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BIANCHI TYPE-I COSMOLOGICAL STIFF FLUID MODEL IN WESSON'S THEORY

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1. Introduction

Wesson [1] proposed a simple formulation of Scale-invariant theory of gravitation incorporating the gauge function $\beta(x^i)$, where x^i are coordinates in the four dimensional space time and the tensor field is identified with Riemannian metric tensor g_{ij}. One can interpreted g_{ij} as a gravitational potential tensor which determines the interaction between matter (or other fields) and graviation. It is pointed out that this theory agrees with general relativity up to the accuracy of the observations made up to now, Dirac [2]; Hoyle and Narlikar [3] and Canuto et. al. [4] have studied earlier several aspects of the Scale-invariant theory of gravitation. But the Wesson's [1] formulated of Scale-invariant theory is sofar the best theory to describe all the interactions between matter and gravitation.

In this theory Mohanty and Daud [5] have already studied the cosmological models governed by vacuum field equations when the space time is described by homogeneous and anisotropic Bianchi type-I metric with different type of gauge functions. However in both the papers it is shown that the models reduce to Kasner model [6] when cosmological constant is zero, but for non-zero cosmological constant the models isotropize as in the Einstein's theory.

In this paper we show that Bianchi type-I cosmological models governed by stiff fluid distribution is compatible in Scale-invariant theory of gravitation [1]. It has been also observed that the presence of gauge field does indicate some distinguishable features in the models compared to those developed only with the perfect fluid as the material source. In Sec.2, we set the field equations of Scale-invariant theory of gravitation. In Sec.3, we obtain the explicit exact solutions of the field equations with the help of energy-momentum tensor for perfect fluid and confine ourselves to the equation of state $p = \rho c^2$ in order to overcome the paucity of field equations. In Sec.4 we study the physical system decribed by the solution interms of dimensionless cosmological constant. Sec.5, bears conclusion.

2. Field equations

The field equations of the Scale-invariant theory of gravitation formulated by Wesson [1] are

$$G_{ij} + \frac{2\beta_{;ij}}{\beta} - \frac{4\beta_{,i}\beta_{,i}}{\beta^2} + \left(\frac{g^{ab}\beta_{,a}\beta_{,b}}{\beta^2} - \frac{2g^{ab}\beta_{;ab}}{\beta}\right) g_{ij} + \Lambda_0\beta^2 g_{ij} = -kT_{ij} \quad (1)$$

where

$$G_{ij} \equiv R_{ij} - \frac{1}{2}g_{ij}R. \quad (2)$$

Here, G_{ij} is the conventional Einstein tensor involving g_{ij}, T_{ij} is the usual stress tensor of the matter; and Comma and semicolon denote partial and covariant differentiations respectively. The cosmological term Λg_{ij} of Einstein's theory is transformed to $\Lambda_0\beta^2 g_{ij}$ in Scale-invariant theory with a dimensionless constant Λ_0 and $k = \frac{8\pi G}{c^4}$. Here we consider the Bianchi type-I metric with gauge function $\beta(x^i)$

$$ds_w^2 = \beta^2(-A^2dx^2 - B^2dy^2 - C^2dz^2 + c^2dt^2), \quad (3)$$

where A, B, C are functions of t only, c is the velocity of light. The energy-momentum tensor for perfect fluid distribution is given by

$$T_{ij} \equiv (p + \rho c^2)U_iU_j - pg_{ij} \quad (4)$$

together with

$$g_{ij}U^iU^j = \frac{1}{\beta^2}, \quad (5)$$

where U^i is the four-velocity vector of the fluid, and p and ρ are the proper pressure and energy density respectively.

By use of comving coordinates the field equations (1) for the metric (3) can he written as

$$\frac{B_{44}}{B} + \frac{C_{44}}{C} + \frac{B_4C_4}{BC} - \frac{\beta_4^2}{\beta^2} + \frac{2\beta_4}{\beta}\left(\frac{B_4}{B} + \frac{C_4}{C}\right) + \frac{2\beta_{44}}{\beta} - \Lambda_0\beta^2c^2 = -kpc^2\beta^2 \quad (6)$$

$$\frac{C_{44}}{C} + \frac{A_{44}}{A} + \frac{C_4A_4}{CA} - \frac{\beta_4^2}{\beta^2} + \frac{2\beta_4}{\beta}\left(\frac{C_4}{C} + \frac{A_4}{A}\right) + \frac{2\beta_{44}}{\beta} - \Lambda_0\beta^2c^2 = -kpc^2\beta^2 \quad (7)$$

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{A_4B_4}{AB} - \frac{\beta_4^2}{\beta^2} + \frac{2\beta_4}{\beta}\left(\frac{A_4}{A} + \frac{B_4}{B}\right) + \frac{2\beta_{44}}{\beta} - \Lambda_0\beta^2c^2 = -kpc^2\beta^2 \quad (8)$$

$$\frac{A_4B_4}{B}+\frac{B_4C_4}{C}+\frac{C_4A_4}{BC}+\frac{3\beta_4^2}{\beta^2}+\frac{2\beta_4}{\beta}\left(\frac{A_4}{B}+\frac{B_4}{C}+\frac{C_4}{C}\right)-\Lambda_0\beta^2c^2=+k\rho c^4\beta^2 \quad (9)$$

where the suffix 4 following an unknown function denotes ordinary differentiation with respect to t. Following Wesson's [1] simple formulation of scale-invariant theory, we take the gauge function in the form

$$\beta=\frac{1}{ct}. \quad (10)$$

Using the gauge function (10) in the equations (6)-(9), we get

$$\frac{B_{44}}{B}+\frac{C_{44}}{C}+\frac{B_4C_4}{BC}+\frac{3}{t^2}-\frac{2}{t}\left(\frac{B_4}{B}+\frac{C_4}{C}\right)-\frac{\Lambda_0}{t^2}=-\frac{kp}{t^2} \quad (11)$$

$$\frac{C_{44}}{C}+\frac{A_{44}}{A}+\frac{C_4A_4}{CA}+\frac{3}{t^2}-\frac{2}{t}\left(\frac{C_4}{C}+\frac{A_4}{A}\right)-\frac{\Lambda_0}{t^2}=-\frac{kp}{t^2} \quad (12)$$

$$\frac{A_{44}}{A}+\frac{B_{44}}{B}+\frac{A_4B_4}{AB}+\frac{3}{t^2}-\frac{2}{t}\left(\frac{A_4}{A}+\frac{B_4}{B}\right)-\frac{\Lambda_0}{t^2}=-\frac{kp}{t^2} \quad (13)$$

$$\frac{A_4B_4}{AB}+\frac{B_4C_4}{BC}+\frac{C_4A_4}{CA}+\frac{3}{t^2}-\frac{2}{t}\left(\frac{A_4}{A}+\frac{B_4}{B}+\frac{C_4}{C}\right)-\frac{\Lambda_0}{t^2}=+\frac{k\rho c^2}{t^2}. \quad (14)$$

There are four equations (11)-(14) with five unknown physical variables A, B, C, p and ρ. For complete determinacy of the system one extra condition is needed. In this paper we therefore confine ourselves to the equation of state

$$p=\rho c^2 \quad (15)$$

representing a stiff-matter field.

Now adding equations (11), (12) and (13) to 3 times of (14) and using (15) we get

$$\frac{(ABC)_{44}}{ABC}-\frac{5}{t}\frac{(ABC)_4}{ABC}-\frac{3(\Lambda_0-3)}{t^2}=0$$

which leads to

$$t^2(ABC)_{44}-5t(ABC)_4-3(\Lambda_0-3)ABC=0. \quad (16)$$

3. Solutions

Equation (16) yields

$$ABC = t^3\left(Mt^{\sqrt{3\Lambda_0}} + Nt^{-\sqrt{3\Lambda_0}}\right), \tag{17}$$

where M and N are constants of integration. The solution of the field equations (11)-(14) inview of (17) may be written as

$$\begin{aligned} A &= t\left(Mt^{\sqrt{3\Lambda_0}} + Nt^{-\sqrt{3\Lambda_0}}\right)^{p_1} \\ B &= t\left(Mt^{\sqrt{3\Lambda_0}} + Nt^{-\sqrt{3\Lambda_0}}\right)^{p_2} \\ C &= t\left(Mt^{\sqrt{3\Lambda_0}} + Nt^{-\sqrt{3\Lambda_0}}\right)^{p_3}, \end{aligned} \tag{18}$$

where $\Lambda_0 > 0$ and p_i's are constants such that

$$\sum_{i=1}^{3} p_i = 1. \tag{19}$$

Here, the above choice for the determination of the field variables A, B and C can be settled by actual substitution of the solution (18) in the field equations (11)-(14).

Now substituting equations (18) in the equations (11)-(14) we get

$$3\Lambda_0\left(\frac{T_1}{T}\right)^2\left(p_2^2 + p_3^2 + p_2p_3 - p_2 - p_3\right) + 3\Lambda_0\left(p_2 + p_3\right) - \Lambda_0 = -kp$$

$$3\Lambda_0\left(\frac{T_1}{T}\right)^2\left(p_3^2 + p_1^2 + p_3p_1 - p_3 - p_1\right) + 3\Lambda_0\left(p_3 + p_1\right) - \Lambda_0 = -kp$$

$$3\Lambda_0\left(\frac{T_1}{T}\right)^2\left(p_1^2 + p_2^2 + p_1p_2 - p_1 - p_2\right) + 3\Lambda_0\left(p_1 + p_2\right) - \Lambda_0 = -kp$$

and

$$3\Lambda_0\left(\frac{T_1}{T}\right)^2\left(p_1p_2 + p_2p_3 + p_3p_1\right) - \Lambda_0 = kp,$$

where

$$T = Mt^{\sqrt{3\Lambda_0}} + Nt^{-\sqrt{3\Lambda_0}}$$

and

$$T_1 = Mt^{\sqrt{3\Lambda_0}} - Nt^{-\sqrt{3\Lambda_0}}.$$

But these equations are consistent only when

$$p_1 = p_2 = p_3 = \frac{1}{3}.$$

Subsequently, we have

$$\sum_{i=1}^{3} p_i^2 = \frac{1}{3}. \tag{20}$$

Thus we get

$$kp(= k\rho c^2) = \Lambda_0 \left(\frac{-4MN}{T^2}\right).$$

Since $\Lambda_0 > 0$. The non-zero constants M and N are of opposite sign for a physically realistic solutions. Again, subject to the condition (20), equation (17) yields the admissible solution

$$A = B = C = t\left(Mt^{\sqrt{3\Lambda_0}} + Nt^{-\sqrt{3\Lambda_0}}\right)^{\frac{1}{3}}. \tag{21}$$

Inview of equation (21), the metric (3) takes the form

$$ds_w^2 = -\left(\frac{Mt^{\sqrt{3\Lambda_0}} + Nt^{-\sqrt{3\Lambda_0}}}{c^3}\right)^{\frac{2}{3}} (dx^2 + dy^2 + dz^2) + \frac{dt^2}{t^2}. \tag{22}$$

However, this metric can be transformed through a proper choice of coordinates to the form

$$ds_w^2 = \beta'^2 \left[-T^{2/3} 3\Lambda_0 (T^2 - 4MN)(dx^2 + dy^2 + dz^2)c^2 dT^2\right], \tag{23}$$

where

$$\beta' = \frac{1}{\sqrt{3\Lambda_0 c^2 (T^2 - 4MN)}}.$$

Thus, in the Scale-invariant theory of gravitation the universe is isotropized due to the presence of gauge function. However, stiff fluid distribution exists for non zero finite cosmological constant.

4. Some physical properties

In this section we intend to study the following physical properties of the model obtained in the preceeding sections.

(i) It may be mention here that the stiff fluid distribution exists in Scale-invariant theory for a non-zero finite dimensionless cosmological constant Λ_0 and $M^2 + N^2 \neq 0$, otherwise matter distribution does not exists.

(ii) For a positive non-zero Λ_0, we have

$$p(=\rho c^2) \to \alpha \text{ as } T \to 0$$
$$\to 0 \text{ as } T \to \alpha$$

and

$$\beta' \to \text{constant as } T \to 0$$
$$\to 0 \qquad \text{as } T \to 0.$$

Here, $T = 0$ yields a singularity which can be interpreted as Bigbang in Scale-invariant theory. However the matric collapses and the gauge function becomes constant at $T = 0$. The model collapses and gauge function vanishes as $T \to \alpha$.
(iii) For $\Lambda_0 = 0$, we have $p(=\rho c^2) = O$, then the metric leads to that of a flat space-time, and the gauge function does not exist.
(iv) The proper volume $V = ABC.\ \beta'^3$ for the model (23) becomes

$$V = \frac{T}{c^3}$$

$$V \to 0 \text{ as } T \to 0$$

and

$$V \to \alpha \text{ as } T \to \alpha.$$

This indicates that the model obtained here represents an open model. The universe starts evolution with zero spatial volume and expands uniformly; and blows up at infinite future.
(v) Following Roychoudhuri [7], the expansion scalar θ and the anisotropy $|\sigma|$ are defined as

$$\theta = \frac{3R_4}{R}$$

and

$$\sigma^2 = \frac{1}{12}\left[\left(\frac{g_{11,4}}{g_{11}} - \frac{g_{22,4}}{g_{22}}\right)^2 + \left(\frac{g_{22,4}}{g_{22}} - \frac{g_{33,4}}{g_{33}}\right)^2 + \left(\frac{g_{33,4}}{g_{33}} - \frac{g_{11,4}}{g_{11}}\right)^2\right].$$

In this model, we have

$$\theta = \frac{I}{T} \text{ and } \sigma^2 = 0.$$

So, the expansion scalar becomes indefinitely large or indefinitely small according as $T \to 0$ or $T \to \alpha$. For non-zero finite T the expansion scalar $\theta > 0$ indicates that the model is expanding in nature. This also supports the analysis done earlier regarding the presence of singularity at $T = 0$.

$\frac{\sigma^2}{\theta^2} = 0$ shows that the spatially homogeneous anisotropic cosmological models in Scale-invariant theory reduce to the spatialy homogeneous isotropic models in presence of stiff fluid.

(vi) The expansion isotropizes the universe and occures uniformly in all the spatial directions at the rate $\frac{T^{2/3}}{c^2}$.

(vii) The matterfield does not exist either for $\Lambda_0 = 0$ or $M = 0$ or $N = 0$ but for $\Lambda_0 \neq 0$ the model collapses.

However, as studied earlier by Mohanty and Daud [5], this model does not leed to Kasner [6] model either for $M = 0$ or $N = 0$. The generalization of Kasner model exists in case of Scalar-tensor theory [8] but not in the Scale-invariant theory.

5. Conclusion

It is intersting to note that the stiff fluid cosmological model obtained here is completely governed finite by a non-zero finite dimensionless cosmological constant Λ_0 whereas vacuum cosmological models studied earlier by Mohanty and Daud [5] exist for $\Lambda_0 \geq 0$. However in both vacuum and stiff fluid models neither $\Lambda_0 = 0$ nor $\Lambda_0 \neq 0$ provide the generalization of Kasner model analogous to the case studied earlier by Reddy and Venkateswarlu [8] in Biometric theory of Rosen [9].

REFERENCES

[1] P.S. Wesson, Astron. Astrophysics, 102, 45-52, (1981).

[2] P.A.M. Dirac, Proc. R. Soc. Lond., A333, 403, (1973). Ibid, A338, 439, (1974).

[3] F.Hoyle and J.V.Narlikar, *Action at a distance in Physics and cosmology*, Freeman, San Francisce, U.S.A.(1974).

[4] V.Canuto,P.J. Adams, Phys. Rev., Ser.3, D16, 1643, (1977a);
S.H.Hsieh and E.Tsiang, V.Canuto, S.H.Hsieh and P.J. Adams,Phys.Rev. Lett., 39, 429, (1977b).

[5] G. Mohanty and S.M. Daud, Acta. Physica Hungarica, (1993a), (Communicated); Astro. Physics and Space Science, U.K. (1993b), (Communicated).

[6] E. Kasner, Am. J. Math., 43, 217, (1921).

[7] A.K. Roychaudhuri, Phys. Rev., 98, 1123, (1955).

[8] D.R.K.Reddy and R. Venkateswarlu, Astrophysics and Space Science, 135, 287-290, 1987; Astrophysics and Space Science, 158, 169-171, 1989.

[9] N. Rosen, Gen. Rel. Grav., 6, 277, 1973; Gen. Rel. Grav., 6, 259, 1975.

BIANCHI TYPE-I COSMOLOGICAL STIFF FLUID MODEL IN WESSON'S THEORY

It is shown that there exists spatially homogeneous and isotropic Bianchi type-I cosmological model in Wesson's [1] Scale-invariant theory of gravitation

when the source of the gravitational field is a perfect fluid characterised by the equation of state $p = \rho c^2$. Consequently the degenerated vacuum cosmological model, in this theory, is presented and the physical properties are stutied.

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