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Anisotropic Brans-Dicke field

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Abstract

A class of solution of B-D field equations are obtained when the space time is described by cylindrically symmetric Jordan-Ehlers metric with three parameters in two variables. It is found that the solutions are physically realistic in the range $0 < r < \infty$ and radiating in nature but do not represent line mass along the axis of symmetry.

1 Introduction

Brans-Dicke [1] have proposed a new theory of gravitation (known as B-D theory) by introducing a long range scalar field. They have modified Einstein theory in such a way that their theory is more suitable to satisfy the requirements of Mach's principle. This theory has two alternative mathematical representations. In the first representation usually known as canonical representation, the field equations may be written as

$$G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij} = \\ = \frac{8\pi}{\Phi}T_{ij} - \frac{W}{\Phi^2}(\Phi_{,i}\Phi_{,j} - \frac{1}{2}g_{ij}\Phi_{,k}\Phi^{,k}) - \frac{1}{\Phi}(\Phi_{;ij} - g_{ij}\Phi^{;k}_{;k}), \qquad (1)$$

and

$$\Phi^{\alpha}_{;\alpha} = \frac{8\pi}{3+2W} T \qquad (2)$$

where Φ is B-D scalar field and the parameter W is a dimensionless constant. Here the comma and semicolon denote partial and co-variant differentiations respectively with respect to g_{ij}.

The second representation of this theory is known as Dicke [3] representation which can be obtained from the canonical representation by unit transformation when the units of length, time and reciprocal mass are changed by a scalar factor. The field equations in this representation can be written as

$$\bar{G}_{ij} \equiv \bar{R}_{ij} - \frac{1}{2}\bar{R}g_{ij} = -8\pi G\bar{T}_{ij} - \frac{1}{2}(2W+3)(\psi_{,i}\psi_{,j} - \frac{1}{2}\bar{g}_{ij}\psi_{,k}\psi^{,k}) \qquad (3)$$

and

$$\bar{g}^{ab}\psi_{:ab} = \frac{8\pi G}{3+2W}T, \qquad (4)$$

where G is the Newtonian Gravitational constant, $\psi = \ln(\Phi/\bar{\Phi})$ and colon represents covariant derivative with respect to $\bar{g}_{ij}$. Eventhough both the representations are physically equivalent, the canonical representation is mathematically suitable to study the equations of motion in B-D theory whereas the Dicke representation is convenient to study the scalar wave (Dicke, [4]). However, in both the cases the field equations of B-D theory become those of Einstein theory when $W \longrightarrow \infty$ and $\Phi \longrightarrow G$. Due to the highly nonlinear nature of B-D field equations Mohanty [7] has considered only two particular cases and derived solutions for the space time described by Jordan-Ehlers-Kundt [5] metric

$$ds^2 = e^{2A-2B}\left(dt^2 - dr^2\right) - r^2 e^{-2B} d\theta^2 - e^{2B}(Cd\theta + dZ)^2, \qquad (5)$$

where $A = A(r,t)$, $B = B(r,t)$ and $C = C(r,t)$ and r, θ, z and t correspond to the coordinates X^1, X^2, X^3 and X^4 respectively. In this paper we obtain solution for the cases other than those studied earlier by Mohanty [7]. We set field equations and obtain their consequences in Sec. 2. In Sec. 3, we solve the field equations for some particular cases in order to avoid the mathematical complexity due to the non-linear nature of the field equations. The nature of the solutions is studied in Sec. 5.

2 B-D field equations

The B-D field equations (1) and (2) with $T_{ij} = 0$ for the metric (5) can be written in the following explicit forms:

$$\frac{1}{4r^2}(C_1^2 + C_4^2)e^{4B} + B_1^2 + B_4^2 - \frac{A_1}{r} = -\frac{W}{2\Phi^2}(\Phi_1^2 + \Phi_4^2) -$$

$$-\frac{1}{\Phi}\left[\Phi_{11} - \Phi_1(A_1 - B_1) - \Phi_4(A_4 - B_4)\right], \tag{6}$$

$$\frac{3C^2}{4r^2}(C_4^2 - C_1^2)e^{3B} + e^{4B}\left[\frac{1}{4}(C_1^2 - C_4^2) + 4C(B_1C_1 - B_4C_4) + c\nabla_2 C + \right.$$

$$\left. + C^2(2\nabla_1 B - B_1^2 + B_4^2 - A_{11} + A_{44})\right] + r^2(B_4^2 - B_1^2 - A_{11} + A_{44}) =$$

$$= \frac{W}{2\Phi^2}(C^2e^{2B} + r^2e^{-2B})e^{2B}(\Phi_1^2 - \Phi_4^2) +$$

$$+ \frac{1}{\Phi}\left[-\Phi_1\left\{Ce^{4B}(C_1 + B_1C) + r(1 - rB_1)\right\} + \right.$$

$$\left. + \Phi_4\left\{C\left(C_4 + CB_4\right)e^{4B} - r^2B_4\right\}\right], \tag{7}$$

$$\frac{3}{4r^2}(C_4^2 - C_1^2)e^{4B} + 2\nabla_1 B - B_1^2 + B_4^2 - A_{11} + A_{44} =$$

$$= \frac{W}{2\Phi^2}(\Phi_1^2 - \Phi_4^2) + \frac{1}{\Phi}\left[-B_1\Phi_1 + B_4\Phi_4\right], \tag{8}$$

$$\frac{1}{4r^2}(C_4^2 + C_1^2)e^{4B} + B_1^2 + B_4^2 - \frac{A_1}{r} =$$

$$= -\frac{W}{2\Phi^2}(\Phi_1^2 + \Phi_4^2) - \frac{1}{\Phi}\left[\Phi_{44} - \Phi_1(A_1 - B_1) - \Phi_4(A_4 - B_4)\right], \qquad (9)$$

$$\frac{C_1 C_4}{2r^2}e^{4B} + 2B_1 B_4 - \frac{A_4}{r} =$$

$$= -\frac{W}{\Phi^2}\Phi_1\Phi_4 - \frac{1}{\Phi}\left[\Phi_{14} - \Phi_1(A_4 - B_4) - \Phi_4(A_1 - B_1)\right], \qquad (10)$$

$$\frac{3C}{4r^2}(C_4^2 - C_1^2)e^{4B} + \frac{1}{2}\nabla_2 C + 2(B_1 C_1 - B_4 C_4) +$$

$$+2C\nabla_1 B - CB_1^2 + CB_4^2 - C(A_{11} - A_{44}) =$$

$$= \frac{WC}{2\Phi^2}(\Phi_4^2 - \Phi_1^2) + \frac{1}{2\Phi}\left[\Phi_4(C_4 + 2CB_4) - \Phi_1(C_1 + 2CB_1)\right], \qquad (11)$$

and

$$\Phi_{11} - \Phi_{44} + \frac{\Phi_1}{r} = 0, \qquad (12)$$

where

$$\nabla_1 = \frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial t^2} + \frac{1}{r}\frac{\partial}{\partial r}, \quad \text{and} \quad \nabla_2 = \frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial t^2} - \frac{1}{r}\frac{\partial}{\partial r}.$$

For the sake of brevity here-afterwards the suffixes 1 and 4 after the field variables represent partial differentiation with respect to r and t respectively.

3 Consequences of the field equations

Now one can easily get from equation (6), (9) and (12) that

$$\Phi_1 = 0. \tag{13}$$

Using equation (13) in equation (12), we have

$$\Phi_{44} = 0, \tag{14}$$

which on integration yields

$$\Phi = at + b, \tag{15}$$

where a and b are two arbitrary constants.

Using equation (15), the field equations (6)-(11) can be put in the following forms [7]

$$\nabla_2 C + 4(B_1 C_1 - B_4 C_4) = \frac{aC_4}{at+b}, \tag{16}$$

$$\nabla_1 B - \frac{C_1^2 - C_4^2}{2r^2} e^{4B} = \frac{aB_4}{at+b}, \tag{17}$$

$$A_{11} - A_{44} + B_1^2 - B_4^2 - \frac{e^{4B}}{4r^2}(C_1^2 - C_4^2) = \frac{a^2 W}{2(at+b)^2} + \frac{aB_4}{at+b}, \tag{18}$$

$$A_1 \left[1 - \left(\frac{ar}{at+b}\right)^2\right] = \frac{e^{4B}}{4r}(C_1^2 + C_4^2) + r(B_1^2 + B_4^2) + + \frac{Wra^2}{2(at+b)^2} +$$
$$+ \frac{arB_4}{at+b} - \frac{aC_1C_4}{2(at+b)} e^{4B} - 2r^2 B_1 B_4 \frac{a}{(at+b)} - \frac{a^2 r^2 B_1}{(at+b)^2}, \tag{19}$$

$$A_1 \left[1 - \left(\frac{ra}{at+b}\right)^2\right] = \frac{C_1 C_4}{2r} e^{4B} + 2rB_1B_4 + \frac{arB_1}{(at+b)} -$$
$$- \frac{a}{4(at+b)}(C_1^2 + C_4^2) e^{4B} - \frac{ar^2}{(at+b)^2}(B_1^2 + B_4^2) - \frac{a^3 W r^2}{2(at+b)^3} - \frac{a^2 r^2 B_4}{(at+b)^2}. \tag{20}$$

Equation (18) is redundant since it can be obtained from rest of the equations. Equation (18) also represents the integrability condition for A. Therefore we drop equation (18) and solve the remaining equations i.e. (16), (17), (19) and (20) for A, B and C.

4 Solutions

Here we intend to derive some more explicitly and exact solutions to the field equations (16), (17), (19) and (20) in order to study the nature of the B-D field. Therefore we solve the field equations in the following cases:

Case (I) $B = 0$,

Case (II) $B = \frac{1}{4}$ in $r + b$ (b is an arbitrary constants),

Case (III) $B_4 = 0 = C_4$,

Case (IV) $B_1 = 0 = C_1$.

Case (I): $B = 0$.

In this case equation (17) reduces to

$$C_1 = \pm C_4. \tag{21}$$

With the help of equations (16) and (21) it can be easily obtained that

$$C = Constant. \tag{22}$$

Subsequently equations (19) and (20) reduce to

$$A_1 = \frac{Wra^2}{2} \frac{1}{[(at+b)^2 - (ra)^2]}, \tag{23}$$

and

$$A_4 = -\frac{Wr^2a^3}{2} \frac{1}{(at+b)[(at+b)^2 - (ra)^2]}. \tag{24}$$

Now integrating (23) and (24), we get

$$A = -\frac{W}{4} \ln\left[1 - \left(\frac{ra}{at+b}\right)^2\right] + K_1 \tag{25}$$

where K_1 is an arbitrary constant of integration. Thus in this case equations (15), (22) and (25) constitute the solution of the B-D field equations.

Case (II): $B = (1/4)\ln r + b$.

In this case equation (17) also yields

$$C_1 = \pm C_4. \tag{26}$$

As before using these equations in (16), we get

$$C = Constant. \tag{27}$$

Substituting the values of B and C in equations (19) and (20), integrating, we get

$$A = -\frac{W}{4}\ln\left[1 - \left(\frac{ra}{at+b}\right)^2\right] + \frac{1}{4}\ln r + f(t), \tag{28}$$

where $f(t)$ is an arbitrary function of t only. Again substituting (28) in (20), we find

$$f(t) = Constant. = K_2, \text{ (say)}.$$

Thus the solution for A is given by

$$A = -\frac{W}{4} In\left[\left[1 - \left(\left(\frac{ra}{at+b}\right)^2\right)\right] + \frac{1}{4} Inr + K_2\right]. \tag{29}$$

Hence in this case the solution is given by equations (15), (27) and (29).

Case (III): $B_4 = 0 = C_4$.

In this case equation (16) reduces to

$$C_{11} - \frac{1}{r}C_1 + 4B_1C_1 = 0. \tag{30}$$

Integrating equation (30), we get

$$C_1 = rK_1e^{-4B}, \tag{31}$$

where K_1 is an arbitrary constant of integration.

Substituting (31) in (17), we get

$$B_{11} + \frac{B_1}{r} = \frac{K_1^2}{2}e^{-4B}, \tag{32}$$

which is equivalent to

$$e^{2B}(e^{2B})_{11} - \left[\left(e^{2B}\right)_1\right]^2 + \frac{1}{4}e^{2B}(e^{2B})_1 = K_1^2. \tag{33}$$

As before equation (33) yields

$$e^{2B} = \frac{K_2}{2}\left[(K_3 r)^2 + 1\right], \tag{34}$$

subject to the condition

$$K_2^2 K_3^2 = K_1^2, \tag{35}$$

where K_2 and K_3 are arbitrary constants. Substituting (34) in (31), we get

$$C_1 = \frac{4rK_1}{K_2^2\left[(K_3 r)^2 + 1\right]^2}. \tag{36}$$

On integration (36) yields

$$C = -\frac{2}{K_1}\frac{1}{(K_3 r)^2 + 1} + K_4. \tag{37}$$

Substituting the values of C and B from (34) and (37) in (19) and integrating, we get

$$A = \frac{1}{2K_3}\ln\left[(K_3 r)^2 + 1\right] - \frac{W}{4}\ln\left[1 - \left(\frac{ra}{at+b}\right)^2\right] + f(t), \tag{38}$$

where $f(t)$ is an arbitrary function of t only. Now substituting (38) in (20) and integrating, we find $f(t) = Constant = K_5$ (say). Thus we find

$$A = \frac{1}{2K_3}\ln\left[(K_3 r)^2 + 1\right] - \frac{W}{4}\ln\left[1 - \left(\frac{ra}{at+b}\right)^2\right] + K_5. \tag{39}$$

In this case equations (15), (34), (37) and (39) constitute solution of B-D field equations.

Case (IV): $B_1 = 0 = C_1$.

This case does not yield any solution for B-D field equations.

5 Conclusions

In order to study analytically the singularities of the solutions obtained in this paper we use Bonnor's [1] criteria. In this criteria we adopt pseudocartesian coordinates in place of cylindrical coordinates in order to avoid the coordinate singularities. It may be verified that all the solutions obtained in this paper are singular on the axis of symmetry and at spatial and temporal infinities. Thus the solutions are regular and physically realistic in the range $0 < r < \infty$ and at $t = 0$, the B-D scalar field becomes constant and subsequently the solutions become that of Einstein's field equations of empty space-time.

The solution representing a field due to line-mass along the axis of symmetry [6] is given by

$$ds^2 = \left(\frac{r}{r_o}\right)^{\frac{q^2+2q}{2}} \left(dt^2 - dr^2\right) - \left(\frac{r}{r_o}\right)^{q} r^2 d\theta^2 - \left(\frac{r}{r_o}\right)^{-q} dz^2.$$

It may be easily verified that none of the solutions obtained here represents line-mass.

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Anizotropno Brans – Dicke polje

Klasa rešenja jednačina B-D polja se dobija za slučaj opisa prostor-vremena Jordan-Elerovom metrikom sa tri parametra u dve promenljive. Pronadjeno je da su rešenja fizički realistična u opsegu $0 < r < \infty$ i radiativne prirode.