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LINEAR STABILITY ANALYSIS OF COMPRESSIBLE HIGH-TEMPERATURE MIXING LAYERS

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1. Introduction

Several flow phenomena in gas dynamics proceed followed by relatively high temperatures. We will mention just a few, like flow phenomena arising in various combustion processes (jet propulsion, internal combustion engines), or in high-speed aerodynamics (re-entry problem). These problems cannot be treated within the frames of classical gas dynamics, because the gas does not behave as calorically perfect. At least its behaviour is one of a thermally perfect gas in which vibrational energies are set up, and C_p and C_v - specific heats at constant pressure and constant volume, respectively are not constants, but depend on the temperature. For example, air at normal pressure behaves roughly as a thermally perfect gas in the range from 600 K to 2500 K [1]. At higher temperatures dissociation of the gas takes place, followed by ionization for even higher temperatures, and C_p and C_v become dependent on the pressure too.

Although several important problems in the context of high-temperature gas dynamics have been solved (s. [2]), this field of fluid flows, due to its complexity, is not nearly as elaborated as the classical gas dynamics. This is particularly true when stability characteristics of various gas flows at high temperatures are concerned. In this paper we attempt to solve a problem of hydrodynamic stability of a high-temperature gas flow. We analyse the linear stability properties of an unbounded mixing layer of a thermally perfect gas. We assume that both velocity and temperature basic profiles, generally independent of each other, are present, so that the basic flow is a nonhomentropic one, and that the gas is non-viscous and non-conductive.

The corresponding problem for the flow of a calorically perfect gas has attracted some attention of researchers in the past. The first paper especially concerned with the stability of compressible homentropic free-shear flows was the one by Blumen [3], and its extension to nonhomentropic flows was performed by Djordjevic and Redekopp [4]. The latter authors performed also a very complex weakly nonlinear stability analysis for viscous and conductive gases and derived a Landau type equation describing the evolution of a neutrally stable disturbances [5], [6]. Their results reveal the important consequences that the Mach number, the excess or the deficit of the temperature in the critical layer, and the symmetry properties of the basic profiles may have upon the stability characteristics of compressible mixing layers.

This paper represents a direct extension of the analysis given in [4] for a thermally perfect gas. Although the single equation for the pressure amplitude, governing stability characteristics, differs from the corresponding one in [4] in that it contains explicitly the ratio $C_p/C_v = \gamma(T_0)$, which now depends in a certain way on the basic temperature profile, the analysis shows that the effect of established vibrational energies in a thermally perfect gas is not apparent, except via the reference Mach number, whereby this effect is stabilizing.

2. Formulation of the problem

We consider linear stability characteristics of a compressible mixing layer with the basic flow shown in Fig.1. Gas is supposed to be thermally perfect. Writting the equations governing such a flow in nondimensional form by introducing the following scales: d - for lengths, V_r - for velocities, d/V_r - for time, T_r - for temperature and p_r - for pressure, and eliminating the density via the equation of state, we obtain:

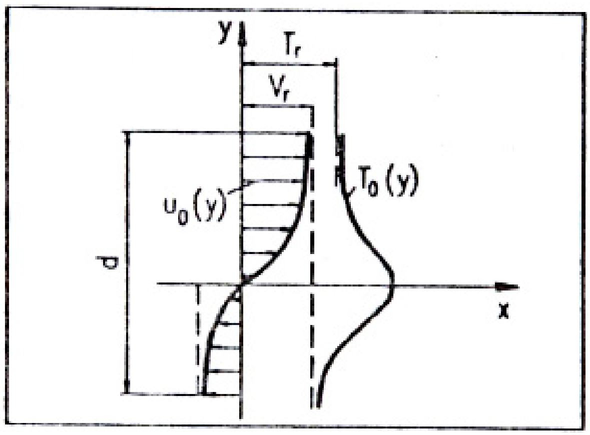

Fig.1. Basic velocity and temperature profiles

$$
\begin{aligned}
&\frac{D\vec{V}}{Dt} + \frac{T}{\gamma_r M_r^2} + grad \;\; \ln p = 0 \\
&\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{V}\nabla \\
&\frac{D \ln p}{Dt} + \gamma(T) div \;\; \vec{V} = 0 \\
&\frac{D \ln p}{Dt} + [\gamma(T) - 1]\, div \;\; \vec{V} = 0
\end{aligned} \tag{1}
$$

Here, as in [4], γ_r and M_r are the ratio of specific heats and the Mach number of the reference state, respectively, and $\vec{V}$ is the velocity vector with components (u, v) in the coordinate directions (x, y). In contrast to [4], $\gamma(T)$ - local ratio of specific heats, is here a temperature dependent quantity. For its evaluation we will use the following formula [1]:

$$\gamma(T) = 1 + \left[\frac{5}{2} + \frac{\left(\frac{\theta_r}{T}\right)^2 e^{\frac{\theta_r}{T}}}{\left(e^{\frac{\theta_r}{T}} - 1\right)^2} \right]^{-1}, \tag{2}$$

where $\theta_r = h\nu/kT_r$, and $h = 6.625 \times 10^{-34}$ Js is Planck's constant, $k = 1.38 \times 10^{-23}$ J/K is Boltzmann's constant and ν is the fundamental vibrational frequency of the molecule (for example $\nu = 7.06 \times 10^{13}$ s^{-1} for nitrogen). Variation of γ with T for some values of the parameter θ_r in the range of practical importance is shown in Fig.2.

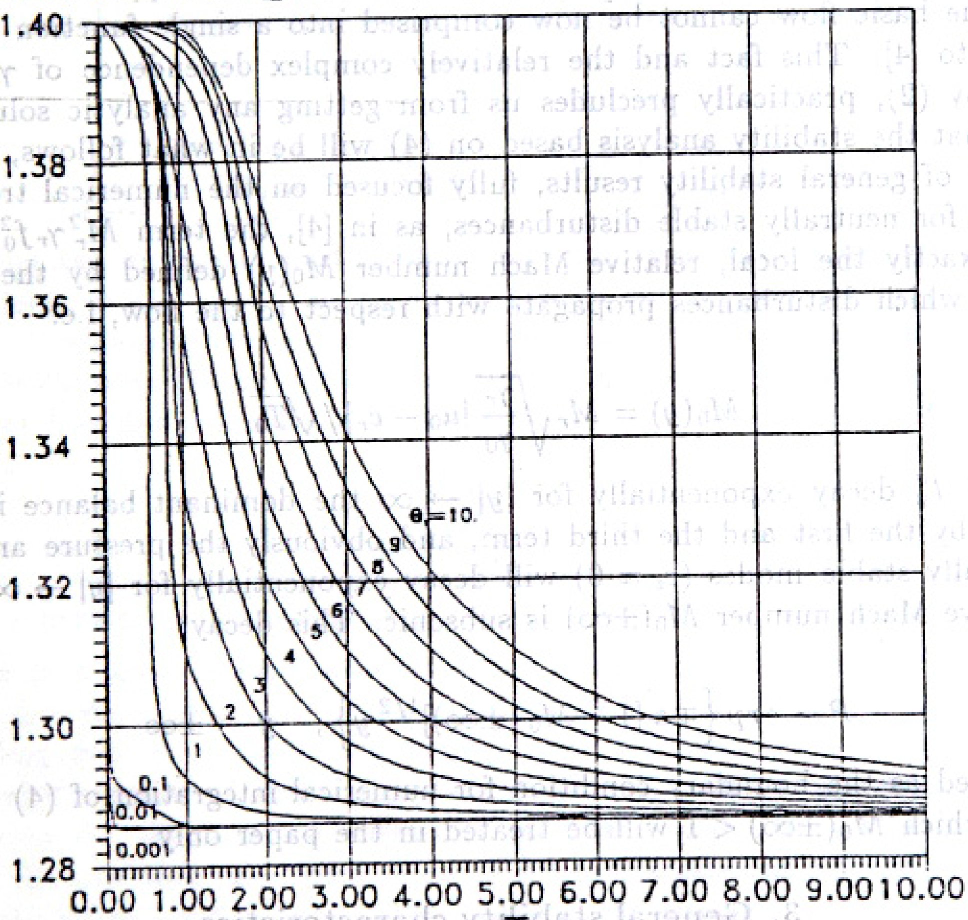

Fig.2. Dependence of γ on T for a thermally perfect gas

For the purposes of linearizing the system (1) about the basic state $u_0(y)$, $T_0(y)$ and $p_0 = 1$ it is necessary to expand $\gamma(T)$ into a Taylor series:

$$\gamma(T_0 + T') = \gamma(T_0) + \left.\frac{d\gamma}{dt}\right|_{T=T_0} T' + ..., \tag{3}$$

where T' is the temperature perturbation. At that, the first term only will participate into a linear stability analysis and will be denoted for convenience by $\gamma_0(y)$, i.e. $\gamma(T_0) = \gamma_0(y)$. Owing to that, linearized versions of the first two equations of (1) are independent of the temperature perturbation, which can be, as in [4], simply evaluated from the third of equations (1), after the complete stability analysis, based exclusively on the first two equations, is performed.

Using normal mode representation with α as wave number and $c = c_r + ic_i$ as complex speed of waves propagating in x direction, linearized versions of the first two equations of (1) can be combined to yield the following single eigenvalue equation for the pressure amplitude P:

$$f_0 D^2 P - 2f_0' DP - \alpha^2 f_0 \left(1 - M_r^2 \frac{\gamma_r}{\gamma_0} f_0^2\right) P = 0, \tag{4}$$

where $D = d/dy$ and $f_0 = (u_0 - c)/\sqrt{T_0}$. Due to explicit appearance of $\gamma_0(y)$ in (4), the basic flow cannot be now comprised into a single function $f_0(y)$, in contrast to [4]. This fact and the relatively complex dependence of γ_0 on T_0, defined by (2), practically precludes us from getting any analytic solutions to (4), so that the stability analysis based on (4) will be in what follows, with the exception of general stability results, fully focused on the numerical treatment. However, for neutrally stable disturbances, as in [4], the term $M_r^2 \gamma_r f_0^2/\gamma_0$ represents exactly the local, relative Mach number $M_0(y)$ defined by the relative speed by which disturbances propagate with respect to the flow, i.e.

$$M_0(y) = M_r \sqrt{\frac{\gamma_r}{\gamma_0}} \left|u_0 - c_r\right| / \sqrt{T_0}. \tag{5}$$

If u_0' and T_0' decay exponentially for $|y| \to \infty$ the dominant balance in (4) is governed by the first and the third term, and obviously the pressure amplitude for neutrally stable modes ($c_i = 0$) will decay exponentially for $|y| \to \infty$ too, if the relative Mach number $M_0(\pm\infty)$ is subsonic. This decay:

$$P \backsim exp\left\{\mp\alpha \left[1 - M_0(\pm\infty)\right]^{1/2} y\right\}, \quad y \to \pm\infty \tag{6}$$

will be used as the boundary condition for numerical integration of (4) and the cases in which $M_0(\pm\infty) < 1$ will be treated in the paper only.

3. General stability characteristics

In spite of the fact that equation (4) differs from the corresponding equation describing stability characteristics of flow of a calorically perfect gas, it is shown that general stability results obtained in [4] hold for a thermally perfect gas also.

That is why we will just quote a couple of them which are the most important, without the detailed proofs.

If f_0 is nowhere zero (i.e. $c_i \neq 0$, or, if $c_i = O$, there are no critical layers) we can divide equation (4) by f_0^3, multiply it by $\overset{*}{P}$ - complex conjugate of P and integrate it over $[-\infty, \infty]$. Then, the integrated term vanishes, since transverse velocity component, and consequently DP vanishes for $|y| \to \infty$, and we obtain:

$$\int_{-\infty}^{\infty} \frac{|DP|^2}{f_0^2} dy + \alpha^2 \int_{-\infty}^{\infty} \frac{1}{f_0^2} \left(1 - M_r^2 \frac{\gamma_r}{\gamma_0} f_0^2\right) |P|^2 \, dy = 0. \tag{7}$$

For $c_i = 0$ this relation, if the definition (5) of $M_0(y)$ is taken into account, can be written in the form:

$$\int_{-\infty}^{\infty} \frac{1}{f_0^2} (1 - M_0^2) |P|^2 \, dy < 0.$$

It follows from here that, since the flow at $|y| \to \infty$ is subsonic, it must be locally supersonic in some portion of the layer.

For unstable disturbances ($c_i \neq 0$) the relation (7) is conveniently written as:

$$\int_{-\infty}^{\infty} \frac{|DP|^2 + \alpha^2 |P|^2}{|f_0|^4} \overset{*}{f_0}{}^2 dy - \alpha^2 M_r^2 \int_{-\infty}^{\infty} \frac{\gamma_r}{\gamma_0} |P|^2 \, dy = 0.$$

Examining the imaginary part of that one finds that there must be at least one critical level in the flow, say y_c so that $u_0(y_c) = c_r$, for the amplified waves to exist. Combining the real and imaginary parts, one can show that Howard's semicircle theorem [7] applies without modification for this flow. Finally, one can show that Lees-Lin criterion [8] holds without modification for this flow also, i.e.

$$\frac{d}{dy} \left(\frac{u_0'}{T_0} \right)_{y=y_c} = 0 \tag{8}$$

is both a necessary and a sufficient condition for the existence of instabilities.

The fact that general stability characteristics for the flow of a thermally perfect gas considered here remain unchanged when compared with the corresponding flow of a calorically perfect gas, and also the fact that the ratio γ_r/γ_0 experiences a relatively small variation over the flow, as revealed by Fig.2, imply that the effect of established vibrational energies in a thermally perfect gas, in the problem considered here, will not be pronounced, and this will be clearly shown in what follows. Consequently, the numerical results obtained can be practically equally used for both calorically and thermally perfect gas. In this regard they represent a substantial extension of the results obtained in [4], because there the attention was focused almost exclusively on the problems which allow exact solutions for neutrally stable waves.

4. Numerical treatment of the problem

In order to reduce the range of integration of equation (4) to a finite one, we will as in [3] introduce new independent and dependent variables, respectively:

$$z = tanh\ y, \quad F(z) = \frac{DP}{P}$$

to obtain the following nonlinear first order differential equation for F:

$$F' = \frac{2f_{0z}}{f_0}F + \frac{\alpha^2\left(1 - M_r^2\frac{\gamma_r}{\gamma_0}f_0^2\right) - F^2}{1-z^2}, \tag{9}$$

with the boundary conditions for neutral modes:

$$F(\pm 1) = \mp\alpha\sqrt{1 - M_r^2\frac{\gamma_r}{\gamma_0(\pm 1)}f_0^2(\pm 1)}.$$

Two classes of basic flow will be treated by numerical integration of (9):

- with symmetric temperature profile:

$$u_0 = tanh\ y, \quad T_0 = 1 + b\ sinh^m y, \quad b > -1, \quad m > 0,$$

and

- with asymmetric temperature variation:

$$u_0 = \sqrt{T_0}tanh\ y, \quad T_0 = a + b\ tanh\ y, \quad a + b = 1, \quad a \geq b.$$

For both, Lees-Lin criterion (8) holds at $y = 0$, so that $c_r = 0$. If the temperature prifile is symmetric, $F(z)$ is an odd function and the integration ought to be performed in the range $-1 \leq z \leq 0$, with $F(0) = 0$, while for asymmetric temperature distribution $F(z)$ is neither odd nor even and the range of integration in this case is $-1 \leq z \leq 1$.

However, the second term on the right of (9) is not determined at $z = \pm 1$, so that some prolems arise when one starts the integration at, say $z = -1$. In order to overcome this problem the L'Hospital's rule may be used. This rule can be straightforwardly applied for asymmetric temperature variations for any values of parameters a and b, and for symmetric temperature profiles for any b and $m \geq 2$. For $m < 2$, $F'(-1)$ is infinite so that an expansion of F near $z = -1$ should be developed for the same purpose. The details of the numerical integration are given in [9] and will not be reproduced here, but simply stated and discussed. As implied earlier, parameter θ_r has minor effects on the stability characteristics, and all results will be presented for $\theta_r = 1$ only.

In the symmetric case the influence of parameters b and m upon stability boundaries is very pronounced, as evidenced from Fig.3. If the critical layer is heated ($b > 0$), increase of m stabilizes the flow because the unstable region is considerably suppressed with m. On the contrary, if the critical layer is cooled ($b < 0$), increase of m destabilizes the flow, and the unstable region is extended. We cannot find any rational physical explanation for this switch-over. On the other hand, if m is fixed, increase of b stabilizes the flow, so that a heated critical layer is always more stable than a cooled one.

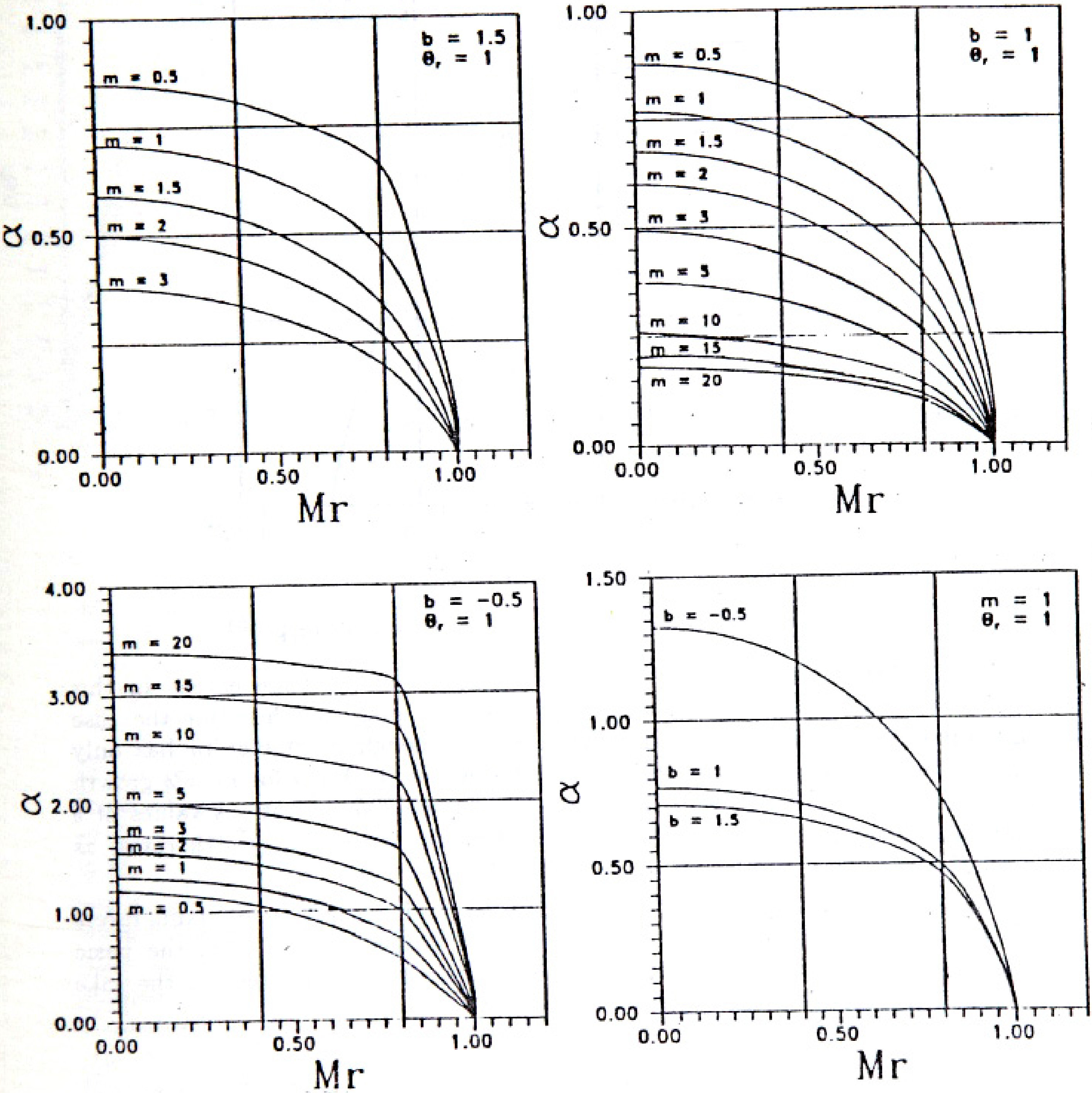

Fig.3. Neutral curves for different values of the governing parameters, for a symmetric temperature profile

In the asymmetric case the effect of parameters a and b upon stability boundary turned out to be much less pronounced. For any combinations of them in the range of physical interest, the neutral curve is very close to the quarter of a circle, as seen in Fig.4.

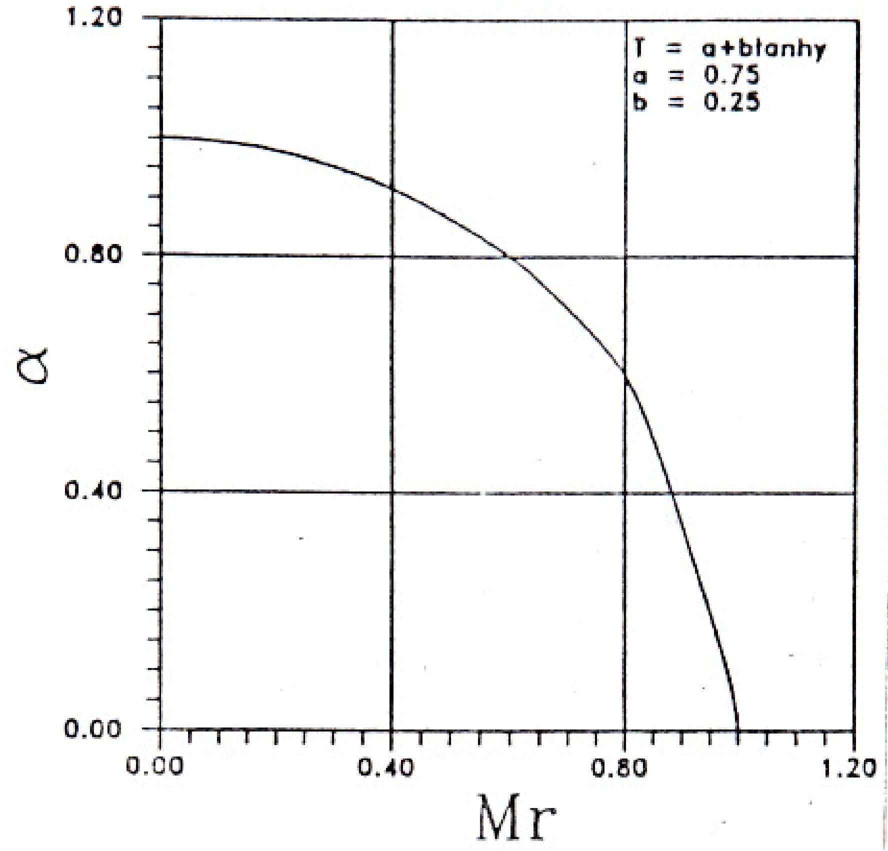

Fig.4. Neutral curve for asymmetric temperature profile

Instability modes with $c_r = 0$ are touched upon to some extent in this paper too (for more details, s. [9]), and will be presented here for the case of symmetric temperature profile. Interestingly enough parameter m has only minor effects on the growth rates of unstable waves. In Fig.5 for $m = 2$ growth rates αc_i versus unstable wave numbers α are presented for various values of b and M_r. Increase of both b and M_r considerably lessens the growth rates, as implied also from stability diagrams in Fig.3.

In [4] a special case of stability of nonhomentropic compressible mixing layers for which $m = M_r^2$ is treated. This case is an artificial one in that the basic temperature profile changes with M_r - the artifact being tolerated for the sake of taking advantage of an exact solution of (4):

$$P = sinh^{\alpha^2} y, \quad \alpha^2 + M_r^2 = 1.$$

In this case a switch-over in the effect of b on the growth rates αc_i was noticed at $M_r \approx 0.74$. For lower values of M_r, αc_i increases with b, while for higher values of M_r it decreases with b. Here αc_i decreases with b for all values of M_r. The switch-over noticed in [4] is obviously attributed to an unnaturally chosen temperature profile.

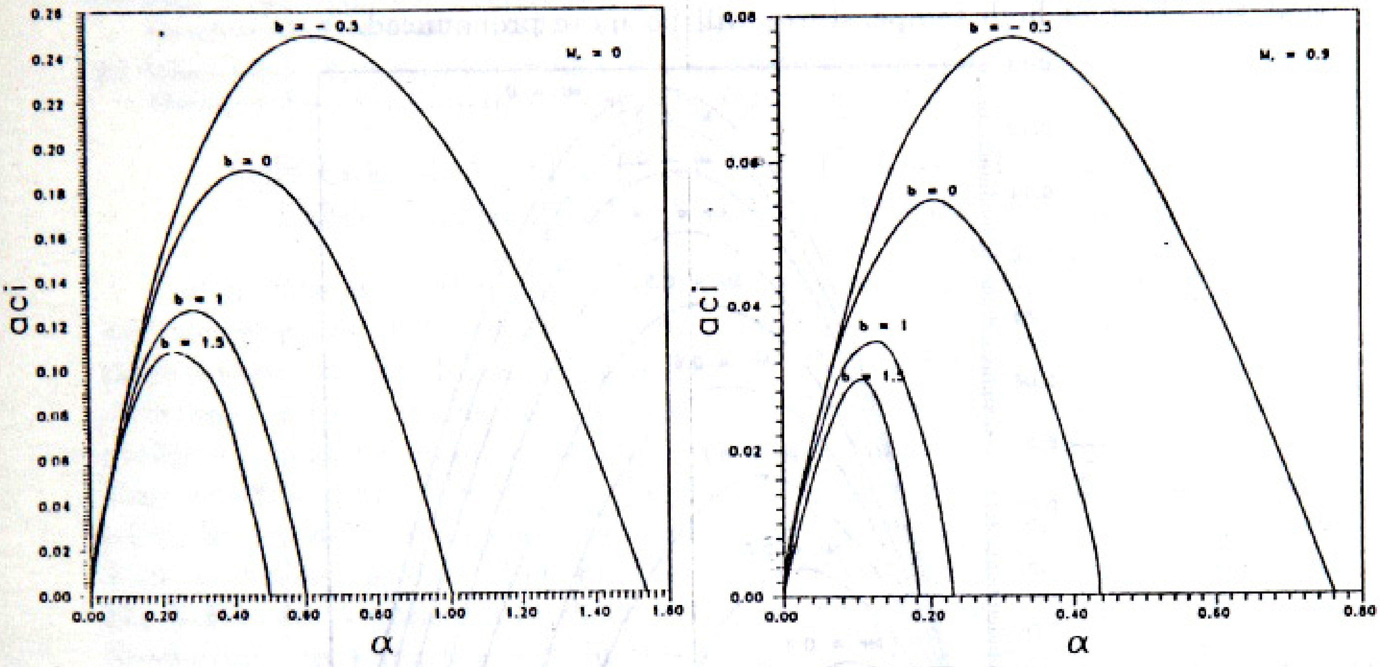

Fig.5. Growth rates curves for $m = 2$ and various b and M_r

Finally, in Fig.6 for $b = 1$ and $m = 2$, αc_i versus α is presented for various M_r. The diagram clearly illustrates the suppressing influence of increasing M_r on the growth rates αc_i

5. Conclusions

The main objective of the paper was a study of linear stability characteristics of compressible mixing layers at high temperatures at which gas behaves as thermally perfect, and consequently $\gamma = C_p/C_v$ is a function of temperature - the effect that enters the governing equation via the nondimensional parameter θ_r. Both, the general stability characteristics and the numerical results did not show any appreciable influences of θ_r upon stability properties of compressible mixing layers. However, this statement is to be accepted with some caution, because the effect of temperatures enters the stability analysis also via the reference Mach number M_r. Since γ_r for a thermally perfect gas is always less than that for a calorically perfect one (s. Fig.2), the corresponding value of M_r for the same velocity and temperature at $y \to +\infty$ will be greater for a thermally perfect gas. The increase of M_r, as clearly shown in the paper, stabilizes the flow because it shrinks the instability region on one hand and lessens the growth rates on the other. Consequently, the main conclusion of the paper is that the effect of high temperatures is stabilizing. This is physically acceptable since the vibrational energies excited at high temperatures are capable of absorbing some energy from the basic flow. In a possible higher order theory - a weakly nonlinear one, as performed for a calorically perfect gas in [5] and [6], it is to be expected that, not only $\gamma(T_0)$, but also $\frac{d\gamma}{dT}\Big|_{T=T_0}$ will be explicitly present in the Landau type equation governing the nonlinear evolution of neutrally stable disturbances, and that the effect of high temperatures will be more pronounced.

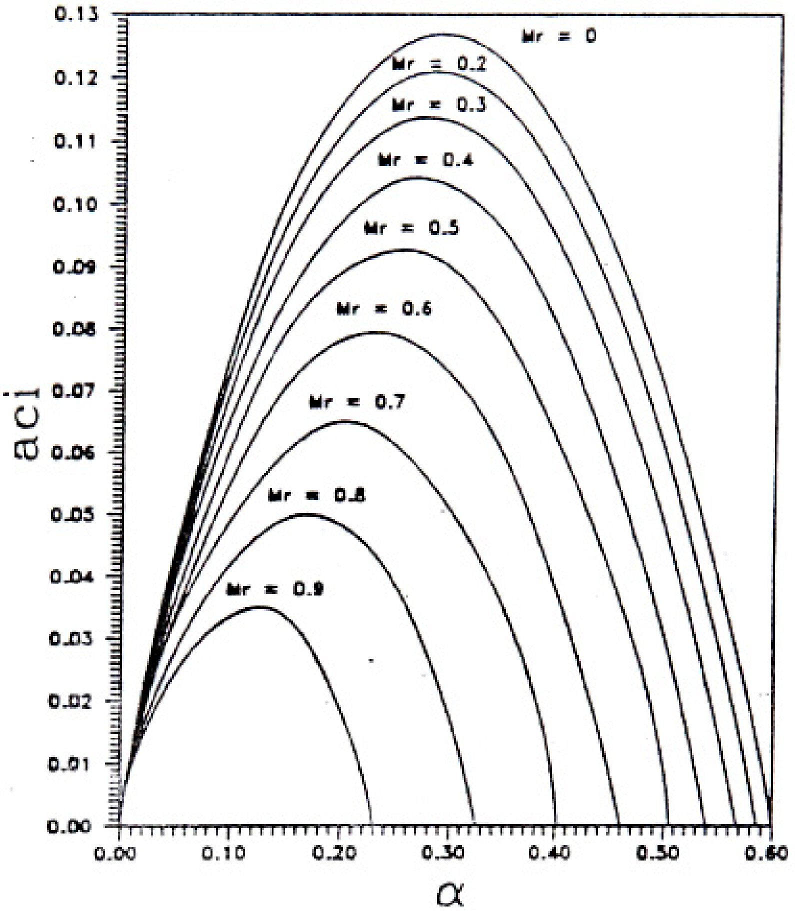

Fig.6. Growth rates curves for $m = 2$, $b = 1$ and various M_r

In addition, the influence of the parameter m for a symmetric temperature profile, that actually defines the thickness of the temperature layer (it decreases with m), was numerically investigated in details. Thickening of the layer destabilizes the flow if we have excess of the temperature in the critical layer, and stabilizes it if we have a temperature deficit. It is difficult to find a physical explanation for such a switch-over.

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LINEARE STABIZITÁTSANALYSE DER KOMPRESSIBLEN TEMPERATURHOHEN VERMISCHUNGSSCHICHTEN

In vorliegender Arbeit wurde die lineare hydrodynamische Stabilitätstheorie der unbegrenzten kompressiblen vermischenden Schichten bei hohen Temperaturen behandelt, bei denen sich das Gas als thermisch ideal verhält. Die Grundströmung ist nicht homentrop, so dass die Geschwindigkeits- und Temperaturprofile voneinander unabhängig sind, wobei auf den Einfluss der Symmetrie bzw. Unsymmetrie dieser Profile auf die Stabilitätseigenschaften näher eingegangen wird. Es wurden die allgemeinen Stabilitätskriterien abgeleitet und einige Probleme numerisch gelöst. Der Einfluss der Grundparameter, zum Beispiel, der Machzahl, der Vermischungsschichtdicke, des Uberschusses oder des Defizits der Temperatur in kritischer Schicht und anderes, auf die Lage der Neutralgrenze und auf den Zuwachsfaktor instabiler Moden wurde untersucht.

LINEARNA ANALIZA STABILNOSTI STIŠLJIVIH, VISOKO-TEMPERATURSKIH MEŠAJUĆIH SLOJEVA

U radu je obradjena linearne teorija hidrodinamičke stabilnosti neograničenih stišljivih mešajućih slojeva na visokim temperaturama, na kojima se gas ponaša kao termički idealan. Osnovno strujanje je nehomentropsko, tako da su profili brzine i temperature medjusobno nezavisni, pri čemu je posebna pažnja posvoćena uticaju simetričnosti, odnosno nesimetričnosti ovih profila na osobine stabilnosti. Formirani su opšti kriterijumi stabilnosti i numerički je rešeno niz primera. Analiziran je uticaj osnovnih parametara, kao što su Mahov broj, debljina mešajućeg sloja, višak ili manjak temperature u kritičnom sloju, i dr. na položaj neutralnih krivih i faktor rasta nestabilnih modova.

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