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Optimal control of the system with subdifferential material law

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Abstract

In this paper the following problem is considered: find such loading on the path of boundary Γ of the body Ω, as near as possible to the desired value F_∂, such that some linear transformation of the displacement field u take desired value h in a certain Hilbert space. The existence result of the optimal loading is given as in [4]. The weak formulation of the problem is in the form of variational inequality because of subdifferential form of the constitutive law in considered system. The existence result of the variational inequality is given as in [3].

1 Introduction

In this paper the mechanical problem is considered where weak formulation of the problem is in the form of variational inequality (principle of virtual work in inequality form) [2], namely, the subdifferential form of the constitutive law, stress-strain law, give rise to have a variational inequality as a weak formulation of the problem. the constitutive laws of the form: $\sigma \in \partial w(\epsilon)$, (see [2], [3]) describes Hooke's elastic materials, the elastic workhardening materials, the elastic-ideally "plastic" materials (Hencky's theory), the materials obeying the law of the deformation theory of plasticity and the elastic locking materials, [4].

For the existence proof compactness arguments combined with monotonicity arguments are use (see [2], [3] and [5]). As for optimal control problem we concern a standard formulation: we minimize the difference $\| Mu - h \|^2_{\mathcal{H}}$, where Mu is linear transformation of the state variable and we add the term $\| F - F_\partial \|^2_U$ for the controls. The existence results for optimal control problem is given as in [4].

2 Variational formulation of the problem and existence result

Let Ω be an open, bounded and connected subset of R^3 which is occupied by a deformable body in its undeformed state. Let Γ be boundary of Ω which is assumed appropriately regular (e.g. a Lipshitzian boundary). Let $\sigma = \{\sigma_{ij}\}$ (resp. $\epsilon = \{\epsilon_{ij}\}$), $i, j = 1, 2, 3$, be the stress (resp. strain) tensor and let $f = \{f_i\}$ (resp. $u = \{u_i\}$) be the volume force (resp. displacement) vector. Let $n = \{n_i\}$ denote the outward unit normal vector to Γ; $S_i = \sigma_{ij} n_j$ (summation convention) are boundary forces. We assume that the boundary is divided into two disjoint open subsets Γ_U and Γ_F, i.e. $\Gamma = \bar{\Gamma}_U \cup \bar{\Gamma}_F$.

In the framework of small deformations and nonlinear monotone elastic behaviour of the body Ω we consider the following problem:

Find u such that

$$\begin{cases} \sigma_{ij,j} + f_i = 0, & \text{in } \Omega, \\ 2\epsilon_{ij} = u_{i,j} + u_{j,i}, & \text{in } \Omega, \\ \sigma \in \partial w\,(\epsilon), & \text{in } \Omega, \\ u_i = 0, & \text{on } \Gamma_U, \\ S_i = F_i, & \text{on } \Gamma_F. \end{cases} \tag{1}$$

Here the comma denotes differentiation, ∂ is the subdifferential of convex analysis [2] and $w(\cdot)$ is a proper, convex and lower semicontinuous (l.s.c.) functional on R^6 (see [2]).

The appropriate expressions of w which correspond to the law in (1) are discussed in [2].

Let us assume further that $u_i,\ v_i \in W^{1,p}(\Omega)$, $p > 3$; $F_i \in L^2(\Gamma_F)$, $f_i \in L^{p'}(\Omega)$, $p' = p/(p-1)$ (see [1] for definition of Sobolev space), and let $(\cdot,\cdot)$ denote the duality pairing between $(L^p(\Omega))^m$ and $\left(L^{p'}(\Omega)\right)^m$ for any m.

The set of kinematically admissible displacements of the points of the body Ω is defined by

$$V_o = \left\{ v \in \left(W^{1,p}(\Omega)\right)^3 \mid v = 0 \text{ on } \Gamma_U \right\}. \tag{2}$$

Multiplying by $(v_i - u_i)$ equilibrium equation in (1) and integrating over Ω (the Green-Gauss theorem is used) the following virtual work equation results

$$\int_\Omega \sigma_{ij}\left(\epsilon_{ij}(v) - \epsilon_{ij}(u)\right) d\Omega = \int_\Omega f_i (v_i - u_i)\, d\Omega + \int_{\Gamma_F} F_i (v_i - u_i)\, d\Gamma, \quad \forall v \in V_o. \tag{3}$$

Relation $\sigma \in \partial w(\epsilon)$ is equivalent to the variational equation ([3])

$$w(\epsilon(v)) - w(\epsilon(u)) \geq \sigma_{ij}\left(\epsilon_{ij}(v) - \epsilon_{ij}(u)\right), \quad \forall \epsilon(v) \in R^6. \tag{4}$$

Further we define the convex, l.s.c. and proper functional $W(\cdot)$ on $(L^p(\Omega))^6$, by the relation ([3])

$$W(\epsilon) = \begin{cases} \displaystyle\int_\Omega w(\epsilon)\, d\Omega & \text{if } w(\epsilon) \in L^1(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{5}$$

For $\sigma_{ij} \in L^{p'}(\Omega)$ and $\epsilon_{ij} \in L^p(\Omega)$ the relation $\sigma \in \partial w(\epsilon)$ is the extension to $(L^p(\Omega))^6 \times \left(L^{p'}(\Omega)\right)^6$ of the relation $\sigma \in \partial w(\epsilon)$, if this holds a.e. in Ω ([3]).

The functional $W(\epsilon)$ have the same properties as $w(\epsilon)$.

After Eqs. (4) and (5) variational equation (3) become variational inequality, i.e.

$$W(\epsilon(v)) - W(\epsilon(u)) \geq l(v-u), \quad \forall v \in V_o,$$

where
$$l(v) = \int_\Omega f_i v_i d\Omega + \int_{\Gamma_F} F_i v_i d\Gamma.$$

Then we consider the following problem:

Find u such that
$$W(\epsilon(v)) - W(\epsilon(u)) \geq l(v-u), \quad \forall v \in V_o. \tag{6}$$

Here we consider the case when the functional $w(\epsilon)$ is not differentiable everywhere. In this case ([4]) a sequence of convex differentiable functionals w_ρ depending of the parameter ρ is considered such that

(*i*) As $\rho \to 0$
$$W_\rho(\epsilon(v)) = \int_\Omega w_\rho(\epsilon(v))\, d\Omega \to W(\epsilon(v)), \quad \forall v \in V_o. \tag{7}$$

(*ii*) If $v_\rho \to v$ weakly in V_o for $\rho \to 0$ and $\int_\Omega w_\rho(\epsilon(v\rho))\, d\Omega < C$, then
$$\liminf_{\rho \to 0} \int_\Omega w_\rho(\epsilon(v_\rho))\, d\Omega \geq W(\epsilon(v)), \tag{8}$$

(*iii*)
$$(\text{grad } W_\rho(\epsilon(v)), \epsilon(v)) \geq C \int_\Omega (\epsilon_{ij}(v)\, \epsilon_{ij}(v))^{\frac{p}{2}}\, d\Omega, \tag{9}$$

where C is independent of ρ.

Now we define the regularized problem:

Find $u_\rho \in V_o$ such that
$$(\text{grad } W_\rho(\epsilon(u_\rho)), \epsilon(v)) = l(v), \quad \forall v \in V_o. \tag{10}$$

In order to discretize (10), let us consider a Galerkin basis $\{w_i\}$ of V_o and let V_n be the corresponding n-dimensional subspace. Then we define the following problem:

Find $u_{\rho n} \in V_n$ such that
$$(\text{grad } W_\rho(\epsilon(u_{\rho n})), \epsilon(v)) = l(v), \quad \forall v \in V_n. \tag{11}$$

The following proposition holds.

Proposition 1 *If w_ρ satisfied (i), (ii) and (iii) then the problem (11) has a solution.*

Proof. According to the trace theorem of Sobolev spaces and (*iii*)

$$\|u_{\rho n}\| < C, \tag{12}$$

where C is independent of ρ and n. Thus, as $n \to \infty$

$$u_{\rho n} \to u_\rho \quad \text{weakly in } V_o. \tag{13}$$

We also have, that as $\rho \to 0$

$$u_\rho \to u \quad \text{weakly in } V_o. \tag{14}$$

From (11) we get that

$$\|\text{grad}\, W_\rho\left(\epsilon\left(u_{\rho n}\right)\right)\|_{\left(L^{p'}(\Omega)\right)^6} \leq C, \tag{15}$$

where C is independent of n and ρ and thus as $n \to \infty$

$$\text{grad}\, W_\rho\left(\epsilon\left(u_{\rho n}\right)\right) \to \Psi_\rho \quad \text{weakly in } \left(L^{p'}(\Omega)\right)^6. \tag{16}$$

From (11) we obtain by passing to the limit, $n \to \infty$, the variational equality

$$\left(\Psi_\rho, \epsilon(v)\right) = l(v), \quad \forall v \in V_o. \tag{17}$$

Further, let us form the nonnegative expression

$$X_n = \left(\text{grad}\, W_\rho\left(\epsilon\left(u_{\rho n}\right)\right) - \text{grad}\, W_\rho\left(\epsilon(w)\right), \epsilon\left(u_{\rho n}\right) - \epsilon(w)\right) \geq 0, \quad \forall w \in V_n, \tag{18}$$

which by means of (11) becomes

$$X_n = l\left(u_{\rho n}\right) - \left(\text{grad}\, W_\rho\left(\epsilon(w)\right), \epsilon\left(u_{\rho n}\right) - \epsilon(w)\right) - \left(\text{grad}\, W_\rho\left(\epsilon\left(u_{\rho n}\right)\right), \epsilon(w)\right) \geq 0, \quad \forall w \in V_n.$$

Due to (13), (16) and (17) we have that $\forall w \in V_o$ as $n \to \infty$

$$\begin{aligned} \lim X_n &= l\left(u_\rho\right) - \left(\Psi_\rho, \epsilon(w)\right) - \left(\text{grad}\, W_\rho\left(\epsilon(w)\right), \epsilon\left(u_\rho\right) - \epsilon(w)\right) \\ &= \left(\Psi_\rho, \epsilon\left(u_\rho\right) - \epsilon(w)\right) - \left(\text{grad}\, W_\rho\left(\epsilon(w)\right), \epsilon\left(u_\rho\right) - \epsilon(w)\right) \geq 0. \end{aligned} \tag{19}$$

Here we have also used the fact that

$$\epsilon(u_{\rho n}) \to \epsilon(u) \quad \text{weakly in } \left(L^p(\Omega)\right)^6,$$

because Korn's inequality implies that $\epsilon : u \to \epsilon(u)$ is continuous linear function from $\left(W^{1,p}(\Omega)\right)^3$ into $\left(L^p(\Omega)\right)^6$.

Now we apply the monotonicity argument or Minty's argument:

Let us set in (19) $\epsilon(u_\rho) - \epsilon(w) = \lambda\epsilon(\theta)$, $\lambda > 0$. We get the expression

$$(\Psi_\rho - \text{grad } W_\rho(\epsilon(u_\rho) - \lambda\epsilon(\theta)), \epsilon(\theta)) \geq 0, \quad \forall\theta \in V_o. \tag{20}$$

Due to the monotonicity of the function

$$\lambda \to (\text{grad } W_\rho(\epsilon(u_\rho) - \lambda\epsilon(\theta)), \epsilon(\theta)), \quad \forall\theta \in V_o,$$

we may take the limit of (20) for $\lambda \to 0$ and find

$$(\Psi_\rho - \text{grad } W_\rho(\epsilon(u_\rho)), \epsilon(\theta)) \geq 0, \quad \forall\theta \in V_o. \tag{21}$$

Substituting $\theta = \pm v$ in (21) we get

$$\Psi_\rho - \text{grad } W_\rho(\epsilon(u_\rho)). \tag{22}$$

Further we take the limit with respect to ρ. From Eqs. (17) and (22) we find, due to the convexity of w_ρ, the relation

$$\int_\Omega (w_\rho(\epsilon(v)) - w_\rho(\epsilon(u_\rho)))\, d\Omega \geq l(v - u_\rho), \quad \forall v \in V_o. \tag{23}$$

Let us choose in Eq. (23) a v such that $W(\epsilon(v)) < \infty$. Then due to (7)

$$\int_\Omega w_\rho(\epsilon(v))\, d\Omega < C,$$

and (23) implies that

$$\int_\Omega w_\rho(\epsilon(u_\rho))\, d\Omega < C. \tag{24}$$

Therefore due to Eq. (14), Eq. (8) holds. From (23) we find

$$\liminf_{\rho\to 0} \int_\Omega w_\rho(\epsilon(v))\, d\Omega \geq \liminf_{\rho\to 0} \left[\int_\Omega w_\rho(\epsilon(u_\rho))\, d\Omega - l(v - u_\rho)\right],$$

for every $v \in V_o$, with $W(\epsilon(v)) < \infty$. But from (7), (8) and (14) we conclude that inequality

$$W(\epsilon(v)) - W(\epsilon(u)) \geq l(v - u), \quad \forall v \in V_o, \quad \text{with } W(\epsilon(v)) < \infty,$$

is satisfied by $u \in V_o$ with $W(\epsilon(u)) < \infty$.□

3 Optimal control problem

Let $F = \{F_i\}$ on Γ_F be the control variable and $U = (L^2(\Gamma_F))^3$ - the space of the controls. Let admissible set of controls be defined by

$$U_\alpha = \left\{ F \in \left(L^2(\Gamma_F)\right)^3 \mid \|F - F_\partial\|_U \leq K \right\}, \tag{25}$$

where $F_\partial \in (U_\partial)$ is given.

With every control $F \in U_\partial$ we associate the cost functional $J(F)$ by

$$J(F) = \frac{\alpha}{2} \|Mu - h\|^2_{\mathcal{H}} + \frac{\beta}{2} \|F - F_\partial\|^2_U, \qquad \alpha, \beta > 0, \tag{26}$$

where $M \in \mathrm{L}(V_o, \mathcal{H})$ is given operator and $\mathcal{H}$ is a Hilbert space of observations; $h \in \mathcal{H}$ is given.

The optimal control problem for the variational inequality (6) is defined as follows:

Find $F^* \in U_\partial$ such that

$$J(F^*) \leq J(F), \qquad \forall F \in U_\partial. \tag{27}$$

The following proposition holds ([5]).

Proposition 2 *For the problem (27) exists at least one optimal control $F^* \in U_\partial$.*

Proof. Let $F^{(n)}$ be a minimizing sequence in U_∂, i.e.

$$J\left(F^{(n)}\right) \to J_* = \inf_{F \in U_\partial} J(F), \tag{28}$$

and let $u^{(n)} = u\left(F^{(n)}\right)$ be the solution of (6). By (26) and (9) the sequences $F^{(n)}$ and $u^{(n)}$ are bounded, i.e., $\left\|F^{(n)}\right\|_U \leq C$ and $\left\|u^{(n)}\right\|_V \leq C$. Accordingly, we may extract subsequences $F^{(m)}$ and $u^{(m)}$ such that

$$F^{(m)} \to \bar{F} \qquad \text{weakly in } U_\partial, \tag{29}$$

$$u^{(m)} \to \bar{u} \qquad \text{weakly in } V_o. \tag{30}$$

Since U_∂ and V_o are convex and closed sets, they are also weakly closed, and thus $\bar{F} \in U_\partial$ and $\bar{u} \in V_o$. We must prove that $\bar{F}$ and $\bar{u}$ satisfy variational inequality (6), i.e. that $\bar{u} = \bar{u}\left(\bar{F}\right)$.

If we form the system (u in (6) is replaced by $u^{(m)}$)

$$W(\epsilon(v)) - (f, v) - \left\langle F^{(m)}, v \right\rangle_{\Gamma_F} \geq$$
$$W\left(\epsilon\left(u^{(m)}\right)\right) - \left(f, u^{(m)}\right) - \left\langle F^{(m)}, u^{(m)} \right\rangle_{\Gamma_F}, \quad \forall v \in V_o. \tag{31}$$

After weakly l.s.c. of $W(\epsilon(v))$, the Sobolev trace theorem $\left(W^{1,p}(\Omega)\right)^3 \to \left(L^q(\Gamma_F)\right)^3$, $q \geq 1$ ([1]), and (29) and (30), (31) become

$$W(\epsilon(v)) - W(\epsilon(\bar{u})) \geq (f, v - \bar{u}) - \langle F, v - \bar{u} \rangle_{\Gamma_F}, \quad \forall v \in V_o, \tag{32}$$

which means that $\bar{u} = \bar{u}(F)$. Now it will be proved that $\bar{F}$ is the optimal control which was denoted by F^*. Functional $J(F)$ is weakly l.s.c. in U and hence

$$\liminf_{n \to \infty} J\left(F^{(n)}\right) \geq J\left(\bar{F}\right). \tag{33}$$

But

$$\liminf_{n \to \infty} J\left(F^{(n)}\right) = \inf_{F \in U_\partial} J(F) = J(F^*), \tag{34}$$

and thus $J\left(\bar{F}\right) = \inf_{F \in U_\partial} J(F) = J(F^*)$, and we may take $\bar{F} = F^*$ in the proposition.□

4 Regularization of the control problem

Here, we replace the problem (27) by the problem:

Find $F_\rho^* \in U_\partial$ such that

$$J_\rho\left(F_\rho^*\right) \leq J_\rho(F), \quad \forall F \in U_\partial, \tag{35}$$

where

$$J_\rho(F) = \frac{\alpha}{2}\left\|M u_\rho(F) - h\right\|_{\mathcal{H}}^2 + \frac{\beta}{2}\left\|F - F_\partial\right\|_U^2, \quad \alpha, \beta > 0, \tag{36}$$

and $u_\rho(F)$ is solution of the problem (10).

The following proposition holds ([4]).

Proposition 3 *There exists at least one solution F_ρ^*, u_ρ of (35) and (10), respectively, and as $\rho \to 0$ there exists a subsequence denoted by F_ρ^* again such that*

$$F_\rho^* \to F^*, \qquad \textit{strongly in } U_\partial, \tag{37}$$

$$u_\rho\left(F_\rho^*\right) \to u\left(F^*\right), \qquad \textit{strongly in } V_o, \tag{38}$$

$$J_\rho\left(F_\rho^*\right) \leq J\left(F^*\right). \tag{39}$$

Proof. The solutions F_ρ^*, u_ρ exists by the proof of proposition 1, and proposition 2, respectively. By proposition 1 we have

$$J_\rho(F) \to J(F), \qquad \text{as } \rho \to 0, \quad \forall F \in U_\partial. \tag{40}$$

Hence

$$J_\rho\left(F_\rho^*\right) \leq J_\rho\left(F^*\right) \to J\left(F^*\right) = \inf_{F \in U_\partial} J(F), \tag{41}$$

and thus

$$\limsup J_\rho\left(F_\rho^*\right) \leq J\left(F^*\right). \tag{42}$$

But,

$$J_\rho\left(F_\rho^*\right) \geq \frac{\beta}{2}\left\|F_\rho^* - F_\partial\right\|_U^2, \tag{43}$$

and thus $\left\|F_\rho^*\right\|_U$ is bounded. Accordingly we may extract a subsequence denoted again by F_ρ^*, such that $F_\rho^* \to F_o^*$ weakly in U; $F_o^* \in U_\partial$ because U_∂ is weakly closed.

It can be proved by taking the limit in the variational inequality (10), that

$$u_\rho\left(F_\rho^*\right) \to u\left(F_o^*\right), \qquad \text{weakly in } V_o, \tag{44}$$

and thus

$$\liminf J_\rho\left(F_\rho^*\right) \geq J\left(F_o^*\right), \tag{45}$$

which together with (42) gives $F^* \to F^*_o$ and (39) is proved.

We have for $\rho \to 0$ that

$$\lim J_\rho\left(F^*_\rho\right) = \lim \left[\frac{\alpha}{2}\left(Mu_\rho\left(F^*_\rho\right), Mu_\rho\left(F^*_\rho\right)\right)_{\mathcal{H}} + \frac{\beta}{2}\left(F^*_\rho, F^*_\rho\right)_U \right.$$
$$\left. -\frac{\alpha}{2}\left(Mu_\rho\left(F^*_\rho\right), h\right)_{\mathcal{H}} + \|h\|^2_{\mathcal{H}}\right] = J(F^*),$$

and from (40) that

$$\lim_{\rho \to 0} J_\rho(v) = J(0). \tag{46}$$

Moreover, there exists constants $C_1 > 0,\ C_2 > 0$, such that

$$C_2 \|F\|^2_U \leq \left(M(u(F) - u(0)), M(u(F) - u(0))\right)_{\mathcal{H}}$$
$$+\tfrac{\beta}{2}(F, F)_U \leq C_1 \|F\|^2_U, \qquad \forall F \in U, \tag{47}$$

which means that

$$\left(M(u(F) - u(0)), M(u(F) - u(0))\right)_{\mathcal{H}} + (\beta/2)(F, F)_U$$

is a norm equivalent to norm $\|F\|_U$. Accordingly, from (46) and (47) it results that already obtained weak convergence in (37) and (38) is indeed strong convergence.□

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Optimalno upravljanje sistemom sa subdiferencijalnim materijalnim zakonom

U ovom radu se razmatra sledeći problem: odrediti opterećenje na jednom delu granice Γ tela Ω, što bliže željenom opterećenju F_∂, tako da odredjena linearna transformacija polja pomeranja u dobije željenu vrednost h u odredjenom Hilbertovom prostoru. Egzistencija rešenja optimalnog opterećenja data je kao u [4]. Slaba formulacija problema je u obliku varijacione nejednakosti zbog subdiferencijalnog oblika konstitutivnog zakona u posmatranom sistemu. Rezultat egzistencije rešenja varijacione nejednakosti je dat kao u [3].