member of constant cross section are derived. Using this kind of operators the problem is solved not only symbolically and formally but also the expressions for stresses and displacements are significantly simplified.

## 1 Introduction

In the calculation of structures by the stiffness method the stiffness matrices are used. In the present paper the stiffness matrix of the fixed-end composite member, which is usually called a member type of "k", will be derived. The elements of the stiffness matrix will be determined without mathematical negligences. The solutions contains only the inevitable approximation concerning the desriptions of the rheological properties of materials. Concrete is considered as an aging linear viscoelastic material. The relaxation of the prestressing steel is taken into account. The expressions are developed for any given concrete creep function and for a member of variable cross section. Following these expressions, as a special case, the expressions for the member of constant cross section are derived.

The mathematical method, using linear integro differential operators, in the theory of aging linear viscoelasticity is given by Mandel. Since the operators obey the algebra laws of ordinary numbers, the problem is solved symbolically and formally so that the mathematical operations are quotated only. Developing the theory of composite structures Lazic introduced linear integral operators for which the laws of algebra of ordinary numbers are valid, too. Using this kind of operators the problem was solved not only symbolically and formally but also the expressions for stresses and displacements were significantly simplified. Their values can be obtained by the least number of mathematical operations. The same procedure will be used in the present paper.

## 2 Force-displacement relationship for the composite member

Consider the fixed-end member is shown in Fig. 1. The bending moments $M_i$ and $M_k$, the shear forces $T_i$ and $T_k$, and axial forces $N_i$ and $N_k$at the ends of the member are introduced as components of vector $[R]$. They present the generalized forces and they are called the nodal forces. Their positive values are shown in Fig. 1a, as is usually done in the stiffness method.

$$[R] = \begin{bmatrix} N_i \\ T_i \\ M_i \\ N_k \\ T_k \\ M_k \end{bmatrix}. \tag{1}$$

The nodal forces in $[R]$ satisfy equilibrium conditions.

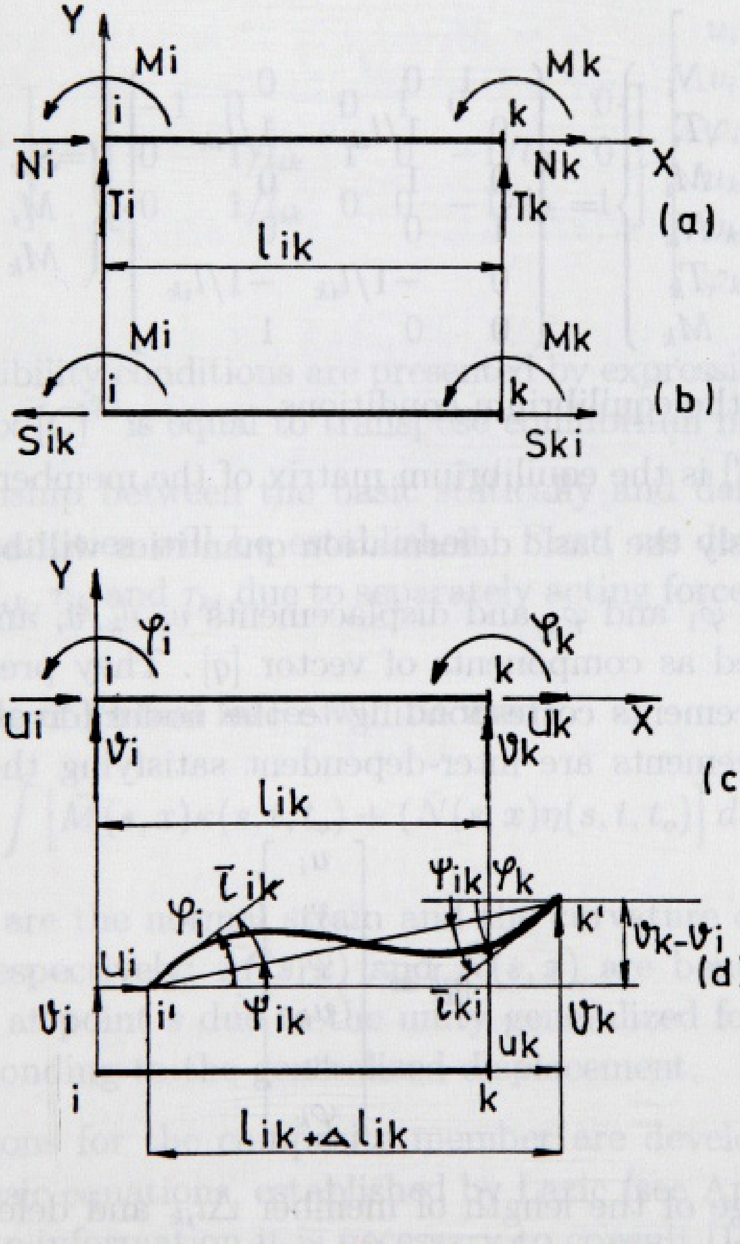


Fig. 1. Fixed-end member under consideration.

The quantities $S_{ik}$, $M_i$ and $M_k$ (Fig. 1b) are basic statically independent ones. They are introduced as components of vector $[S]$:

$$[S] = \begin{bmatrix} S_{ik} \\ M_i \\ M_k \end{bmatrix}. \tag{2}$$

The relationship between vector $[R]$ and vector $[S]$ is:

$$[R] = [C][S],$$

$$\begin{Bmatrix} N_i \\ T_i \\ M_i \\ N_k \\ T_k \\ M_k \end{Bmatrix} = \begin{Bmatrix} -1 & 0 & 0 \\ 0 & 1/l_{ik} & 1/l_{ik} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1/l_{ik} & -1/l_{ik} \\ 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} S_{ik} \\ M_i \\ M_k \end{Bmatrix}, \tag{3}$$

representing the equilibrium conditions.

Matrix $[C]$ is the equilibrium matrix of the member.

Analogously the basic deformation quantities will be defined.

Rotations $\varphi_1$ and $\varphi_k$ and displacements $v_i, v_k, u_i$ and $u_k$ (Fig. 1.c) are introduced as components of vector $[q]$. They present the generalized displacements corresponding to the nodal forces in $[R]$. These nodal displacements are inter-dependent satisfying the compatibility conditions.

$$[q] = \begin{bmatrix} u_i \\ v_i \\ \varphi_i \\ u_k \\ v_k \\ \varphi_k \end{bmatrix}. \tag{4}$$

The change of the length of member $\Delta l_{ik}$ and deformation angles $\tau_{ik}$ and $\tau_{ki}$ (Fig. 1d) are basic deformation independent quantities of the member. They are introduced as components of vector $[\delta]$:

$$[\delta] = \begin{bmatrix} \Delta l_{ik} \\ \tau_{ik} \\ \tau_{ki} \end{bmatrix}. \tag{5}$$

The relationship between the basic deformationally independent quantities $[\delta]$ and nodal displacements $[q]$ is given in a matrix form as follows:

$$[\delta] = [C]^T [q],$$

$$\begin{bmatrix} \Delta l_{ik} \\ \tau_{ik} \\ \tau_{ki} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1/l_{ik} & 1 & 0 & -1/l_{ik} & 0 \\ 0 & 1/l_{ik} & 0 & 0 & -1/l_{ik} & 1 \end{bmatrix} \begin{bmatrix} u_i \\ v_i \\ \varphi_i \\ u_k \\ v_k \\ \varphi_k \end{bmatrix}. \tag{6}$$

The compatibility conditions are presented by expression (6). Compatibility matrix $[C]^T$ is equal to transpose equilibrium matrix $[C]$.

The relationship between the basic statically and deformationally independent quantities will be established. First, we develop the expressions for $\Delta l_{ik}, \tau_{ik}$ and $\tau_{ki}$ due to separately acting forces $S_{ik}$, $M_i$ and $M_k$.

The principle of virtual forces will be used:

$$\xi = \int \left[\bar{M}(s,x)\kappa(s,t,t_o) + (\bar{N}(s,x)\eta(s,t,t_o)\right] ds, \tag{7}$$

where $\eta$ and $\kappa$ are the normal strain and the curvature change of the member axis respectively; $\bar{M}(s,x)$ and $\bar{N}(s,x)$ are bending moment and axial force at point $s$ due to the unity generalized force $\bar{P} = 1$ at point $x$ corresponding to the generalized displacement.

All expressions for the composite member are developed through the so-called basic equations, established by Lazic (see Appendix). For a more complete information it is necessary to consult [1], [2].

Substituting (A11) into (7), we obtain:

$$\begin{aligned}\Delta^* &= \Delta^*(s,t,t_o) = E_u J_u \xi = \int_L \bar{M}(s,x)\hat{F}'_{22}(s,t,\tau)M(s,\tau,t_o)\frac{J_u}{J_i(s)}ds + \\ &+\frac{J_u}{F_u}\int_L \bar{N}(s,x)\hat{F}'_{11}(s,t,\tau)N(s,\tau,t_o)\frac{F_u}{F_i(s)}ds + \\ &+\int \hat{F}'_{12}(s,t,\tau)\left[\bar{M}(s,x)N(s,\tau,t_o)+\bar{N}(s,x)M(s,\tau,t_o)\right]\frac{J_u}{S_i(s)}ds,\end{aligned} \tag{8}$$

where statical and deformation quantities are time functions.

We observe that the unity force $S_{ik} = S_{ik}(t,t_o) = 1^*$, i.e., $M_i = M_i(t,t_o) = 1^*$, i.e., $M_k = M_k(t,t_o) = 1^*$ is acting from the time $t_o$. Function $1^* = 1^*(t,t_o)$ is Heaviside function. Then forces $N(s)$ and $M(s)$ due to these influences, are only the functions of coordinate $s$:

$$N \equiv N(s)1^*(t,t_o), \qquad M \equiv M(s)1^*(t,t_o). \tag{9}$$

The change of the length $\Delta l_{ik}$ and deformation angles $\tau_{ik}$ and $\tau_{ki}$ due to force $S_{ik}(t,t_o) = 1^*$, when $M_i, M_k$ and the external load are equal to zero, will be denoted as $\Delta l_{ik-s}, \tau_{ik-s}$ and $\tau_{ki-s}$. Using (8) and (9) these quantities are:

$$\begin{aligned}\Delta l_{ik-s}(t,t_o) &= \int_L \bar{N}(s)F^*_{11}(s,t,\tau)N(s)\frac{1}{E_u F_i(s)}ds, \\ \tau_{ik-s}(t,t_o) &= -\int_L \bar{M}_i(s)F^*_{12}(s,t,\tau)N(s)\frac{1}{E_u S_i(s)}ds, \\ \tau_{ki-s}(t,t_o) &= \int_L \bar{M}_k(s)F^*_{12}(s,t,\tau)N(s)\frac{1}{E_u S_i(s)}ds.\end{aligned} \tag{10}$$

Besides two time arguments $t$ and $\tau$ functions $F^*_{hl}(h,l=1,2)$ depend on the coordinate $s$, because, in the integral case, the geometrical properties of the member cross section change along the axis. The member model with constant cross section in a finite number of intervals is adopted. To the observed interval (a) correspond length $L^{(a)}$, functions

$F_{h1}^{*(a)}$ and so on. Then, the following notations are introduced:

$$
\begin{aligned}
N_{ik}^{(a)} &= \frac{1}{E_u F_i^{(a)}} \int_{L^{(a)}} \bar{N}(s) N(s) ds, \\
l_{ik,s}^{(a)} &= \frac{1}{E_u S_i^{(a)}} \int_{L^{(a)}} \bar{M}_i(s) N(s) ds, \qquad (11) \\
l_{ki,s}^{(a)} &= \frac{1}{E_u S_i^{(a)}} \int_{L^{(a)}} \bar{M}_k(s) N(s) ds
\end{aligned}
$$

and functions:

$$
\begin{aligned}
\delta_{ik-s}^{*}(t,t_o) &= \sum_{(a)} N_{ik}^{(a)} F_{11}^{*(a)}(t,t_o), \\
\alpha_{ik-s}^{*}(t,t_o) &= \sum_{(a)} \ell_{ik,s}^{(a)} F_{12}^{*(a)}(t,t_o), \qquad (12) \\
\alpha_{ki-s}^{*}(t,t_o) &= \sum_{(a)} \ell_{ki,s}^{(a)} F_{12}^{*(a)}(t,t_o).
\end{aligned}
$$

These functions present the change of length of the member and deformation angles due to $S_{ik}(t,t_o) = 1^*$ and when $M_i, M_k$ and external load are equal to zero.

The corresponding operators [1] are as follows:

$$
\begin{aligned}
\hat{\delta}_{ik-s}'(t,\tau) &= \sum_{(a)} N_{ik}^{(a)} \hat{F}_{11}'^{(a)}(t,\tau), \\
\hat{\alpha}_{ik-s}'(t,\tau) &= \sum_{(a)} \ell_{ik-s}^{(a)} \hat{F}_{12}'^{(a)}(t,\tau), \qquad (13) \\
\hat{\alpha}_{ki-s}'(t,\tau) &= \sum_{(a)} \ell_{ki-s}^{(a)} \hat{F}_{12}'^{(a)}(t,\tau), \qquad \tau \geq t_o.
\end{aligned}
$$

The change of the length and the deformation angles due to $S_{ik} = S_{ik}(t,t_o)$, are shown in the following manner:

$$
\Delta l_{ik-s}(t,t_o) = \hat{\delta}_{ik-s}'(t,\tau) S_{ik}(\tau,t_o),
$$

$$\tau_{ik-s}(t,t_o) = -\hat{\alpha}'_{ik-s}(t,\tau)S_{ik}(\tau,t_o), \quad (14)$$

$$\tau_{ki-s}(t,t_o) = \hat{\alpha}'_{ki-s}(t,\tau)S_{ik}(\tau,t_o).$$

Analogously the change of the length and deformation angles due to the bending moment $M_i(t,t_o)$ as a time function, is defined:

$$\Delta l_{ik-M_i}(t,t_o) = -\hat{\delta}'_{ik-M_i}(t,\tau)M_i(\tau,t_o),$$

$$\tau_{ik-M_i}(t,t_o) = \hat{\alpha}'_{ik-s}(t,\tau)M_i(\tau,t_o), \quad (15)$$

$$\tau_{ki-M_i}(t,t_o) = -\hat{\beta}'_{ki}(t,\tau)M_i(\tau,t_o),$$

where, using expression (8) it is:

$$\hat{\delta}'_{ik-M_i}(t,\tau) = \hat{\alpha}'_{ik-s}(t,\tau), \quad (16)$$

$$\hat{\alpha}'_{ik}(t,\tau) = \sum_{(a)} \mathcal{M}^{(a)}_{ik} \hat{F}'^{(a)}_{22}(t,\tau), \quad \mathcal{M}^{(a)}_{ik} = \frac{1}{E_u J_i^{(a)}} \int_{L^{(a)}} \bar{M}_i(s)M_i(s)ds,$$

$$\hat{\beta}'_{ik}(t,\tau) = \sum_{(a)} \overline{\mathcal{M}}^{(a)}_{ik} \hat{F}'^{(a)}_{22}(t,\tau), \quad \overline{\mathcal{M}}^{(a)}_{ik} = \frac{1}{E_u J_i^{(a)}} \int_{L^{(a)}} \bar{M}_k(s)M_i(s)ds.$$

Due to $M_k(t,t_o)$ :

$$\Delta l_{ik-M_k}(t,t_o) = \hat{\delta}'_{ik-M_k}(t,\tau)M_k(\tau,t_o),$$

$$\tau_{ik-M_k}(t,t_o) = -\hat{\beta}'_{ik}(t,\tau)M_k(\tau,t_o), \quad (17)$$

$$\tau_{ik-M_k}(t,t_o) = \hat{\alpha}'_{ik}(t,\tau)M_k(\tau,t_o),$$

where, using expression (8) it is:

$$\hat{\delta}'_{ki-M_k}(t,\tau) = \hat{\alpha}'_{ki-s}(t,\tau),$$

$$\hat{\beta}'_{ik}(t,\tau) = \hat{\beta}_{ki}(t,\tau), \tag{18}$$

$$\hat{\alpha}'_{ki}(t,\tau) = \sum_{(a)} \mathcal{M}^{(a)}_{ki} \hat{F}'^{(a)}_{22}(t,\tau), \quad \mathcal{M}^{(a)}_{ki} = \frac{1}{E_u J^{(a)}_i} \int_{L^{(a)}} \bar{M}_k(s) M_k(s) ds.$$

Applying the principle of superposition and based on (14), (15) and (17) the final expressions for the change of length and deformation angles of the member can be defined:

$$\begin{aligned} \Delta l_{ik} &= \hat{\delta}'_{ik-s} S_{ik} - \hat{\alpha}'_{ik-s} M_i + \hat{\alpha}'_{ki-s} M_k, \\ \tau_{ik} &= -\hat{\alpha}'_{ik-s} S_{ik} + \hat{\alpha}'_{ik} M_i - \hat{\beta}'_{ki} M_k, \\ \tau_{ki} &= \hat{\alpha}'_{ki-s} S_{ik} - \hat{\beta}'_{ik} M_i + \hat{\alpha}'_{ki} M_k. \end{aligned} \tag{19}$$

The relationship between the basic deformationally independent quantities (5) and basic statically independent quantities (2) are represented by expression (19). This relationship may be also written in the operator matrix form:

$$[\delta] = \left[\hat{f}'\right][S]. \tag{20}$$

where:

$$\left[\hat{f}'\right] = \begin{bmatrix} \hat{\delta}'_{ik-s} & -\hat{\alpha}'_{ik-s} & \hat{\alpha}'_{ki-s} \\ -\hat{\alpha}'_{ik-s} & \hat{\alpha}'_{ik} & -\hat{\beta}'_{ik} \\ \hat{\alpha}'_{ki-s} & -\hat{\beta}'_{ik} & \hat{\alpha}'_{ki} \end{bmatrix}, \tag{21}$$

is the symmetric operator flexibility matrix. The elements of this matrix are operators having the commutative property. Following these expressions, as a special case, the elements of operator flexibility matrix for the member of constant cross section, will be in the form:

$$\hat{\delta}'_{ik-s}(t,\tau) = \frac{l_{ik}}{E_u F_i} \hat{F}'_{11}(t,\tau),$$

$$\hat{\alpha}'_{ik}(t,\tau) = \frac{l_{ik}}{2E_u S_i}\hat{F}'_{12}(t,\tau) = \hat{\alpha}'_{ki-s}(t,\tau),$$

(22)

$$\hat{\alpha}'_{ik}(t,\tau) = \frac{l_{ik}}{3E_u J_i}\hat{F}'_{22}(t,\tau) = \hat{\alpha}'_{ki}(t,\tau),$$

$$\hat{\beta}'_{ik}(t,\tau) = \frac{l_{ik}}{6E_u J_i}\hat{F}'_{22}(t,\tau).$$

# 3 Operator stiffness matrix

First, we develop the basic operator stiffness matrix. We recall that for linear integral operators the laws of algebra of ordinary numbers are valid including, in our case, the commutative law. Owing to that the solutions of equation (20) may be formally obtained as a solution of a system of linear algebraic equations:

$$[S] = \left[\hat{f}'\right]^{-1}[\delta]\,. \tag{23}$$

The inverse matrix of the operator flexibility matrix is the symmetric operator matrix, which is called the basic operator stiffness matrix:

$$\left[\hat{K}'_o\right] = \left[\hat{f}'\right]^{-1} = \begin{bmatrix} \hat{N}'_{ik} & \hat{S}'_{ik} & \hat{S}'_{ki} \\ \hat{S}'_{ik} & \hat{A}'_{ik} & \hat{B}'_{ik} \\ \hat{S}'_{ki} & \hat{B}'_{ik} & \hat{A}'_{ki} \end{bmatrix}. \tag{24}$$

Then relation (23) will be of the form:

$$[S] = \left[\hat{K}'_o\right][\delta]\,. \tag{25}$$

The elements of the basic operator stiffness matrix are operators $\hat{N}'_{ik}$, $\hat{S}'_{ik}$, $\hat{S}'_{ki}$, $\hat{A}'_{ik}$, $\hat{B}'_{ik}$ and $\hat{A}'_{ki}$ corresponding to functions $N^*_{ik}(t,t_o)$, $S^*_{ik}(t,t_o)$, $S^*_{ki}(t,t_o)$, $A^*_{ik}(t,t_o)$, $B^*_{ik}(t,t_o)$ and $A^*_{ki}(t,t_o)$. It is known that these functions present the generalized forces at the ends of the member $i$ and $k$ due to:$\Delta l_{ik} = 1^*(t,t_o)$, $\tau_{ik} = \tau_{ki} = 0$, i.e.: $\Delta l_{ik} = 0$, $\tau_{ik} = 1^*(t,t_o)$, $\tau_{ki} = 0$, i.e.:$\Delta l_{ik} = 0$, $\tau_{ik} = 0$, $\tau_{ki} = 1^*(t,t_o)$ respectively.

## 3.1 The fixed-end member of variable cross section

The elements of matrix $\left[\hat{K}'_o\right]$ will be derived as elements of the inverse matrix (21).

$$\left[\hat{K}'_o\right] = (\det \hat{f}')^{-1} \operatorname{adj} \hat{f}'. \tag{26}$$

The determinant of operator matrix $\left[\hat{f}'\right]$ is operator $\hat{D}'_k$ and it can be written in the form:

$$\hat{D}'_k = \hat{\delta}'_{ik-s}\left(\hat{\alpha}'_{ik}\hat{\alpha}'_{ki} - \hat{\beta}'_{ik}\hat{\beta}'_{ik}\right) + \hat{\alpha}'_{ik-s}\left(-\hat{\alpha}'_{ik-s}\hat{\alpha}'_{ki} + \hat{\alpha}'_{ki-s}\hat{\beta}'_{ik}\right) +$$

$$+\hat{\alpha}'_{ki-s}\left(\hat{\alpha}'_{ik-s}\hat{\beta}'_{ik} - \hat{\alpha}'_{ki-s}\hat{\alpha}'_{ik}\right), \tag{27}$$

elements of basic operator stiffness matrix $\left[\hat{K}'_o\right]$ being:

$$\hat{N}'_{ik} = \left(\hat{\alpha}'_{ik}\hat{\alpha}'_{ki} - \hat{\beta}'_{ik}\hat{\beta}'_{ik}\right)\left(\hat{D}'_k\right)^{-1},$$

$$\hat{A}'_{ik} = \left(\hat{\delta}'_{ik-s}\hat{\alpha}'_{ki} - \hat{\alpha}'_{ki-s}\hat{\alpha}'_{ki-s}\right)\left(\hat{D}'_k\right)^{-1},$$

$$\hat{A}'_{ki} = \left(\hat{\delta}'_{ik-s}\hat{\alpha}'_{ik} - \hat{\alpha}'_{ik-s}\hat{\alpha}'_{ik-s}\right)\left(\hat{D}'_k\right)^{-1}, \tag{28}$$

$$\hat{B}'_{ik} = \left(\hat{\delta}'_{ik-s}\hat{\beta}'_{ik} - \hat{\alpha}'_{ik-s}\hat{\alpha}_{ki,s}\right)\left(\hat{D}'_k\right)^{-1} = \hat{B}'_{ki},$$

$$\hat{S}'_{ik} = \left(-\hat{\alpha}'_{ki-s}\hat{\beta}'_{ik} + \hat{\alpha}'_{ki-s}\hat{\alpha}'_{ki}\right)\left(\hat{D}'_k\right)^{-1},$$

$$\hat{S}'_{ki} = \left(\hat{\alpha}'_{ik-s}\hat{\beta}'_{ik} - \hat{\alpha}'_{ki-s}\hat{\alpha}'_{ik}\right)\left(\hat{D}'_k\right)^{-1}.$$

To obtain the operator stiffness matrix, relating nodal forces $[R]$ and nodal displacements $[q]$, we substitute (25) and $(6)_1$ into $(3)_1$. Then:

$$[R] = [C]\,[S] = [C]\left[\hat{K}'_o\right][\delta] = [C]\left[\hat{K}'_o\right][C]^T[q], \tag{29}$$

i.e.:

$$[R] = \left[\hat{K}'\right][q], \tag{30}$$

where:

$$\left[\hat{K}'\right] = [C]\left[\hat{K}'_o\right][C]^T. \tag{31}$$

The elements of the operator stiffness matrix of the fixed-end member can be expressed through the elements of the basic operator stiffness matrix, (24) as follows:

$$\left[\begin{array}{cccccc} \hat{N}'_{ik} & -\frac{1}{l}\left(\hat{S}'_{ik}+\hat{S}'_{k}\right) & -\hat{S}'_{ik} & -\hat{N}'_{ik} & \frac{1}{l}\left(\hat{S}'_{ik}+\hat{S}'_{ki}\right) & -\hat{S}'_{ki} \\ & \frac{1}{l^2}\left(\hat{C}'_{ik}+\hat{C}'_{k}\right) & \frac{1}{l}\hat{C}'_{ik} & \frac{1}{l}\left(S'_{ik}+S'_{ki}\right) & -\frac{1}{l^2}\left(\hat{C}'_{ik}+\hat{C}'_{ki}\right) & \frac{1}{l}\hat{C}'_{ki} \\ & & \hat{A}'_{ik} & \hat{S}'_{ik} & -\frac{1}{l}\hat{C}'_{ik} & \hat{B}'_{ik} \\ & & & \hat{N}'_{ik} & -\frac{1}{l}\left(\hat{S}'_{ik}+\hat{S}'_{ki}\right) & \hat{S}'_{ki} \\ & \text{symmetrical} & & & \frac{1}{l^2}\left(\hat{C}'_{ik}+\hat{C}'_{ki}\right) & -\frac{1}{l}\hat{C}'_{ki} \\ & & & & & \hat{A}'_{ki} \end{array}\right] \tag{32}$$

where:

$$\hat{A}'_{ik}+\hat{B}'_{ik}=\hat{C}'_{ik}$$

$$\hat{A}'_{ki}+\hat{B}'_{ik}=\hat{C}'_{ki}, \qquad \hat{B}'_{ik}=\hat{B}'_{ki}, \tag{33}$$

$$l=l_{ik}$$

## 3.2 The fixed-end member of constant cross section

The elements of the basic operator stiffness matrix of the fixed-end member of constant cross section will be developed:

Introducing (22) into (27) the determinant of operator matrix $\left[\hat{f}'\right]$ may be represented in the following form:

$$\hat{D}'_k=\frac{l^3_{ik}}{12E^3_u F_i J_i}\hat{F}'_{22}\left(\hat{F}'_{11}\hat{F}'_{22}-\hat{F}'_{12}\hat{F}'_{12}\right). \tag{34}$$

Based on expression (A18) and with the next notation:

$$d_k=\frac{l^3_{ik}}{12E^3_u F_i J_i}, \tag{35}$$

we arrive at:

$$\hat{D}'_k = d_k \hat{F}'_{22} \hat{F}'_1 \hat{F}'_2. \tag{36}$$

Operator $\left(\hat{D}'_k\right)$ has inverse operator $\left(\hat{D}'_k\right)^{-1}$ satisfying equations:

$$\left(\hat{D}'_k\right)^{-1} \hat{D}'_k = \hat{1}', \tag{37}$$

$$\left(\hat{D}'_k\right)^{-1} = \frac{1}{d_k}\left(\hat{F}'_{22}\right)^{-1}\left(\hat{F}'_1 \hat{F}'_2\right)^{-1}. \tag{38}$$

Operator $\hat{I}'_{22}$ is introduced. It satisfies the relation:

$$\hat{F}'_{22} \hat{I}'_{22} = \hat{1}', \tag{39}$$

where operator $\hat{I}'_{22}$ may be expressed in the following way [4]:

$$\hat{I}'_{22} = \frac{\gamma'_1 \gamma'_2}{\gamma'_{11}} \hat{1}' + \frac{\gamma_1 \gamma_2}{\gamma_{11}} \hat{R}' + \left(1 - \frac{\gamma'_1 \gamma'_2}{\gamma'_{11}} - \frac{\gamma_1 \gamma_2}{\gamma_{11}}\right) \hat{\mathcal{B}}'_{11}. \tag{40}$$

Using expressions (A14), (A15) and (38) expression (37) becomes:

$$\left(\hat{D}'_k\right)^{-1} = \frac{1}{d_k} \hat{I}'_{22} \hat{R}'_1 \hat{R}'_2. \tag{41}$$

According to (28), (22), (41) and (A17) the elements of the basic operator stiffness matrix for the fixed-end member of constant cross section may be finally represented in the following form:

$$\hat{N}'_{ik} = \frac{E_u F_i}{l_{ik}} \gamma'_{11} \hat{1}' + \frac{E_u F_i}{l_{ik}} \gamma_{11} \hat{R}',$$

$$\hat{A}'_{ik} = \frac{E_u J_i}{l_{ik}} \left(\gamma'_{22} + \frac{3\gamma'_1 \gamma'_2}{\gamma'_{11}}\right) \hat{1}' + \frac{E_u J_i}{l_{ik}} \left(\gamma_{22} + \frac{3\gamma_1 \gamma_2}{\gamma_{11}}\right) \hat{R}' +$$

$$+ \frac{3E_u J_i}{l_{ik}} \left(1 - \frac{\gamma'_1 \gamma'_2}{\gamma'_{11}} - \frac{\gamma_1 \gamma_2}{\gamma_{11}}\right) \hat{\mathcal{B}}'_{11} = \hat{A}'_{ki},$$

$$\hat{B}'_{ik} = \frac{E_u J_i}{l_{ik}} \left( -\gamma'_{22} + \frac{3\gamma'_1 \gamma'_2}{\gamma'_{11}} \right) \hat{1}' + \frac{E_u J_i}{l_{ik}} \left( -\gamma_{22} + \frac{3\gamma_1 \gamma_2}{\gamma_{11}} \right) \hat{R}' +$$

$$+ \frac{3E_u J_i}{l_{ik}} \left( 1 - \frac{\gamma'_1 \gamma'_2}{\gamma'_{11}} - \frac{\gamma_1 \gamma_2}{\gamma_{11}} \right) \hat{\mathcal{B}}'_{11} = \hat{B}'_{ki},$$

$$\hat{S}'_{ik} = \frac{E_u S_i}{l_{ik}} \gamma_{12} \hat{1}' - \frac{E_u S_i}{l_{ik}} \gamma_{12} \hat{R}' = -\hat{S}'_{ki}, \tag{42}$$

where expressions (40) and (A9) are used.

Using (32), (33) and (42) the operator stiffness matrix for the fixed-end member of constant cross section may be represented as follows:

$$\begin{bmatrix} \hat{N}'_{ik} & 0 & -\hat{S}'_{ik} & -\hat{N}'_{ik} & 0 & \hat{S}'_{ik} \\ & \frac{2}{l_{ik}^2} \hat{C}'_{ik} & \frac{1}{l_{ik}} \hat{C}'_{ik} & 0 & -\frac{2}{l_{ik}^2} \hat{C}'_{ik} & \frac{1}{l_{ik}} C'_{ki} \\ & & \hat{A}'_{ik} & \hat{S}'_{ik} & -\frac{1}{l_{ik}} \hat{C}'_{ik} & \hat{B}'_{ik} \\ & & & \hat{N}'_{ik} & 0 & -\hat{S}'_{ik} \\ & \text{symmetrical} & & & \frac{2}{l_{ik}^2} \hat{C}'_{ik} & -\frac{1}{l_{ik}} \hat{C}'_{ik} \\ & & & & & \hat{A}'_{ki} \end{bmatrix}, \tag{43}$$

where:

$$\hat{C}'_{ik} = \hat{A}'_{ik} + \hat{B}'_{ik} = \hat{C}'_{ki}. \tag{44}$$

Based on expression (42) operator $\hat{C}'_{ik}$ may be expressed in the following form:

$$\hat{C}'_{ik} = \frac{6E_u J_i}{l_{ik}} \left[ \frac{\gamma'_1 \gamma'_2}{\gamma'_{11}} \hat{1}' + \frac{\gamma_1 \gamma_2}{\gamma_{11}} \hat{R}' + \left( 1 - \frac{\gamma'_1 \gamma'_2}{\gamma'_{11}} - \frac{\gamma_1 \gamma_2}{\gamma_{11}} \right) \hat{\mathcal{B}}'_{11} \right]. \tag{45}$$

The function $\mathcal{B}^*_{11}$, corresponding to operator $\hat{\mathcal{B}}'_{11}$, is the solution of the inhomogeneous integral equation:

$$\hat{\mathcal{K}}'_{11} \mathcal{B}^*_{11} = 1^*. \tag{46}$$

Function $\mathcal{K}^*_{11}$ linearly depend on the concrete creep function $F^*$ and on the reduced geometrical properties of the composite cross section:

$$\mathcal{K}^*_{11} = \gamma_{11} 1^* + \gamma'_{11} F^*. \tag{47}$$

Expressions (46) and (47) are analogous to expressions (A22) and (A23).

Comparing expressions (46) and (A3) we conclude that the solutions of the parametric inhomogeneous integral equation:

$$\mathcal{K}^*(\lambda,t,t)\,\mathcal{B}^*(\lambda,t,t_o)-\int_{t_o}^{t}\frac{\partial\mathcal{K}^*(\lambda,t,\theta)}{\partial\theta}\mathcal{B}^*(\lambda,\theta,t_o)\,d\theta=1^*, \qquad (48)$$

provide function $\mathcal{B}^*_{11}$ for $\lambda=\gamma_{11}$ and $R^*$ for $\lambda=1$.

From expressions (28) and (41) it can be seen that the elements of the operator stiffness matrix represent the five-fold operator products. That means that, for each element, it is necessary to carry out five-fold integrations with the displacement function. Using the properties of the linear integral operators it was possible to reduce these expressions to the simplest form. The elements of the operator stiffness matrix (42) and (45), are represented as linear combinations of the operators $\hat{R}'$ and $\widehat{\mathcal{B}}'_{11}$, where corresponding functions $R^*$ and $\mathcal{B}^*_{11}$ are the solutions of equation (48) for each concrete creep function $F^*$.

# 4 Conclusion

The elements of the operator stiffness matrix for the fixed-end composite member are derived using linear integral operator procedure. This mathematical approach was applied because it was possible, by simple mathematical operations, to develop the expressions for these elements. They refer to any given concrete creep function and to the member of variable cross section. Using operator transformations the complex integrations appearing in these expressions are reduced to the simplest form. The elements for the member of constant cross section are presented in that way. They are linear combinations of two operators, whose corresponding functions are the solutions of only one parametric inhomogeneous integral equation.

In the calculation of composite structures by the stiffness method the stiffness matrices were applied. Then the integrations of their elements occur. Because of that it is very important that the elements of the operator stiffness matrix have the simplest form.

# References

[1] Lazic J., Lazic V.: General Theory of Composite and Prestressed Structures, The Serbian Academy of Sciences and Arts, Monographs, DXLII, No. 22, Beograd (1982). (in Serbian)

[2] Lazic J., Lazic V.: Stresses and Displacement of Composite and Prestressed Structures, Fundamental Research on Creep and Shrinkage of Concrete, Edited by F.H. Wittmann, Martinus Nijhoff Publ. The Hague, 1982. str. 413-425.

[3] Sekulovic M.: Matrix Structural Analysis, Gradjevinska knjiga, Beograd, 1991. (in Serbian)

[4] Deretic-Stojanovic B.: The Calculation of Composite Structures by the Slope Deflection Method, Doctoral dissertation, Beograd (1992). (in Serbian)

### Appendix A: The basic expressions of the composite cross section

The composite member in the general case, contains concrete ($b$), prestressing steel ($p$), steel member ($n$) and reinforced steel ($m$).

The stress strain relationship for concrete is integral and may be symbolically written in the operator form [1]:

$$\varepsilon = \frac{1}{E_{bo}} \hat{F}' \sigma_b. \tag{A1}$$

The solution of equation (A1) is:

$$\sigma_b = E_{bo} \hat{R}' \varepsilon. \tag{A2}$$

Operators $\hat{F}'$ and $\hat{R}'$ are inverse. They satisfy the relation:

$$\hat{R}'\hat{F}' = \hat{F}'\hat{R}' = \hat{1}'. \tag{A3}$$

The concrete creep and relaxation functions $F^*$ and $R^*$ represent the following integrals [1]:

$$F^* = \hat{F}'1^*, \qquad R^* = \hat{R}'1^*. \tag{A4}$$

The stress strain relationship for prestressing steel has the form:

$$\delta_p = E_p\hat{R}'_p\varepsilon, \tag{A5}$$

where relaxation function $R^*_p$ linearly depends on concrete relaxation function $R^*_p$ linearly depends on concrete relaxation function $R^*$ [2].

$$R^*_p = \hat{R}'_p1^* = (1-\rho)\,1^* + \rho R^*. \tag{A6}$$

Other kinds of steel: steel member $(n)$ and reinforcing steel $(m)$ obey Hooke's law.

$$\sigma_k = E_k\varepsilon, \qquad k = n, m \tag{A7}$$

Starting form the assumption that for the composite cross-section Bernoulli's hypothesis of plain cross section is valid, then from the equilibrium conditions of external and internal forces in the cross-section and from relations (A2), (A5), (A7) the system of inhomogeneous integral equations is obtained:

$$E_uF_i\hat{R}'_{11}\eta + E_uS_i\hat{R}'_{12}\kappa = N, \qquad E_uF_i\hat{R}'_{12}\eta + E_uS_i\hat{R}'_{22}\kappa = M, \tag{A8}$$

where $\eta = \eta\,(x,t,t_o)$ is a normal strain, $\kappa = \kappa\,(x,t,t_o)$ is the curvature change of the member axis, $E_u$ is the relative modulus of elasticity, $F_i$ and $J_i$ are transformed cross-section area and moment of inertia of this area;

$$S_i = \sqrt{F_iJ_i}.$$

The elements of operator matrix $\left[\hat{R}'_{hl}\right]_{2,2}$ are defined in the following form:

$$\hat{R}'_{hl} = (\delta_{hl} - \gamma_{hl})\, 1' + \gamma_{hl}\hat{R}', \qquad \delta_{hl} = \begin{cases} 1 \text{ za } h = 1 \\ 0 \text{ za } h \neq 1 \end{cases}, \quad h = 1,2, \quad \text{(A9)}$$

and their principal values are:

$$\hat{R}'_h = \gamma'_h \hat{1}' + \gamma_h \hat{R} \qquad h = 1,2, \qquad \text{(A10)}$$

$\gamma_{hl}$ are the elements of symmetric matrix $[\gamma_{hl}]_{2,2}$ of reduced geometrical properties, and $\gamma_h$ $(h = 1,2)$ are the principal values; $(\gamma'_h = 1 - \gamma_h)$. The geometrical properties depend on the rheological properties of materials, too.

Since the operators have the commutativity property, the system (A8) is solved by Crammers rule. The solution is:

$$\eta = \frac{1}{E_u F_i}\hat{F}'_{11} N + \frac{1}{E_u S_i}\hat{F}'_{12} M, \qquad \kappa = \frac{1}{E_u S_i}\hat{F}'_{12} N + \frac{1}{E_u J_i}\hat{F}'_{22} M, \quad \text{(A11)}$$

where:

$$\hat{F}'_{11} = \left(\hat{D}'\right)^{-1} \hat{R}'_{22}, \qquad \hat{F}'_{22} = \left(\hat{D}'\right)^{-1} \hat{R}'_{11}, \qquad \hat{F}_{12} = \hat{F}'_{21} = -\left(\hat{D}'\right)^{-1} \hat{R}'_{12}. \quad \text{(A12)}$$

The principal values of operator matrix $\left[\hat{F}'_{hl}\right]_{2,2}$ are $\hat{F}'_h$ $(h = 1,2)$, and the next relation is valid:

$$\hat{R}'_h \hat{F}'_h = \hat{F}'_h \hat{R}'_h = \hat{1}', \qquad \text{(A13)}$$

The determinant of operator matrix $\left[\hat{R}'_{hl}\right]_{2,2}$ is presented as operator $\hat{D}'$:

$$\hat{D}' = \hat{R}'_{11}\hat{R}'_{22} - \hat{R}'_{12}\hat{R}'_{12} = \hat{R}'_1\hat{R}'_2 = \hat{R}'_2\hat{R}'_1. \qquad \text{(A14)}$$

On the basis of expressions (A13) and (A14) inverse operator $\left(\hat{D}'\right)^{-1}$ is given by:

$$\left(\hat{D}'\right)^{-1} = \hat{F}'_1\hat{F}'_2 = \hat{F}'_2\hat{F}'_1, \qquad \text{(A15)}$$

and according to expression (A12) the operators $F'_{hl}$ have the following form:

$$\hat{F}'_{11} = \hat{F}'_1\hat{F}'_2\hat{R}'_{22}, \qquad \hat{F}'_{22} = \hat{F}'_1\hat{F}'_2\hat{R}'_{11}, \qquad \hat{F}'_{12} = \hat{F}'_{21} = -\hat{F}'_1\hat{F}'_2\hat{R}'_{12}. \tag{A16}$$

Using expressions (A13) and (A16), we obtain:

$$\hat{R}'_{11} = \hat{R}'_1\hat{R}'_2\hat{F}'_{22}, \qquad \hat{R}'_{22} = \hat{R}'_1\hat{R}'_2\hat{F}'_{11}, \qquad \hat{R}'_{12} = \hat{R}'_{21} = -\hat{R}'_1\hat{R}'_2\hat{F}'_{12}. \tag{A17}$$

The next relation is used:

$$\hat{F}'_{11}\hat{F}'_{22} - \hat{F}'_{12}\hat{F}'_{12} = \hat{F}'_1\hat{F}'_2. \tag{A18}$$

The following operator is introduced:

$$\hat{B}'_h = \hat{R}'\hat{F}'_h \qquad h = 1, 2, \tag{A19}$$

which linearly depends on operator $\hat{F}'_h$:

$$\hat{B}'_h = \frac{1}{\gamma_h}1' - \frac{\gamma'_h}{\gamma_h}\hat{F}'_h \qquad h = 1, 2. \tag{A20}$$

Functions $B^*_h$ given by:

$$B^*_h = B^*_h(\gamma_h, t, t_o) = \hat{B}'_h 1^* = \hat{R}'F^*_h = \hat{F}'_h R^* \qquad h = 1, 2, \tag{A21}$$

are the basic functions of the composite cross-section [1]. The determination of function $B^*_h$ is reduced to the solution of the parametric inhomogeneous integral equations:

$$\hat{K}'_h B^*_h = 1 \qquad h = 1, 2. \tag{A22}$$

The functions $K^*_h$ linearly depend on the creep function $F^*$ and on the reduced geometrical properties of cross section:

$$K^*_h = K^*_h(\gamma_h, t, t_o) = \hat{K}'_h 1^* = \gamma_h 1^* + \gamma'_h F^* \qquad h = 1, 2. \tag{A23}$$

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**Operatorska matrica krutosti spregnutog stapa tipa "k"**

U ovom radu je izvedena operatorska matrica krutosti za spregnut štap obostrano kruto vezan, tj. štap tipa "k". Elementi operatorske matrice krutosti su odredjeni bez ikakvih matematičkih zanemarenja. Rešenja sadrže samo neizbežne aproksimacije vezane za opis reoloških osobina materijala. Beton se tretira kao linearno viskoelastičan materijal sa starenjem, uzeta je u obzir i relaksacija čelika za prethodno naprezanje. Izvedeni izrazi važe za proizvoljnu funkciju puzanja betona, za štap proizvoljno promenljivog poprečnog preseka. Koriste se linearni integralni operatori. Ovakav matematički prilaz omogućava da se odgovarajućim operatorskim transformacijama višestruke integracije, koje se pri izvodjenju pojavljuju, svedu na najjednostavniji oblik. Elementi operatorske matrice krutosti za štap konstantnog poprečnog preseka dati su na taj nacin, tj. prikazani su kao linearne kombinacije samo dva operatora.

Operatorske matrice krutosti se koriste u metodi deformacija, gde se vrše naknadne integracije, pa je od posebne važnosti da elementi ovih matrica budu što jednostavniji.