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Vacuum cosmological models in Wesson's theory

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Abstract

Vacuum cosmological models in the simple formulation of scale-invariant theory proposed by P. S. Wesson [1] with a gauge function $\beta(x)$ are obtained through the solutions of the field equations for three different cases when the space time is described by Bianchi type-I metric. Some physical properties of the models are also discussed.

1 Introduction

Wesson [1] formulated a Scale-invariant theory of gravitation using a gauge function $\beta(\bar{x})$ where $\bar{x}$ are coordinates in the four dimensional space time and the tensor field is identified with the metric tensor g_{ij}. His theory is both coordinate and scale-invariant in nature. The field equations formulated by Wesson for the combined scalar and tensor fields are

$$G_{ij} + \frac{2\beta_{;ij}}{\beta} - \frac{4\beta_{,i}\beta_{,j}}{\beta^2} + \left\{ \frac{g^{ab}\beta_{,a}\beta_{,b}}{\beta^2} - \frac{2g^{ab}\beta_{;ab}}{\beta} \right\} g_{ij} + \Lambda_o \beta^2 g_{ij} = -k \cdot T_{ij} \tag{1}$$

Here, G_{ij} is the conventional Einstein tensor involving g_{ij} and semicolon (;) means covariant differentiation with respect to g_{ij} and comma (,) means partial differentiation with respect to coordinates. All raising and lowering operations are carried out by using g_{ij}. The cosmological term Λg_{ij} of Einstein's theory transformed to $\Lambda_o \beta^2 g_{ij}$ in scale-invariant theory with a dimensionless constant Λ_o, T_{ij} is the energy-momentum tensor of the matter field and $k = (8\Pi G/c^4)$.

Omote [2]; Dirac [3] ; Hoyle and Narlikar [4]; and Canuto *et al.* [5]; are some of the authors who have studied to establish a new scale-invariant theory in order to have a good match with the experimental evidences. A comprehensive review of scale-invariant theories of both gravity and particle Physics has been done by Wesson [6]. But the Wesson's [1] formulation of scale-invariant theory is so far the best theory to describe all the interactions of Physics with the gravitational interaction.

In this paper, we study the Bianchi type-I spatially homogeneous and anisotropic cosmological model in Wesson's [1] simple formulation of scale-invariant theory of gravitation when the source of the matter field $T_{ij} = 0$. In Sec. 2, we formulate the vacuum field equations. In Sec. 3, we obtain explicit exact solutions of vacuum field equations for three different cases in order to overcome the mathematical complexities. We find that the model in the first case ($\Lambda_o = 0$) degenerates the well known Kasner model in Einstein's theory of gravitation. In other two cases, the models reduce to spatially-homogeneous isotropic space time in Einstein's theory. In Sec. 4, some physical properties of the solutions are studied. The model in the first case, starts from a singularity (may be Bigbang) of zero proper volume and increases indefinitely with time. In this section it is also shown that for the last two cases, the universe starts expanding from a constant volume and become indefinitely large. The anisotropy as well as the expansion scalar are also indefinitely large near the singularity in the first model. The other two models are expanding in nature and do not have initial singularity. Sec. 5, being last section includes conclusion.

2 Field equations

We consider the Bianchi type-I metric with gauge function β

$$ds_w^2 = \beta^2\left[-A^2dx^2 - B^2dy^2 - C^2dz^2 + c^2dt^2\right], \tag{2}$$

where A, B, C are functions of time t, c is the velocity of light. This ensures that the model is spatially-homogeneous. With the metric (2) the field equations (1) in vacuum may be written as

$$\frac{B_{44}}{B} + \frac{C^2_{44}}{C} + \frac{B_4C_4}{BC} - \frac{\beta_4}{\beta^2} + 2\frac{\beta_4}{\beta}\left\{\frac{B_4}{B} + \frac{C_4}{C}\right\} + 2\frac{\beta_{44}}{\beta} - \Lambda_o\beta^2c^2 = 0, \tag{3}$$

$$\frac{C_{44}}{C} + \frac{A^2_{44}}{A} + \frac{C_4A_4}{CA} - \frac{\beta_4^2}{\beta^2} + 2\frac{\beta_4}{\beta}\left\{\frac{C_4}{C} + \frac{A_4}{A}\right\} + 2\frac{\beta_{44}}{\beta} - \Lambda_o\beta^2c^2 = 0, \tag{4}$$

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4B_4}{AB} - \frac{\beta_4^2}{\beta^2} + 2\frac{\beta_4}{\beta}\left\{\frac{A_4}{A} + \frac{B_4}{B}\right\} + 2\frac{\beta_{44}}{\beta} - \Lambda_o\beta^2c^2 = 0, \tag{5}$$

$$\frac{A_4B_4}{AB} + \frac{B_4C_4}{BC} + \frac{C_4A_4}{CA} + \frac{3\beta_4^2}{\beta^2} + \frac{2\beta_4}{\beta}\left\{\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C}\right\} - \Lambda_o\beta^2c^2 = 0, \tag{6}$$

where the suffix 4, following an unknown function denotes ordinary differentiation with respect to t.

Following Wesson's [1] simple formulation of scale-invariant theory we take the gauge function in the form

$$\beta = \frac{1}{ct} \tag{7}$$

Now using the gauge function (7) in the equation (3)-(6) we get

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4C_4}{BC} - \frac{2}{t}\left\{\frac{B_4}{B} + \frac{C_4}{C}\right\} = \frac{\Lambda_o - 3}{t^2}, \tag{8}$$

$$\frac{C_{44}}{C} + \frac{A_{44}}{A} + \frac{C_4A_4}{CA} - \frac{2}{t}\left\{\frac{C_4}{C} + \frac{A_4}{A}\right\} = \frac{\Lambda_o - 3}{t^2}, \tag{9}$$

$$\frac{A_{44}}{B} + \frac{B_{44}}{C} + \frac{A_4B_4}{AB} - \frac{2}{t}\left\{\frac{A_4}{A} + \frac{B_4}{B}\right\} = \frac{\Lambda_o - 3}{t^2}, \tag{10}$$

$$\frac{A_4B_4}{AB} + \frac{B_4C_4}{BC} + \frac{C_4A_4}{CA} - \frac{2}{t}\left\{\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C}\right\} = \frac{\Lambda_o - 3}{t^2}. \tag{11}$$

Adding equations (8), (9), (10) to 3 times of (11), we get

$$\frac{(ABC)_{44}}{ABC} - \frac{5(ABC)_4}{t \cdot ABC} - \frac{3(\Lambda_o - 3)}{t^2} = 0,$$

which leads to

$$t^2(ABC)_{44} - 5t(ABC)_4 - 3(\Lambda_o - 3)ABC = 0. \tag{12}$$

3 Solutions

To solve the field equations for exact and explicitly solutions, here we consider the following three cases:

Case (i): $\Lambda_o = 0$.

Since A, B, C are non-zero, the equations (12) yields

$$ABC = t^3 \left[M \ln t + N\right], \tag{13}$$

where M ($\neq 0$) and N are real constants. Thus the solution of the field equations (8)-(11) in view of equation (13) may be written as

$$A = t\left[M \ln t + N\right]^{p_1},$$

$$B = t\left[M \ln t + N\right]^{p_2}, \tag{14}$$

$$C = t\left[M \ln t + N\right]^{p_3}.$$

where p_i 's are constants and related by

$$\sum_{i=1}^{3} p_i = 1. \tag{15}$$

Here the overdeterminacy for determination of three unknowns A, B, C from equations (8)-(11) can be settled by actual substitution of the solution (14) in the field equations. Then we get another relation

$$\sum_{i<j} p_i p_j = 0, \quad i, j = 1, 2, 3. \tag{16}$$

Now equation (15) and (16) lead to a relation

$$\sum_{i=1}^{3} p_i^2 = 1. \tag{17}$$

Therefore the model for an anisotropic homogeneous universe in the scale-invariant theory can be put in the form

$$ds^2 = \beta^2 \left[-t^2 \{M \ln t + N\}^{2p_1} dx^2 - t^2 \{M \ln t + N\}^{2p_2} dy^2 - \right.$$
$$\left. - t^2 \{M \ln t + N\}^{2p_3} dz^2 + c^2 dt^2 \right], \tag{18}$$

with $\beta = 1/ct$. This model may be transformed through a proper choice of coordinates to the form

$$ds^2 = \beta'^2 \left[-T_1^{2p_1} dX^2 - T_1^{2p_2} dY^2 - T_1^{2p_3} dZ^2 + c^2 dT_1^2 \right], \tag{19}$$

where $\beta^2 = (1/Mc) = constant$.

Again, further transformation of coordinates reduces the metric (19) to the form

$$ds^2 = -T_1^{2p_1} dX^2 - T_1^{2p_2} dY^2 - T_1^{2p_3} dZ^2 + dT_1^2. \tag{20}$$

In this case, we observe that the model in the scale-invariant theory reduces to that of Einstein with proper choice of coordinates in absence of matter field that is $T_{ij} = 0$. Moreover, the relations (15) and (17) reveal that the Bianchi type-I vacuum model in the framework of simple formulation of scale-invariant theory proposed by Wesson (1981) reduces to well known Kasner [7] model.

Case (ii): $\Lambda_o = 3$.

In this case, equation (12) yields

$$ABC = \frac{M}{6} t^6 + N, \tag{21}$$

where M $(\neq 0)$ and N are constants.

Subsequently, the solution of the field equations (8)-(11) in view of equation (21) may be written as

$$A = \left[\frac{M}{6}t^6 + N\right]^{p_1},$$

$$B = \left[\frac{M}{6}t^6 + N\right]^{p_2}, \tag{22}$$

$$C = \left[\frac{M}{6}t^6 + N\right]^{p_2},$$

where p_i 's are constants and related by

$$\sum_{i=1}^{3} p_i = 1. \tag{23}$$

As before, the overdeteminacy for determination of three unknowns A, B, C from equations (8)-(11) can be settled by actual substitution of the solution (22) in the field equations. Hence we get

$$\sum_{i<j} p_i p_j = 2\left[\frac{\frac{M}{6}t^6 + N}{Mt^6}\right], \quad i, j = 1, 2, 3. \tag{24}$$

But, $\sum_{i<j} p_{ij}$ must be constant, since p_i 's are constants. To avoid this contradiction for a real physical situation, we must have $N = 0$ and therefore,

$$\sum_{i<j} p_i p_j = \frac{1}{3}, \tag{25}$$

and

$$\sum_{i=1}^{3} p_i^2 = \frac{1}{3}.$$

Thus subject to restriction (25), equation (21) yields the following admissible solution

$$A = B = C = \left[\frac{M}{6}\right]^{1/3} t^2. \tag{26}$$

In view of equation (26) the metric (2) takes the form

$$ds^2 = -(M/6c^3)^{2/3}t^2\left[dx^2 + dy^2 + dz^2\right] + dt^2/t^2. \tag{27}$$

However, this metric can be transformed through a proper choice of coordinates to the form

$$ds^2 = -(M/6c^3)^{2/3}\exp(2T_2)\left[dX^2 + dY^2 + dZ^2\right] + dT_2^2 \tag{28}$$

This indicates that the model (28) is spatially-homogeneous isotropic space time in Einstein's theory and which is the degenerated form of spatially-homogeneous anisotropic Bianchi type-I space time in the simple formulation of scale-invariant theory.

Case (iii): $\Lambda_o \neq 3$.

In this case equation (12) yields

$$ABC = \left[Mt^{\sqrt{3\Lambda_o}} + Nt^{-\sqrt{3\Lambda_o}}\right] \tag{29}$$

which $M(\neq 0)$ and N are constants.

Therefore, the solution of the field equations (8)-(11) in view of equation (29) may be written as

$$\begin{aligned} A &= t\left[Mt^{\sqrt{3\Lambda_o}} + Nt^{-\sqrt{3\Lambda_o}}\right]^{p_1}, \\ B &= t\left[Mt^{\sqrt{3\Lambda_o}} + Nt^{-\sqrt{3\Lambda_o}}\right]^{p_2}, \\ C &= t\left[Mt^{\sqrt{3\Lambda_o}} + Nt^{-\sqrt{3\Lambda_o}}\right]^{p_3}, \end{aligned} \tag{30}$$

where $\Lambda_o \geq 0$ and p_i 's are constants related by

$$\sum_{i=1}^{3} p_i = 1. \tag{31}$$

As before, the overdeteminacy for determination of three unknowns A, B, C from equation (8)-(11) can be settled by actual substitution of the solution (30) in the field equations. Hence we obtain the another condition.

$$\sum_{i<j} p_i p_j = \frac{1}{3}\left[\frac{Mt^{\sqrt{3\Lambda_o}} - Nt^{-\sqrt{3\Lambda_o}}}{Mt^{\sqrt{3\Lambda_o}} + Nt^{-\sqrt{3\Lambda_o}}}\right]^2, \qquad i,j = 1,2,3. \tag{32}$$

Since p_i 's are constants, then

$$\sum_{i<j} p_i p_j, \qquad \text{must be constant,} \tag{33}$$

which is true only when $N = 0$.

Therefore $\sum_{i<j} p_i p_j = 1/3$ and

$$\sum_{i=1}^{3} p_i^2 = \frac{1}{3},$$

subsequently, subject to restriction of equation (33); equation (29) yields the following admissible solution

$$A = B = C = M^{1/3} t^{1+\sqrt{3\Lambda_o}/3}. \tag{34}$$

In view of equation (34) the metric (2) takes the form

$$ds^2 = -(M/c^3)^{2/3} t^{2\sqrt{3\Lambda_o}/3}\left[dx^2 + dy^2 + dz^2\right] + dt^2/t^2 \tag{35}$$

However, this metric can be transformed through a proper choice of coordinates to the form

$$ds^2 = -(M/c^3)^{2/3} \exp((2\sqrt{3\Lambda_o/3})T_2)\left[dX^2 + dY^2 + dZ^2\right] + dT_2^2. \tag{36}$$

This case also indicates that the model (36) is spatially-homogeneous isotropic space time in Einstein's theory and which is the degenerated form of spatially-homogeneous anisotropic Bianchi type-I space time in the simple formulation of Scale-invariant theory.

4 Some physical properties

In this section, we intend to study some physical properties of the solutions obtained in this paper.

(i) *Expansion Scalar and Anisotropy:*

Following Roychoudhuri [8], the expansion scalar θ and the anisotropy [6] are defined as

$$\theta = (3R_4/R), \tag{37}$$

and

$$\sigma^2 = \frac{1}{12}\left[\left\{\frac{g_{11,4}}{g_{11}} - \frac{g_{22,4}}{g_{22}}\right\}^2 + \left\{\frac{g_{22,4}}{g_{22}} - \frac{g_{33,4}}{g_{33}}\right\}^2 + \left\{\frac{g_{33,4}}{g_{33}} - \frac{g_{11,4}}{g_{11}}\right\}^2\right], \tag{38}$$

where the volume element is defined as

$$R^3 = ABC\beta^3. \tag{39}$$

Now, for the model (20) we have

$$\theta = \frac{M}{T_1 \exp\frac{T_1 - N}{M}} > 0,$$

and

$$\sigma^2 = \frac{(2/3)M^2}{T_1^2 \exp\left[\frac{2(T_1-N)}{M}\right]} \neq 0.$$

So both these scalars become indefinitely large or indefinitely small according $T_1 \dashrightarrow 0$ or $T_1 \dashrightarrow \infty$ respectively, but for the ratio or anisotropy to expansion, we have

$$\lim_{T_1 \to 0} \frac{\sigma^2}{\theta^2} = \frac{2}{3} \neq 0.$$

Hence there is a singularity at $T_1 = 0$ and the model is not isotropy for finite T_1 and does not approach isotropy for large values of T_1 where as the universe is expanding as $\theta > 0$.

Now for the models (28) and (36), we have

$$\theta \rightarrow constant, \qquad \text{as } T_2 \rightarrow 0,$$

and

$$\theta \rightarrow 0 \qquad \text{as } T_2 \dashrightarrow \infty.$$

In both the cases, the expansion scalar being positive indicates that the models are expanding in nature. However in both the cases, we find $\sigma^2 = 0$ which shows that the spatially-homogeneous anisotropic vacuum cosmological models in Scale-invariant theory reduce to spatially-homogeneous isotropic models. Both the models have no singularity at $T_2 = 0$.

(ii) Proper Spatial Volume:

The Proper volume $R^3 = ABC \cdot \beta^3$ for the model (19) becomes

$$R^3 = \frac{T_1}{M^3 c^3},$$

which expands with T_1 indicating that the model starts expanding from zero volume.

Again, for the models (28) and (36), we have

$$R^3 = \frac{M}{6c^3} \exp(3T_2),$$

and

$$R^3 = \frac{M}{c^3} \exp(\sqrt{3\Lambda_o T_2}) \qquad \text{respectively,}$$

In these two cases, the universe starts expanding from a constant volume and become indefinitely large as $T_2 \rightarrow \infty$.

(iii) Volume element:

The volume element for the model (19) is

$$V = \sqrt{-g} = \frac{T_1}{M^4 c^3} \rightarrow 0 \text{ or } \infty \text{ according as } T_1 \rightarrow 0 \text{ or } \infty.$$

This shows that the universe expands with time, which supports the analysis done earlier. From (15) and (17) it follows that at least one of p_1, p_2 and p_3 must be negative for nontrivial case (i.e. $p_i \neq 0$, $i = 1, 2, 3$). This also indicates the existence of singularity at $T_1 = 0$ in the model (19).

For the volume elements for the model (28) and (36) we have

$$V = \sqrt{-g} = \frac{M}{6c^3} \exp(2T_2) \to \infty, \quad \text{as } T_2 \to \infty,$$

and

$$V = \sqrt{-g} = \frac{M}{c^3} \exp(\sqrt{3\Lambda_o T_2}) \to \infty \quad \text{as } T_2 \to \infty,$$

respectively.

Here, also indicate that the universe expands with time T_2. However in these cases, the model does not have singularity at $T_2 = 0$.

(iv) Kasner Model:

It is interesting to note that the vacuum spatially-homogeneous anisotropic model (20) in Scale-invariant theory represent well known Kasner [7] model with $\sum p_i = 1$ and $\sum p_i^2 = 1$, whereas other two cases do not represent so, but they may represent Mixmaster Universe for $N \neq 0$ near the singularity corresponding to $T_2 = 0$ where the Kasner exponents p_i can become functions of T_2. Then the exponents p_i can be described in terms of Lifshitz-Khalatnikov [9] parameter U as

$$p_1 = \frac{U}{1 + U + U^2},$$

$$p_2 = \frac{1 + U}{1 + U + U^2},$$

$$p_3 = \frac{U(1 + U)}{1 + U + U^2}.$$

For fixed U the models reveal Kasner like behavior whereas for large values of U the geometry of Mixmaster Universe is closed to Kasner model with $p_1 = 1$ and $p_2 = 0$ and $p_3 = 0$. In the present situation all the cases, the metrics lead to the form

$$ds^2 = e^{2\eta}(-d\eta^2 + dx^2) + dy^2 + dz^2,$$

where $\eta = \ln T$.

(v) Rate of expansion:

In case of model (20) the distances parallel to x, y, z-axes expand anisotropically at the rates $T_1^{p_1}, T_1^{p_2}, T_1^{p_3}$ respectively, but for other two cases the model being isotropic in nature expands uniformly in all the spatial directions.

5 Conclusion

It is interesting to note that one of the cosmological models obtained here [i.e. equation (20)] in case of Scale-invariant theory the space time remains anisotropy and represents Kasner model in Einstein's theory; however the generalization of Kasner model does not exist in vacuum Scale-invariant theory; which is similar to the case studied earlier by Reddy and Venkateswarlu [10] in Biometric theory of Rosen [11]. The other interesting part of the investigation is that the vacuum Scale-invariant theory in the last two cases [i.e. equation (28) & (36)] isotropizes the vacuum models provided that the exponents p_i's are constants.

The models obtained here are quite relevant to Scale-invariant theory as they completely depend on time dependent gauge function governing the theory.

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Vakum kosmološki modeli u teoriji Vesona

Vakum kosmološki modeli u jednostavnoj formulaciji Vesona sa "gauge" funkcijom $\beta(x)$ su dobijeni pomoću rešenja jednačina polja za tri različita slučaja metrike Bjankijevog tipa. Analizirana su neka fizička svojstva modela.