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Generalization of Noether's theorem for systems with variable mass

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Abstract

In this article it is demonstrated that the generalization of Noether's theorem to the systems with variable mass can be effectuated directly, starting form the total variation of action, which here has one additional specific term and employing the corresponding Lagrangian equations. In this manner the direct Noether's theorem for such systems is obtained in a more general form, which represents a certain generalization of this obtained by L. Cveticanin in indirect way from d'Alembert-Lagrange's principle. In addition, the corresponding generalized Killing's equations and the inverse Noether's theorem are formulated, including the generate systems also. The found results are illustrated by a simple, but characteristic example.

1 Introduction

As it is well known, Emmy Noether's theorem ([1], see f.e. [2],[3]) asserts that to every r-parametric transformation group of generalized coordinates and time which conserves the action invariant corresponds a set of r independent first integrals of Lagrange's equations. For these systems the inverse theorem exists also: to every set of r first integrals corresponds only one transformation group with as many independent parameters. This theorem is valid only for the conservative systems with constant mass, as well as for those the motion of which can be described by a variational Hamilton's principle. So formulated theorem can be employed as the basis for obtaining all the conservations laws in the mechanics and mathematical physics [4] and the solutions of diverse mechanical problems. The equivalent results can be found on the ground of the general invariant relations by J. Logan [5], utilizing the corresponding Killing's equations without employing the calculus of variations.

The extension of Noether's theorem to nonconservative systems is effectuated in another way by B. Vujanovic and Dj. Djukic [6], [8], which is exposed also in the cited monograph [3]. This has been realized with the aid of some additional suppositions on the noncommutativity of the operations δ and d/dt, or starting from d'Alembert-Lagrange's principle transformed to the central Lagrange's equation. Thus, the corresponding direct and inverse Noether's theorem for the nonconservative systems are obtained and the finding of the first integrals is reduced to finding at least one particular solution of the generalized Killing's equations.

The equivalent and more general results are found also on the basis of the modern differential geometry, utilizing the calculus of manifolds, where the systems are determined in the terms of the vector fields and the differential forms on the tangent bundle of a manifold. This method is specially developed by W. Sarlet and F. Cantrijn [9] and later extended also to the dissipative systems [10]. In this manner the corresponding connection between the so-called generalized symmetries and the first integrals is found, and the corresponding prototype of Noether's theorem in the geometrical language as the special case is formulated.

However, so far the examination of the conservation laws for the systems with variable mass has been very little elaborated. This branch of the mechanics is based on the Meshchersky's fundamental equation of motion (see. f.e. [11]) and developed by many authors, which is of special interest for the diverse problems of technical machinery, as well as of modern rocket technics. In the development of the mechanics with variable mass on this basis these equations are expressed also in the generalized coordinates by A.Kosmodemiyansky [12] and later developed by several authors [13]. So, the corresponding Lagrangian and Hamiltonian equations, as well as the general principles of mechanics are formulated , whereby the analytical mechanics of the systems with variable mass is founded. In the applications, inter alia, these Lagrangian equations are extended by A. Bessonov [14] to the case when the masses are dependent also on the positions and velocities of all the particles, and employed in the diverse problems concerning certain machines with variable mass.

But, in spite of the great importance of this branch of mechanics, only a little number of papers is devoted to the conservation laws of such systems. Several examples of this type appear in the cited and other papers (see f.e. [15]), however a general method for finding the first integrals, i.e. the corresponding Noether's theorem has only recently been found by L. Cveticanin [16]. Generalizing the cited method of Vujanovic for obtaining the associated Noether's theorem on the basis of d'Alembert-Lagrange's principle [8], she started too from this principle including the reactive force also. By introduction of the nonsimultaneous variations into the associated central Lagrangian equation, one obtained a relation, from which immediately the corresponding Noether's theorem for these systems with variable mass follows. The obtained results are illustrated by two examples: a nonlinear vibrating machine and a rotor with time variable mass, and by application of this Noether's theorem the corresponding energy-like conservation laws are found.

In this paper we shall demonstrate that this generalization of Noether's theorem to the systems with variable mass can be effectuated directly, starting from the total variation of action and extending the method given by the author [17] to this case. In this way we shall obtain the direct and inverse Noether's theorem for such systems and the corresponding generalized Killing's equations, including the degenerate systems.

2 Equations of motion of a system with variable mass

Let us consider a mechanical system of N particles with k nonstationary constraints $f_j(\mathbf{r}_\nu, t) = 0, (j = 1, 2, ..., k)$, whose masses in the general case are dependent on the time, their positions and possibly velocities. The equations of this system can be obtained departing from the corresponding d'Alembert-Lagrange's principle

$$(\mathbf{F}_\nu + \mathbf{R}_\nu^* + \boldsymbol{\phi}_\nu - m_\nu \mathbf{a}_\nu) \cdot \delta \mathbf{r}_\nu = 0, \qquad (\nu = 1, 2, ..., N), \tag{1}$$

where $\mathbf{F}_\nu$ represents the active force, $\mathbf{R}_\nu^*$ the nonideal reaction one and $\boldsymbol{\phi}_\nu$ the Meshchersky's reactive force acting on the ν-th particle, here the summation over the repeated indices is understood. In a manner analogous to the habitual case with permanent mass, passing to the generalized coordinates and extending it to the case with variable mass, one obtains

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = Q_i^* + R_i^*, \qquad (i = 1, 2, ..., n). \tag{2}$$

Here L denotes the Lagrangian $L = T - U$ of the system, $\tilde{Q}_i^*$ is the sum of the generalized nonpotential force and nonideal reaction one

$$\tilde{Q}_i^* = \mathbf{F}_\nu^* \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^i} + \mathbf{R}_\nu^* \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^i} = Q_i^* + R_i^*, \tag{3}$$

and P_i is the specific generalized force arising from the change of the masses

$$P_i = \frac{dm_{\nu\sigma}}{dt}\mathbf{u}_{\nu\sigma} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^i} + \frac{d}{dt}\left(\frac{\partial m_\nu}{\partial q^i} \cdot \frac{1}{2}\mathbf{v}_\nu^2\right) - \frac{\partial m_\nu}{\partial q^i} \cdot \frac{1}{2}\mathbf{v}_\nu^2, \tag{4}$$

$\mathbf{u}_{\nu\sigma}$ $(\sigma = 1, 2)$ being the velocities of separation and annection of certain mass concerning the ν-th particle. These are the general equations of motion for the systems with variable mass in generalized coordinates, the corresponding Lagrangian equations obtained by Bessonov [14], where the term P_i expresses the influence of the separation and annection of certain masses to the motion of the system.

3 Total variation of action

Firstly, let us find the total variation of action for the systems with variable mass. By analogy with the proof in the habitual case (see f.e. [2], p. 142, [3], p. 81), the total variation of action is defined by

$$\Delta W \overset{def}{=} \int\limits_{\bar{t}_o}^{\bar{t}_1} L\left(\bar{q}^i\left(\bar{t}\right), \frac{d\bar{q}^i}{d\bar{t}}, \bar{t}\right) d\bar{t} - \int\limits_{t_o}^{t_1} L\left(q^i, \frac{dq^i}{dt}, t\right) dt. \qquad (5)$$

Putting here

$$\bar{t} = t + \Delta t, \qquad \Delta\bar{t} = dt + d\left(\Delta t\right) = dt\left[1 + \frac{d}{dt}\left(\Delta t\right)\right],$$

this expression passes into

$$\Delta W = \int\limits_{t_o}^{t_1} \left\{L\left(q^i{+}\Delta q^i, \dot{q}^i{+}\Delta\dot{q}^i, t{+}\Delta t\right)\left[1 + \frac{d}{dt}\left(\Delta t\right)\right] - L\left(q^i, \dot{q}^i, t\right)\right\} dt. \qquad (6)$$

If the masses of particles on the varied path are not varied, the value of Lagrangian on it in the moment $t + \Delta t$ differs from its value on the direct path in the moment t only because of its explicit dependence on the time

$$L\left(q^i + \Delta q^i, \dot{q}^i + \Delta\dot{q}^i, t + \Delta t\right) = L\left(q^i, \dot{q}^i, t\right) +$$
$$+\Delta q^i\left(\frac{\partial L}{\partial q^i}\right) + \Delta\dot{q}^i\left(\frac{\partial L}{\partial \dot{q}^i}\right) + \Delta t\left(\frac{\partial L}{\partial t}\right) + \ldots$$

Neglecting in (6) the infinitesimal quantities of higher order, the total variation of action will approximately be given by

$$\Delta W = \int\limits_{t_o}^{t_1} \left[\frac{\partial L}{\partial q^i}\Delta q^i + \frac{\partial L}{\partial \dot{q}^i}\Delta\dot{q}^i + \frac{\partial L}{\partial t}\Delta t + L\frac{d}{dt}\left(\Delta t\right)\right] dt, \qquad (7)$$

or, passing from the total variations to the simultaneous ones

$$\Delta q^i = \delta q^i + \dot{q}^i\Delta t, \qquad \Delta\dot{q}^i = \delta\dot{q}^i + \ddot{q}^i\Delta t,$$

one obtains

$$\Delta W = \int_{t_o}^{t_1} \left[L\frac{d}{dt}(\Delta t) + \left(\frac{\partial L}{\partial q^i}\dot{q}^i + \frac{\partial L}{\partial \dot{q}^i}\ddot{q}^i + \frac{\partial L}{\partial t} \right) \Delta t + \left(\frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\delta \dot{q}^i \right) \right] dt. \tag{8}$$

However, in the considered case the dependence of Lagrangian on the time arises not only from its explicit dependence on ti, but also from the change of masses of particles in the course of time, so that

$$\frac{dL}{dt} = \frac{\partial L}{\partial q^i}\dot{q}^i + \frac{\partial L}{\partial \dot{q}^i}\ddot{q}^i + \frac{\partial L}{\partial t} + \frac{\partial L}{\partial m_\nu}\frac{dm_\nu}{dt}, \tag{9}$$

by which is determined the first term in parenthesis in (8). If one transforms the last term as

$$\frac{\partial L}{\partial \dot{q}^i}\delta \dot{q}^i = \frac{\partial L}{\partial \dot{q}^i}\frac{d}{dt}\left(\delta q^i\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\delta q^i\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\delta q^i,$$

the previous expression for ΔW can be represented in the form

$$\Delta W = \int_{t_o}^{t_1} \left[L\frac{d}{dt}(\Delta t) + \left(\frac{dL}{dt} - \frac{\partial L}{\partial m_\nu}\frac{dm_\nu}{dt} \right) \Delta t + \frac{\partial L}{\partial q^i}q^i + \right.$$

$$\left. + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\delta q^i\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\delta q^i \right] dt,$$

or, by grouping the similar terms and passing again to the total variations

$$\Delta W = \int_{t_o}^{t_1} \left\{ \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^i}\left(\Delta q^i - \dot{q}^i \Delta t\right) + L\Delta t \right] + \right.$$

$$\left. + \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} \right) \cdot \left(\Delta q^i - \dot{q}^i \Delta t\right) - \frac{\partial L}{\partial m_\nu}\frac{dm_\nu}{dt}\Delta t \right\} dt. \tag{10}$$

This expression differs from the usual one by one additional term $(\partial L/\partial m_\nu)\,(dm_\nu/dt)\,\Delta t$, which arises from the implicit dependence of Lagrangian of the time through the variable masses. Let us remark here that this result is surprising enough, since one would expect that the habitual expression for the total variation of action does remain unaltered, because on the varied path here the masses are not varied.

4 Generalized Emmy Noether's theorem

In order to formulate the corresponding statement concerning the first integrals, let us suppose that the Lagrangian equations (2) are satisfied and according to these substitute the variational derivative in the last term by $Q_i^* + P_i$, where by the variability of the masses is included in consideration. In addition, if in the integrand one adds and subtracts the term $d\Lambda/dt$, where Λ so-called gauge function can be any function of $q^k, \dot{q}^k$ and t, this expression passes into

$$\Delta W = \int_{t_o}^{t_1} \left\{ \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^i}\left(\Delta q^i - \dot{q}^i \Delta t\right) + L\Delta t + \Lambda \right] \right.$$

$$- \left[\left(Q_i^* + P_i\right)\left(\Delta q^i - \dot{q}^i \Delta t\right) + \right.$$

$$\left. \left. + \frac{\partial L}{\partial m_\nu}\frac{dm_\nu}{dt}\Delta t + \frac{d\Lambda}{dt} \right] \right\} dt. \qquad (11)$$

Let us consider the transformations of generalized coordinates and time in the form of a r-parametric group with r infinitesimal parameters ε_α $(\alpha = 1, 2, ..., r)$

$$\Delta q^i \stackrel{def}{=} \bar{q}^i(t) - q^i(t) = \varepsilon_\alpha \xi_i^\alpha \left(q^k, \dot{q}^k, t\right),$$

$$\Delta t \stackrel{def}{=} \bar{t} - t = \varepsilon_\alpha \xi_o^\alpha \left(q^k, \dot{q}^k, t\right), \quad (\alpha = 1, .., r). \qquad (12)$$

In such a manner, through these parameters the total variation of action (11) becomes dependent on this transformation group. On the other hand, the total variation of action can be transformed directly in the following way

$$\Delta W = \int_{t_o}^{t_1} \Delta (L dt) = \int_{t_o}^{t_1} \left[\Delta L + L \frac{d}{dt}(\Delta t) \right] dt. \qquad (13)$$

By inserting this expression into (11), after grouping the similar terms, one can present this relation in the form

$$\int_{t_o}^{t_1} \left\{ \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^i}\Delta q^i + \left(L - \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i\right)\Delta t + \Lambda \right] - \left[\Delta L + L\frac{d}{dt}(\Delta t) + \right. \right.$$

$$+\left(\tilde{Q}_i^* + P_i\right)\left(\Delta q^i - \dot{q}^i \Delta t\right) + \frac{\partial L}{\partial m_\nu}\frac{dm_\nu}{dt}\Delta t + \frac{d\Lambda}{dt}\bigg]\bigg\} dt = 0. \tag{14}$$

From here we can deduce the following conclusion: if

$$\Delta L + L\frac{d}{dt}(\Delta t) + \left(\tilde{Q}_i^* + P_i\right)\left(\Delta q^i - \dot{q}^i \Delta t\right) + \frac{\partial L}{\partial m_\nu}\frac{dm_\nu}{dt}\Delta t + \frac{d\Lambda}{dt} = 0, \tag{15}$$

the relation (14) is reduced to

$$\int_{t_o}^{t_1} \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^i}\Delta q^i + \left(L - \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i\right)\Delta t + \Lambda\right] dt = 0,$$

and since the time interval (t_o, t_1) is arbitrary, the integrand must be equal to zero, from where follows

$$I \equiv \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i + \left(L - \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i\right)\Delta t + \Lambda = const. \tag{16}$$

By interesting here the expressions (12) and putting $\Lambda = \varepsilon_\alpha \Lambda^\alpha$, one obtains

$$I = \varepsilon_\alpha \left\{\frac{\partial L}{\partial \dot{q}^i}\xi_i^\alpha + \left(L - \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i\right)\xi_o^\alpha + \Lambda^\alpha\right\},$$

and the parameters ε_α being mutually independent, hence follow r relations of the form

$$I^\alpha \equiv \frac{\partial L}{\partial \dot{q}^i}\xi_i^\alpha + \left(L - \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i\right)\xi_o^\alpha + \Lambda^\alpha = const, \quad (\alpha = 1, 2, ..., r)\,. \tag{17}$$

Consequently, to every transformation of the generalized coordinates and time (12) and for each gauge function Λ^α which satisfy the condition (15) correspond r mutually independent first integrals (17). This condition represents here the generalized basic Noether's identity (see [3], p. 119).

So formulated statement represents the generalized Emmy Noether's theorem for the arbitrary systems with variable mass. For oneparametric transformations ($r = 1$) in general it coincides with this theorem obtained by L.Cveticanin [16] via d'Alembert-Lagrange's principle, but differs from her results by a specific additional term $(\partial L/\partial m_\nu)(dm_\nu/dt)$ Δt. In the special case when all the works along the virtual displacements vanish i.e. $(Q_i^* + R_i^*)\,\delta q^i = 0,\ P_i\delta q^i = 0$ and in addition $(\partial L/\partial m_\nu)(dm_\nu/dt) = 0$ the condition (15) is simplified to

$$\Delta(Ldt) + d\Lambda = 0, \tag{18}$$

and the total variation of action (11) then obtains the form

$$\Delta W = \int_{t_o}^{t_1} \Delta(Ldt) = -\int_{t_o}^{t_1} \frac{d\Lambda}{dt}dt, \qquad \text{if } \frac{d\Lambda}{dt} = 0 : \Delta W = 0. \tag{19}$$

Therefore, so generalized Noether's theorem includes as a special case the usual formulation, in the habitual or extended sense. Let us remark that the form of these first integrals is the same as for conservative systems with constant mass, and the influence of the variability of the mass is expressed only through the specific terms P_i and $(\partial L/\partial m_\nu)(dm_\nu/dt)$ in the condition (15) for the existence of these first integrals.

5 Corresponding generalized Killing's equations

In order to find such transformations (12) which satisfy the condition (15), let us find the equations which must be satisfied by the functions ξ_i^α and ξ_o^α, generalizing to this case the method of Vujanovic and Djukic [7] who have for the first time formulated these as the means to find the first integrals. In this aim, we must start from this condition, put here

$$\Delta L = \frac{\partial L}{\partial q^i}\Delta q^i + \frac{\partial L}{\partial \dot{q}^i}\Delta \dot{q}^i + \frac{\partial L}{\partial t}\Delta t,$$

and apply the relation between the total variation of time derivative and the time derivative of total variation (see f.e. [18], p. 11)

$$\Delta \dot{q}^i = \frac{d}{dt}\left(\Delta q^i\right) - \dot{q}^i \frac{d}{dt}(\Delta t).$$

In this manner, the condition (15) can be written in the form

$$\frac{\partial L}{\partial q^i}\Delta q^i + \frac{\partial L}{\partial \dot q^i}\left[\frac{d}{dt}\left(\Delta q^i\right) - \dot q^i \frac{d}{dt}(\Delta t)\right] + \left(\frac{\partial L}{\partial t} + \frac{\partial L}{\partial m_\nu}\frac{dm_\nu}{dt}\right)\Delta t +$$

$$+ L\frac{d}{dt}(\Delta t) + \left(\tilde Q_i^* + P_i\right)\left(\Delta q^i - \dot q^i \Delta t\right) + \frac{d\Lambda}{dt} = 0. \qquad (20)$$

If one substitutes Δq^i and Δt by the corresponding expressions (12), because of independence of the parameters ε_α from here follows

$$\frac{\partial L}{\partial q^i}\xi_i^\alpha + \frac{\partial L}{\partial \dot q^i}\left(\frac{d\xi_i^\alpha}{dt} - \dot q^i \frac{\xi_o^\alpha}{dt}\right) + \left(\frac{\partial L}{\partial t} + \frac{\partial L}{\partial m_\nu}\frac{dm_\nu}{dt}\right)\xi_o^\alpha + L\frac{\xi_o^\alpha}{dt} +$$

$$+ \left(\tilde Q_i^* + P_i\right)\left(\xi_i^\alpha - \dot q^i \xi_o^\alpha\right) + \frac{d\Lambda^2}{dt} = 0, \quad (\alpha = 1, 2, ..., r). \qquad (21)$$

There are the corresponding generalized Killing's equations and their explicit form can be found by explicit writing all the total time derivatives of these functions and grouping the similar terms

$$\frac{\partial L}{\partial q^i}\xi_i^\alpha + \frac{\partial L}{\partial \dot q^i}\left(\frac{\partial \xi_i^\alpha}{\partial q^k}\dot q^k + \frac{\partial \xi_i^\alpha}{\partial t} - \frac{\partial \xi_o^\alpha}{\partial q^k}\dot q^i \dot q^k - \frac{\partial \xi_o^\alpha}{\partial t}\dot q^i\right) +$$

$$+ L\left(\frac{\partial \xi_o^\alpha}{\partial q^k}\dot q^k + \frac{\partial \xi_o^\alpha}{\partial t}\right) + \ddot q^k \left[\frac{\partial L}{\partial \dot q^i}\left(\frac{\partial \xi_i^\alpha}{\partial \dot q^k} - \frac{\partial \xi_o^\alpha}{\partial \dot q^k}\dot q^i\right) + L\frac{\partial \xi_o^\alpha}{\partial \dot q^k} + \frac{\partial \Lambda^\alpha}{\partial \dot q^k}\right] +$$

$$+ \left(\frac{\partial L}{\partial t} + \frac{\partial L}{\partial m_\nu}\frac{dm_\nu}{dt}\right)\xi_o^\alpha + \left(\tilde Q_i^* + P_i\right)\left(\xi_i^\alpha - \dot q^i \xi_o^\alpha\right) + \frac{\partial \Lambda^\alpha}{\partial q^k}\dot q^k + \frac{\partial \Lambda^\alpha}{\partial t} = 0.$$

Since the functions $\xi_i^\alpha, \xi_o^\alpha$ and Λ^α are independent on $\ddot q^k$ this relation will be satisfied only if all the terms close to $\ddot q^k$ and the remained part are equal to zero

$$\frac{\partial L}{\partial \dot q^i}\left(\frac{\partial \xi_i^\alpha}{\partial \dot q^k} - \frac{\partial \xi_o^\alpha}{\partial \dot q^k}\dot q^i\right) + L\frac{\partial \xi_o^\alpha}{\partial \dot q^k} + \frac{\partial \Lambda^\alpha}{\partial \dot q^k} = 0, \quad \begin{pmatrix} \alpha = 1, 2, ..., r \\ k = 1, 2, ..., n \end{pmatrix}, \qquad (22)$$

and

$$\frac{\partial L}{\partial q^i}\xi_i^\alpha + \frac{\partial L}{\partial \dot{q}^i}\left(\frac{\partial \xi_i^\alpha}{\partial q^k}\dot{q}^k + \frac{\partial \xi_i^\alpha}{\partial t} - \frac{\partial \xi_o^\alpha}{\partial q^k}\dot{q}^i\dot{q}^k - \frac{\partial \xi_o^\alpha}{\partial t}\dot{q}^i\right) +$$

$$+\left(\frac{\partial L}{\partial t} + \frac{\partial L}{\partial m_\nu}\frac{dm_\nu}{dt}\right)\xi_o^\alpha + L\left(\frac{\partial \xi_o^\alpha}{\partial q^k}\dot{q}^k + \frac{\partial \xi_o^\alpha}{\partial t}\right) +$$

$$+\left(\tilde{Q}_i^* + P_i\right)\left(\xi_i^\alpha - \dot{q}^i\xi_o^\alpha\right) + \frac{\partial \Lambda^\alpha}{\partial q^k}\dot{q}^k + \frac{\partial \Lambda^\alpha}{\partial t} = 0, \ (\alpha = 1, 2, ..., r). \quad (23)$$

Therefore, if at least one particular solution of this system of $r\,(n+1)$ generalized Killing's equations exists, the condition (15) for existence of the first integrals is satisfied and to every transformation group (12) correspond r mutually independent first integrals (17). These equations represent a extension of the generalized Killing's equations for nonconservative systems obtained by Vujanovic [8] to the systems with variable mass and for $r = 1$ differ from these only by the additional terms P_i and $(\partial L/\partial m_\nu)(dm_\nu/dt)$, characteristic for these systems.

6 Generalized inverse Noether's theorem

In order to formulate the corresponding inverse Noether's theorem, let us suppose that we know a set of r independent first integrals (17), which can be written in the form

$$I^\alpha = \frac{\partial L}{\partial \dot{q}^i}\bar{\xi}_i^\alpha + L\xi_o^\alpha + \Lambda^\alpha, \qquad (\alpha = 1, 2, ..., r), \quad (24)$$

where

$$\bar{\xi}_i^\alpha = \xi_i^\alpha - \dot{q}^i\xi_o^\alpha. \quad (25)$$

Extending the habitual method (see [2], p. 159) to the systems with variable mass, we shall depart from the total variation of action (10) and equalize this with the expression (13), where follows

$$\int_{t_o}^{t_1}\left\{\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^i}\Delta q^i + \left(L - \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i\right)\Delta t\right] - \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i}\right)\left(\Delta q^i - \dot{q}^i\Delta t\right) - \right.$$

$$-\frac{\partial L}{\partial m_\nu}\frac{dm_\nu}{dt}\Delta t-\Delta L-L\frac{d}{dt}(\Delta t)\Big\}\,dt=0.$$

Since the time interval (t_o, t_1) is arbitrary, this integrand must be equal to zero, where we can add and subtract the term $d\Lambda/dt$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^i}\Delta q^i+\left(L-\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i\right)\Delta t+\Lambda\right]-\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}-\frac{\partial L}{\partial q^i}\right)\left(\Delta q^i-\dot{q}^i\Delta t\right)-$$

$$-\left[\Delta L+L\frac{d}{dt}(\Delta t)+\frac{\partial L}{\partial m_\nu}\frac{dm_\nu}{dt}\Delta t+\frac{d\Lambda}{dt}\right]=0. \qquad (26)$$

The expression under the symbol $d(\cdot)/dt$ represents, according to (17) the first integral I, and the expression in the last parenthesis can be replaced from the condition (15) for the existence of the first integrals by $-\left(\tilde{Q}_i^*+P_i\right)\left(\Delta q^i-\dot{q}^i\Delta t\right)$

$$\frac{dI}{dt}-\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}-\frac{\partial L}{\partial q^i}\right)\left(\Delta q^i-\dot{q}^i\Delta t\right)+\left(\tilde{Q}_i^*+P_i\right)\left(\Delta q^i-\dot{q}^i\Delta t\right)=0. \qquad (27)$$

If one substitutes here Δq^i and Δt by (12), because of the independence of these parameters ε_α this relation decomposes into r independent relations:

$$\frac{dI^\alpha}{dt}-\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}-\frac{\partial L}{\partial q^i}\right)\bar{\xi}_i^\alpha+\left(\tilde{Q}_i^*+P_i\right)\bar{\xi}_i^\alpha=0, \qquad (\alpha=1,2,...,r), \qquad (28)$$

where $\bar{\xi}_i^\alpha$ is given by (25). These equations establish a connection between the first integrals and the associated transformation group in the differential form and represent a generalization of the corresponding equations from the cited textbook [2], to which are reduced for $\tilde{Q}_i^*=0$ and $P_i=0$.

If one writes explicitly all the total time derivatives, bearing in mid that L and I^α are the functions of $q^k, \dot{q}^k$and t, after grouping the terms with $\ddot{q}^k$ and the remained ones, we yield

$$\left(\frac{\partial I^\alpha}{\partial \dot{q}^k}-\frac{\partial^2 L}{\partial \dot{q}^i\partial \dot{q}^k}\bar{\xi}_i^\alpha\right)\ddot{q}^k+\left[\frac{\partial I^\alpha}{\partial q^k}\partial \dot{q}^k+\frac{\partial I^\alpha}{\partial t}-\right.$$

$$-\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial q^k}\partial \dot{q}^k + \frac{\partial^2 L}{\partial \dot{q}^i \partial t} - \frac{\partial L}{\partial q^i}\right)\bar{\xi}_i^\alpha + \left(\tilde{Q}_i^* + P_i\right)\bar{\xi}_i^\alpha\Bigg] = 0.$$

Since the expression between parenthesis close to $\ddot{q}^k$ and the remained one are not dependent on $\ddot{q}^k$, this relation will satisfied only if this first expression along each trajectory of the system is equal to zero, i.e.

$$W_{ik}\bar{\xi}_i^\alpha = \frac{\partial I^\alpha}{\partial \dot{q}^k}, \quad \begin{pmatrix} k = 1, 2, ..., n \\ \alpha = 1, 2, ..., r \end{pmatrix}, \tag{29}$$

where

$$W_{ik} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^k}. \tag{30}$$

These equations represent a system of nr linear equations which determine the quantities $\bar{\xi}_i^\alpha$ and if $|W_{ik}| \neq 0$, this is always possible. In this case the inverse matrix W^{-1} exists, and multiplying these by W_{kj}^{-1} and summing over the $W_{ik}W_{kj}^{-1}\bar{\xi}_i^\alpha = W_{kj}^{-1}\frac{\partial I^\alpha}{\partial \dot{q}^k}$, repeated indices since $W_{ik}W_{kj}^{-1} = \delta_{ij}$, one obtains

$$\bar{\xi}_i^\alpha = W_{kj}^{-1}\frac{\partial I^\alpha}{\partial \dot{q}^k} \qquad \begin{pmatrix} j = 1, 2, ..., n \\ \alpha = 1, 2, ..., r \end{pmatrix}. \tag{31}$$

The remained quantity ξ_o^α can be found immediately from the relation (24)

$$\xi_o^\alpha = \frac{1}{L}\left(I^\alpha - \frac{\partial L}{\partial \dot{q}^i}\bar{\xi}_i^\alpha - \Lambda^\alpha\right), \tag{32}$$

by which are also determined the quantities $\xi_i^\alpha = \bar{\xi}_i^\alpha + \dot{q}^i\xi_o^\alpha$, and in this manner the transformation group (12) as well.

Consequently, if the considered system is regular, i.e. $|W_{ik}| \neq 0$, to every set of r first integrals (24) and a gauge function Λ^α uniformly corresponds one r-parametric transformation group (12) of generalized coordinates and time. This statement represents a generalization of the usual inverse Noether's theorem to the arbitrary systems with variable mass. Let us here remark that this proof is independent of the fact whether the system is conservative, nonconservative or with variable mass, since the terms characteristic for these systems do not appear in the equations (29).

7 The case of degenerate systems

In the case when $\Delta \equiv |W_{ik}| = 0$ the systems of equations (29) cannot be completely solved with respect to the quantities $\bar{\xi}_i^\alpha$ and such physical systems are degenerate in the sense of Dirac [19], [20]. In this case too the Lagrange's variables (in contrast to the canonical variables) are independent, so that the direct Noether's theorem expressed in this way remains unchanged. Thus, under the condition (15) for the degenerate systems also to every r-parametric transformation group (12) of generalized coordinates and time correspond r independent first integrals (17).

In order to formulate the inverse Noether's theorem for these systems, let us write the equation (29) concisely as

$$Y_k^\alpha = W_{ik}\bar{\xi}_i^\alpha, \qquad Y_k^\alpha \equiv \frac{\partial I^\alpha}{\partial \dot{q}^k}, \quad \begin{pmatrix} k = 1, 2, ..., n \\ \alpha = 1, 2, ..., r \end{pmatrix} \tag{33}$$

and suppose that the rank of Hessian $|W_{ik}|$ is $R < n$. Then, according to the known theorem on implicit functions, the functions Y_k^α are no independent, between them exists a set of the relations of the form

$$F_p\left(Y_1^1, Y_2^1, ..., Y_n^r; q^k, \dot{q}^k, t\right) = 0, \qquad (p = 1, 2, ..., n - R). \tag{34}$$

They do not contain the variables $\bar{\xi}_i^\alpha$, their number is equal to the degree of degeneracy of the system $P = n - R$, and the appearance of such constraints is one of the characteristics of the degenerate systems.

These relations can be found in the following manner. If we solve the first R equations (33) with respect to the variables $\bar{\xi}_i^\alpha$, one yields them as functions of the first R quantities Y_k^α and the other $\bar{\xi}_i^\alpha$ for $i > R$

$$\bar{\xi}_\rho^\alpha = \bar{\xi}_\rho^\alpha\left(Y_{\sigma'}^\alpha, \bar{\xi}_\sigma^\alpha; q^k, \dot{q}^k, t\right), \qquad (\rho = 1, 2, ..., ; \sigma = R + 1, ..., n). \tag{35}$$

By inserting these functions into the remained $n - R$ equations (33), splitting these expressions to two corresponding parts for $i \leq R$ and for $i > R$, one obtains

$$Y_\sigma^\alpha = W_{i\sigma}\bar{\xi}_i^\alpha = W_{i\sigma}\bar{\xi}_\rho^\alpha\left(Y_{\rho'}^\alpha, \bar{\xi}_\sigma^\alpha, ...\right) + W_{\sigma\sigma'}\bar{\xi}_{\sigma'}^\alpha. \tag{36}$$

However, these equations cannot contain the variables $\bar{\xi}^{\alpha}_{\sigma}$, since in the opposite case it would be possible to solve the system of equations (33) with respect to more than R of the variables $\bar{\xi}^{\alpha}_{i}$, in contradiction with our supposition on the rank of Hessian. So obtained relations between the variables Y^{α}_{k} represent here in the explicit form the cited relations (34).

Therefore, if the physical system is degenerate, i.e. if $\Delta \equiv |W_{ik}| = 0$ with the rank $R < n$, from (33) one can determine only R variables $\bar{\xi}^{\alpha}_{\sigma}$ remain completely arbitrary. Thus, to every set of r first integrals (17) correspond infinitely many r-parametric transformation groups (12) with $n - R$ arbitrary functions $\bar{\xi}^{\alpha}_{\sigma}$. Between the first integrals I^{α} exist certain relations (34) in the differential form, the number of which is equal to the number of the degree of degeneracy, and they do not contain the variables $\bar{\xi}^{\alpha}_{i}$. This is the corresponding inverse Noether's theorem for degenerate systems, which differs from the habitual one by the appearance of certain arbitrary functions, also characteristic for the degenerate systems.

8 An example

Let us illustrate the obtained results by a simple, but characteristic example: a linear harmonic oscillator in a damped medium with variable mass because of its separation. This system is determined by

$$L = \frac{1}{2}m\dot{x}^2 mk^2x^2, \qquad Q^* = -m\beta\dot{x}, \tag{37}$$

and by the law of mass variation. Suppose that the mass of this oscillator decreases exponentially in the course of time and that the velocity of separation of the mass is proportional to the velocity of the considered oscillator

$$m = m_o e^{-\alpha t}, \qquad u = \lambda\dot{x}. \tag{38}$$

The corresponding term P, according to (4) and (38) will be

$$P = \frac{dm}{dt}\mathbf{u}\cdot\frac{\partial\mathbf{r}}{\partial x} = -\alpha m_o e^{-\alpha t}u_x,$$

therefore proportional to the velocity of oscillator

$$P=-\alpha m\lambda\dot{x}, \tag{39}$$

so that the Lagrangian equation (2) for this oscillator is

$$\frac{d}{dt}(m\dot{x})+mk^2x=-m\beta_1\dot{x}, \qquad \beta_1=\beta+\alpha\lambda. \tag{40}$$

In order to formulate this problem in a variational form, let us consider any onedimensional nonconservative system with variable mass, whose motion is determined by

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{x}}-\frac{\partial L}{\partial x}=\tilde{Q}^*+P, \tag{41}$$

and attempt to find a such Lagrangian

$$\tilde{L}(x,\dot{x},t)=f(t)\,L(x,\dot{x},t)\,, \tag{42}$$

so that this equation passes into Euler-Lagrange's equation

$$\frac{d}{dt}\frac{\partial\tilde{L}}{\partial\dot{x}}-\frac{\partial\tilde{L}}{\partial x}=0. \tag{43}$$

By inserting this expression in (43), after grouping the similar terms one obtains

$$\frac{df}{dt}\frac{\partial L}{\partial\dot{x}}+f\left(\frac{df}{dt}\frac{\partial L}{\partial\dot{x}}-\frac{\partial L}{\partial x}\right)=0,$$

where we can substitute this variational derivative according to (41) by $\tilde{Q}^*+P$

$$\frac{\partial L}{\partial\dot{x}}\frac{df}{dt}+\left(\tilde{Q}^*+P\right)f=0. \tag{44}$$

In our case, on the basis of (37) this equation receives the form

$$m\dot{x}\frac{df}{dt}-m\beta_1\dot{x}f=0, \tag{45}$$

with β_1 given by (40), and after separation of variables one yields

$$\frac{df}{f}=\beta_1 dt\Longrightarrow f(t)=e^{\beta_1 t},\quad (c=1)\,,$$

so that the new Lagrangian (42) will be

$$\tilde{L}\left(x,\dot{x},t\right)=e^{\beta_1 t}\left(\frac{1}{2}m\dot{x}^2-\frac{1}{2}mk^2x^2\right). \tag{46}$$

Bearing in mind that according to (38) $m = m_o e^{-\alpha t}$, this expression can be written also in an equivalent manner

$$\tilde{L}\left(x,\dot{x},t\right)=e^{\beta_1' t}\left(\frac{1}{2}m_o\dot{x}^2-\frac{1}{2}m_o k^2x^2\right), \tag{47}$$

where

$$\beta_1'=\beta_1-\alpha=\beta+\alpha\left(\lambda-1\right), \tag{48}$$

in accordance with the presentation of such differential equations in the variational form [21].

Since the considered oscillator in this way is described by such Lagrangian which satisfies the Euler-Lagrange's equation (43), the generalized Noether's theorem here is applicable in the usual form. In this case the specific term for such systems is, on the basis of (46)

$$\frac{\partial\tilde{L}}{\partial m}\frac{dm}{dt}=e^{\beta_1 t}\left(\frac{1}{2}\dot{x}^2-\frac{1}{2}k^2x^2\right)\left(-\alpha m_o e^{-\alpha t}\right),$$

or, according to (38)

$$\frac{\partial\tilde{L}}{\partial m}\frac{dm}{dt}=-\alpha e^{\beta_1' t}\left(\frac{1}{2}m_o\dot{x}^2-\frac{1}{2}k^2x^2\right) \tag{49}$$

i.e. proportional to the relative change of mass in the time unit. Therefore, the corresponding generalized Killing's equations (22) and (23) for this Lagrangian (47), where one must put $\tilde{Q}^*=0$ and $P=0$, here have the form

$$e^{\beta_1' t}m_o\dot{x}\left(\frac{\partial\xi_1}{\partial\dot{x}}-\frac{\partial\xi_o}{\partial\dot{x}}\dot{x}\right)+e^{\beta_1' t}\left(\frac{1}{2}m_o\dot{x}^2-\frac{1}{2}m_o k^2x^2\right)\frac{\partial\xi_o}{\partial\dot{x}}+\frac{\partial\Lambda}{\partial\dot{x}}=0, \tag{50}$$

and

$$e^{\beta_1' t}\left(-m_o k^2 x\right)\xi_1+e^{\beta_1' t}m_o\dot{x}\left(\frac{\partial\xi_1}{\partial x}\dot{x}+\frac{\partial\xi_1}{\partial t}-\frac{\partial\xi_o}{\partial x}\dot{x}^2-\frac{\partial\xi_o}{\partial t}\dot{x}\right)+$$

$$+\beta_1' e^{\beta_1' t}\left(\frac{1}{2}m_o\dot{x}^2-\frac{1}{2}m_o k^2x^2\right)\xi_o-\alpha e^{\beta_1' t}\left(\frac{1}{2}m_o\dot{x}^2-\frac{1}{2}m_o k^2x^2\right)\xi_o+$$

$$+e^{\beta_1' t}\left(\frac{1}{2}m_o\dot{x}^2-\frac{1}{2}m_o k^2x^2\right)\left(\frac{\partial\xi_o}{\partial x}\dot{x}+\frac{\partial\xi_o}{\partial t}\right)+\frac{\partial\Lambda}{\partial x}\dot{x}+\frac{\partial\Lambda}{\partial t}=0. \quad (51)$$

The analogous example for an oscillator in a damped medium with constant mass was studied by Vujanovic [22] and one particular solution is found with the aid of the generalized Killing's equations for $m = const$ (in our notations):$\xi_o = 1, \xi_1 = -0.5\beta x$ and $\Lambda = 0$. In this manner is obtained, for the first time for such an oscillator, for which the usual energy conservation law is not valid, a first integral of the form

$$I_o = e^{\beta t}\left(\frac{1}{2}\dot{x}^2+\frac{1}{2}k^2x^2+\frac{1}{2}\beta x\dot{x}\right)=const, \quad (52)$$

which here plays the role of energy-like conservation law.

This suggests us that in the considered case we can try to find such a particular solution of corresponding equations in the form

$$\xi_o = 1, \qquad \xi_1 = -\frac{1}{2}(\beta + a)\,x, \qquad \Lambda = 0, \quad (53)$$

where a is a constant. By inserting these expressions into (50) and (51) it follows that the first of these equations is satisfied identically, because ξ_o and ξ_1 are independent on $\dot{x}$. The second of these equations, after canceling the exponential factor and substituting β_1' by (48) will be reduced to

$$m_o k^2 x\cdot\frac{1}{2}(\beta+a)\,x-m_o\dot{x}\cdot\frac{1}{2}(\beta+a)\,\dot{x}+\beta\left(\frac{1}{2}m_o\dot{x}^2-\frac{1}{2}m_o k^2x^2\right)+$$

$$+\alpha(\Lambda-1)\left(\frac{1}{2}m_o\dot{x}^2-\frac{1}{2}m_o k^2x^2\right)-\alpha\left(\frac{1}{2}m_o\dot{x}^2-\frac{1}{2}m_o k^2x^2\right)=0, \quad (54)$$

or, putting it in order

$$\frac{1}{2}m_o k^2x^2\left[a-\alpha(\Lambda-1)+\alpha\right]-\frac{1}{2}m_o\dot{x}^2\left[a-\alpha(\Lambda-1)+\alpha\right]=0,$$

and this relation will be satisfied identically only if

$$a = \alpha(\Lambda-2). \quad (55)$$

From here results the new coefficient in (53)

$$\beta + a = \beta + \alpha(\Lambda - 2) = \beta_1' - \alpha,$$

so that such a particular solution of these generalized Killing's equations exists and has the form

$$\xi_o = 1, \qquad \xi_1 = -\frac{1}{2}\beta_1'' x_1, \qquad \Lambda = 0, \qquad (\beta_1'' = \beta_1' - \alpha). \qquad (56)$$

The corresponding first integral, associated to this Lagrangian will be given by (17)

$$I = \frac{\partial \tilde{L}}{\partial \dot{x}}\xi_1 + \left(\tilde{I} - \frac{\partial \tilde{L}}{\partial \dot{x}}\dot{x}\right)\xi_o \qquad (57)$$

In this manner, by inserting (47) and (56) into this expression, one obtains

$$I = e^{\beta_1' t} m_o \dot{x}\left(\frac{1}{2}\beta_1'' x\right) + e^{\beta_1' t}\left(\frac{1}{2}m_o \dot{x}^2 - \frac{1}{2}m_o k^2 x^2\right) - \left(e^{\beta_1' t} m_o \dot{x}\right)\dot{x},$$

i.e. a first integral, which can be written in the form

$$I_1 \equiv e^{\beta_1' t}\left[\left(\frac{1}{2}m_o \dot{x}^2 + \frac{1}{2}m_o k^2 x^2\right) + \frac{1}{2}\beta_1'' x\dot{x}\right] = const, \qquad (58)$$

where β_1' is given by (48). This here plays the role of energy-like conservation law, which differs from the corresponding one for this oscillator with permanent mass (52) found by Vujanovic only by the altered constant terms β_1' and $\beta_1'' = \beta_1' - \alpha$ instead of β, and for $\alpha = 0$ will be reduced to it.

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Generalizacija Noether-ine teoreme na sisteme

sa promenljivom masom

U ovom radu je pokazano da se generalizacija Noether-ine teoreme na sisteme sa promenljivom masom može izvesti direktno, polazeći od

totalne varijacije dejstva, koja ovde ima jedan dopunski specifični član, i koristeći odgovarajuće Lagrange-ove jednačine. Na taj način dobijena je direktna Noether-ina teorema za takve sisteme u opštijem obliku, što predstavlja generalizaciju ove teoreme koju je dobila Livija Cvetićanin na indirektan nacin iz d'Alembert-Lagrange-ovog principa. Sem toga, formulisane su odgovarajuće Killing-ove jednačine i inverzna Noether-ina teorema, uključujući i degenerisane sisteme. Dobijeni rezultati su ilustrovani na jednom prostom, ali karakterističnom primeru.