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Non-periodic solutions to relativistic field equations: Elliptic case

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Abstract

Einstein's field equations are solved for a linearly coupled system of massless scalar and source free electromagnetic fields in a cylindrically symmetric space-time of one degree of freedom and the nature of solution is discussed.

1 Introduction

Scalar fields are of considerable interest in particle physics and general relativity. Massless scalar fields have been extensively studied in general relativity, both from their own standpoint and also as means of studying the gravitational effects of an irrotational stiff fluid which has been shown to be equivalent to the former by Tabensky and Taub [1], Latelier and Tabensky [2] have solved the field equations for such equivalent scalar fields for a space-time geometry described by the cylindrically symmetric metric

$$ds^2 = e^A \left(dt^2 - dr^2\right) - e^B \left(r^2 d\theta^2 + dz^2\right), \tag{1}$$

where A and B are functions of r and t only.

Under the same space-time geometry we solve the field equations for interacting system of massless attractive scalar fields and source free electromagnetic fields. Of the two possible choices of the later, the one considered here has four non-vanishing components of the field tensor derivable from two non-vanishing four-potential components that are shown in Sect. 2 to be functionally related in the general case. In Sect. 3 we obtain non-periodic solutions for A and B that are valid within an elliptical region, as shown in Sect. 4. In Sect. 5, solutions to the Klein-Gordon equation of the scalar fields that are valid within the elliptic region, are obtained. The nature of the scalar fields and electromagnetic fields is studied in Sect. 6 with reference to their singularities and the nullity and uniformity of the latter fields. In conclusion (Sect. 7) we indicate how the solution may have a bearing on gravitational waves.

2 Field equations

The geometrodynamic equations of Einstein we wish to solve are

$$G_{ij} = -T_{ij} - \frac{1}{4\pi}T'_{ij} \tag{2}$$

where G_{ij} is the Einstein tensor and T_{ij} and T'_{ij} are respectively the scalar and electromagnetic stress-energy tensors given by

$$T_{ij} = V_{,i}V_{,j} - \frac{1}{2}g_{ij}V_{,k}V^{,k}, \tag{3}$$

and

$$T'_{ij} = -F_{is}F^{s}_{j} + \frac{1}{4}g_{ij}F^{k1}F_{k1}. \tag{4}$$

The units are so chosen that the velocity of light is unity and the gravitational constant equals to $1/8\pi$. The scalar and electromagnetic fields satisfy the equations:

$$g^{ij}V_{;ij} = 0, \tag{5}$$

and

$$F^{ij}_{;j} = 0. \tag{6}$$

If A_i are the four-potential components, we also have

$$F^{ij} = A_{i,j} - A_{j,i}. \tag{7}$$

We have followed throughout the sign convention of Bergmann [3], semi-colons and comas representing covariant and ordinary differentiation respectively. We introduce the characteristic coordinates

$$u = t - r, \qquad v = t + r, \tag{8}$$

and let (v, θ, z, u) correspond to (x^1, x^2, x^3, x^4). Because of cylindrically symmetry, A_i are independent of θ and z. Therefore we have

$$F_{23} = 0. \tag{9}$$

Four of Eqs. (4) for $G_{ij} = 0$ give

$$F_{12}F_{14} = F_{13}F_{14} = F_{24}F_{14} = F_{34}F_{14} = 0. \tag{10}$$

These give two cases of which we choose the one in which $F_{14} = 0$. It follows from this and (9) that the four-potential is of the form

$$A_i = (0, M, N, 0).$$

Denoting differentiation with respect to v and u by the subscripts 1 and 4 respectively, the fifth one of Eqs. (2) for which $G_{ij} = 0$ gives:

$$N_1 M_4 + M_1 N_4 = 0. \tag{11}$$

The Eq.(6) now gives

$$M_{14} + \frac{1}{4r}(M_1 - M_4) = 0, \tag{12}$$

and

$$N_{14} - \frac{1}{4r}(N_1 - N_4) = 0. \tag{13}$$

In terms of a new potential defined by the relations

$$L_1 = \frac{M_1}{r}, \qquad L_4 = -\frac{M_4}{r}, \tag{14}$$

Eqs. (11) and (12) can be expressed as

$$L_1 N_4 = N_1 L_4 \tag{15}$$

$$L_{14} + \frac{1}{4r}(L_4 - L_1) = 0. \tag{16}$$

Three cases now arise.

Case I. By (16) L_1 and L_4 can only be simultaneously zero. Then (15) is identically satisfied and the field equations involve N only.

Case II. In a similar manner, $N_1 = N_4 = 0$ with the field equations involving L only and (15) satisfied.

Case III. In general when L_1, L_4, N_1, N_4 are all non-zero, Eq. (15) means that Jacobian $\partial(L, N)/\partial(u, v) = 0$. Then L is a function of N, say $L = f(N)$. Substituting in (16) and using (12), we get

$$N_1 N_4 \cdot \frac{d^2 f}{dN^2} = 0.$$

Thus we have

$$L = aN + b. \tag{17}$$

In this and in what follows, all lower case latin letters except r, t, u, v, x, y and z represents constants. The remaining field equations and Eq. (5) can be put in the following forms

$$B_{11} + \frac{1}{2}B_1^2 - A_1 B_1 + \frac{1}{2r}(B_1 - A_1) = -V_1^2 - \frac{k^2}{4\pi}e^{-B}N_1^2, \tag{18}$$

$$B_{44} + \frac{1}{2}B_4^2 - A_4 B_4 - \frac{1}{2r}(B_4 - A_4) = -V_4^2 - \frac{k^2}{4\pi}e^{-B}N_4^2, \tag{19}$$

$$2A_{14} + 2B_{14} + B_1 B_4 = -2V_1 V_4 - \frac{k^2}{2\pi}e^{-B}N_1 N_4, \tag{20}$$

$$B_{14} + B_1 B_4 = -\frac{k^2}{2\pi}e^{-B}N_1 N_4, \tag{21}$$

$$B_{14} + B_1 B_4 = \frac{1}{2r}(B_1 - B_4), \tag{22}$$

$$2V_{14} + \left(\frac{1}{2r} + B_1\right) V_4 + \left(-\frac{1}{2r} + B_4\right) V_1 = 0, \tag{23}$$

where $k^2 = a^2 + 1$. If $a = 0$, these equations reduce to those for case I. Now, solving (17) for N and substituting in Eqs. (18) to (23), we get the field equations involving L, which have the same form as these equations but with a different constant $k'^2 = 1/a^2 + 1$. For $a \to \infty$, case II is retrieved.

3 Elliptic solutions

The Eqs. (13) and (22) can be solved in a variety of ways. for example, changing over to (r, t) coordinates and separating the variables as a sum, we obtain solutions of the form

$$e^B = \frac{1}{2} m' t^2 + n' t + \frac{1}{6} m' r^2 + l', \tag{24}$$

and

$$N = \frac{\sqrt{\pi}}{k} \left(\frac{1}{2} m t^2 + n t + \frac{1}{4} m r^2 + l \log r + p\right). \tag{25}$$

These solutions satisfy (21) for all r and t if $m = l = 0$ and $m' = -3n^2/4$. The Eqs. (24) and (25) then become

$$e^B = \frac{1}{8} \left(8l' + 4n'u + 4n'v - n^2u^2 - n^2uv - n^2v^2\right), \tag{26}$$

and

$$N = \frac{\sqrt{\pi}}{k} (nt + p), \tag{27}$$

where e^B has been written in a symmetric form in (u, v) coordinates.

We introduce the notations

$$\langle u, v \rangle \ = (u - v)\, e^B, \tag{28}$$

$$\frac{d}{dv} \langle v, u \rangle \ = \langle v \rangle, \qquad \frac{d}{du} \langle v, u \rangle \ = - \langle u \rangle. \tag{29}$$

Then (23) becomes

$$2 \langle v, u \rangle\ V_{41} + \langle v \rangle\ V_4 - \langle u \rangle\ V_1 = 0, \tag{30}$$

and represents the integrating condition for Eqs. (18) and (19). The integration of the latter equations is straight forward. The result is

$$e^{A} = \{\langle v\rangle \langle u\rangle\}^{\frac{3}{4}} \exp\left(-\frac{1}{2}B + W\right), \tag{31}$$

where

$$W = \int \langle v,u\rangle \left\{\frac{V_1^2}{\langle v\rangle}dv - \frac{V_4^2}{\langle u\rangle}\right\} du. \tag{32}$$

A complete solution is obtained when (30) is solved for V and the results are substituted in (31) and (32).

It can easily verified that Eqs. (26) , (27) and (30) to (32) identically satisfy (20), thus setting the question of overdeterminancy of the field equations.

4 Region of validity

Since $e^B > 0$, by (26) we get

$$n^2u^2 + n^2uv + n^2v^2 - 4n'u - 4n'v - 8l' < 0.$$

Now the equation

$$n^2u^2 + n^2uv + n^2v^2 - 4n'u - 4n'v - 8l' = 0, \tag{33}$$

represents an ellipse in the uv−plane with centre at the point

$$(u,v) = \left(\frac{4n'}{3n^2}, \frac{4n'}{3n^2}\right),$$

and with the semimajor and semiminor axes having the respective lengths

$$\frac{4}{n}\sqrt{l' + \frac{2n'}{3n^2}}, \qquad \frac{4}{\sqrt{3}n}\left(l' + \frac{2n'}{3n^2}\right), \tag{34}$$

and the major axis oriented at $+135^o$ to the u-axis. The ellipse is real if and only if

$$l' + \frac{2n'}{3n^2} > 0. \tag{35}$$

On the boundary of the ellipse $e^B \to 0$. The left hand side of the inequality is the value of e^B at the centre (u, v) of the ellipse which is, for real ellipses, greater than the value at the boundary. Moreover throughout the ellipse the partial derivatives

$$\left(e^B\right)_u = 2n^2u + n^2v - 4n', \qquad \left(e^B\right)_v = 2n^2v + n^2u - 4n' \tag{36}$$

are continuous. Therefore, there exists an absolute maximum value of e^B at some point of the ellipse at which the expressions (36) vanish [4]. This point is indeed the centre of the ellipse as can be easily verified. Thus, e^B is positive throughout the interior of the ellipse with a maximum value of

$$l' + \frac{2n'}{3n^2},$$

at its centre. Therefore the ellipse represents the region of validity of e^B.

5 The scalar field equation

The Eq. (30) for V can be written as

$$\{\langle v, u\rangle V_1\}_4 + \{\langle v, u\rangle V_4\}_1 = 0, \tag{37}$$

which is equivalent to the pair of equations

$$\langle v; u\rangle V_1 = E(v - u) + G(v), \tag{38}$$

$$\langle v, u\rangle V_4 = E(v - u) + H(u), \tag{39}$$

where E, G and H are arbitrary functions of their arguments. Therefore, we have

$$V = \int \frac{E(v-u)}{\langle v, u\rangle} d(u+v) + \int \frac{G(v)}{\langle v, u\rangle} dv + \int \frac{H(u)}{\langle v, u\rangle} du. \tag{40}$$

The first integral, denoted by I_1 can be reduced to the standard form

$$I_1 = -\frac{2E\left(\sqrt{2}y\right)}{3n^2y} \int \frac{dx}{[x^2 - J^2(y)]}, \tag{41}$$

where

$$J(y) = \frac{4}{3n}\sqrt{l' + \frac{2n'^2}{3n^2} - \frac{n^2y^2}{16}}, \tag{42}$$

by means of the transformations

$$u = \frac{(x-y)}{\sqrt{2}} + \frac{4n'}{3n^2}, \qquad v = \frac{(x+y)}{\sqrt{2}} + \frac{4n'}{3n^2}. \tag{43}$$

If, in addition we assume that

$$G(v) = \langle v \rangle, \qquad H(u) = -\langle u \rangle, \tag{44}$$

then the full integral for V viz., (40) becomes, within the region of validity

$$V = \frac{E\left(\sqrt{2}y\right)}{n^2yJ(y)} \log\left[J(y) + \frac{x}{J(y)-x}\right]$$

$$+ \log\left\{\frac{3}{\sqrt{2}}n^2y\left[J^2(y) - x^2\right]\right\} + P(y), \tag{45}$$

where P is an arbitrary function.

With the help of Eqs. (38), (39) and (44), Eq. (31) can be reduced to

$$e^A = \frac{\{\langle v \rangle \langle u \rangle\}^{\frac{3}{4}}}{(v-u)} \exp\left(W' + 2V - \frac{3B}{2}\right), \tag{46}$$

where

$$W' = \int \frac{E^2}{\langle v, u \rangle}\left(\frac{dv}{\langle v \rangle} - \frac{du}{\langle u \rangle}\right). \tag{47}$$

The coordinates (x, y) defined by (43) are related to the (r, t) coordinates by a simple shift of origin along with a change in scale represented by the relations

$$r = \frac{1}{\sqrt{2}y}, \qquad t = \frac{1}{\sqrt{2}x}, \tag{48}$$

between the unit intervals in the respective directions as can be verified from (8). The corresponding axes coincide with the elliptic region of validity.

6 Nature of the fields

The electromagnetic field components calculated from (7), (8), (14), (17) and (27) are

$$F_{r\theta} = -\frac{\sqrt{\pi}}{k} nar, \qquad F_{zt} = \frac{\sqrt{\pi}}{k} n, \tag{49}$$

the other components being zero. Both the electromagnetic field and the magnetic fields are in the z-direction. Therefore there is no electromagnetic radiation. The electric field is uniform, such as that obtains between two equally but oppositely charged conducting plates of infinite dimensions held perpendicular to the z-axis, with a surface charge density proportional to n, and placed in a medium of permittivity proportional to k. The electromagnetic field is however not uniform because the magnetic field is not. This can also be seen from the fact that the tensor $F_{ij;k} = 0$ for at least

$$F_{13;4} = \frac{1}{4} n \sqrt{\frac{\pi}{(a^2+1)}} B_4 \neq 0. \tag{50}$$

The energy density of the electromagnetic field is found to be

$$T'_{tt} = \frac{\pi n^2}{2} e^{-B}, \tag{51}$$

which has a singularity at the boundary of the elliptic region of validity. The dual electromagnetic field components defined by

$$F^{*ij} = -\frac{1}{2} (-g)^{-\frac{1}{2}} E_{ijkl} F_{kl}, \tag{52}$$

where $-(-g)^{-1/2} E_{ijkl}$ is the fourth order Levi-Civita tensor density, are

$$F^*_{r\theta} = \sqrt{\frac{\pi}{a^2+1}} nr, \qquad F^*_{zt} = -\sqrt{\frac{\pi}{a^2+1}} na, \tag{53}$$

whence the nullity function defined by [5]

$$Q = \sqrt{(F_{ij}F^{ij})^2 + (F_{ij}F^{*ij})^2}, \tag{54}$$

takes the form

$$Q = 2\pi n^2 e^{-B-A}, \tag{55}$$

which by (46), becomes

$$Q = 2\pi n^2 \frac{(v-u)}{\{\langle v\rangle \langle u\rangle\}^{\frac{3}{4}}} e^{-W' - 2V + \frac{B}{2}}. \tag{56}$$

This shows that the electromagnetic field is null i.e., $Q = 0$ when $e^B = 0$, because under this condition both W' and V are infinite as seen from (40) and (47).

Thus both the scalar fields and metric potential A are singular at the boundary of the region of validity. The hyper-surface $r = 0$ is a singularity which is inherent in the metric, and therefore need not be considered. It may however be pointed out that on this hypersurface, any function of time could satisfy (30) and the electromagnetic field would be null, according to (56). As shown in [1] irrotational stiff fluids are equivalent to the scalar fields V with pressure and energy density both equal to $V_{,s}V^{,s}$. It can be seen by contracting (2) that $R = -V_{,s}V^{,s}$. Thus the pressure of the irrotational fluid has the same behaviour as R, without being affected by the electromagnetic fields.

Therefore the pressure-cum-density is, by Eqs. (38), (39) and (44)

$$p = \rho = \frac{[4(E + \langle v\rangle)(E - \langle u\rangle)]}{(v-u)^2} e^{-A-2B}, \tag{57}$$

which has singularities at $r = 0$ and the boundary of the region of validity.

7 Conclusion

We have chosen a cylindrically symmetric space-time because, unlike plane [6] and spherically symmetric [7] cases, gravitational waves can occur therein [8]. Indeed, the solution of (24) for e^B which represents the single degree of freedom of the space-time is in general of the form

$$e^B = \frac{1}{2r} S(t+r) + \frac{1}{2r} U(t-r), \tag{58}$$

which, within the region in which $e^B > 0$, represents a forward and a backward moving wave pattern with an amplitude that is attenuated proportional to r. This paper deals with a particular case of this fact consistent with the type of electromagnetic field we have employed here, having four non-vanishing components. This is one of the two types of electromagnetic fields that satisfy the field equations. A particular complete solution for the field equations has been obtained for the scalar and electromagnetic fields and also for the metric potentials A and B. Therefore every field component has been obtained in terms of r and t only via the single degree of freedom represented by B. It has also been seen that the solutions are equivalent to those of an irrotational stiff fluid with pressure given by (57) which also depends upon B only, not withstanding the presence of electromagnetic fields. If we choose our arbitrary function E so that $p = \rho$ in (57) is positive, it is physically viable and corresponds to the entropy level of the initial stage of the universe that is not chaotic but quiescent [9].

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Neperiodična rešenja jednačina teorije polja u relativizmu: Eliptičan slučaj

Ajnštajnove jednačine teorije polja su rešene za linearno povezan sistem, za skalar bez mase i elektromagnetsko polje bez izvora u cilindrično simetričnom prostor-vremenu, a za jedan stepen slobode. Diskutovana je i priroda rešenja .