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ON THE HAMILTON'S EQUATIONS FOR RELATIVE MOTION

M. Lukačević, O. Jeremić

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1. Introduction

The Hamilton's equations of relative motion of a dynamical system are deduced in this paper on the same way as it was done in the case of absolute motion in [1]. It means that we start from the first form of the Hamilton's principle for relative motion (see, e.g.,[2]), and introduce the relative generalized momenta components instead of the genaralized velocities into the integrand of the Hamiltonian action. In such a way we obtain the constrained variational problem, with the relations between generalized velocities and relative generalized momenta components as the equations of constraints, which leads to the Hamilton's equations for relative motion of our system. We further establish the relations between some of the quantities defined for the relative motion and the corresponding absolute quantities, what gives the possibility to compare the method of deriving the Hamilton's equations proposed in this paper with the one exposed in [3], which is based on the theory of the canonical transformations.

2. Hamilton's equations

We consider a holonomic system of N particles M_i, with masses m_i, acted on by the potential forces only. The position of the particles M_i at time t with respect to the moving rectangular frame of reference $A\xi\eta\zeta$ (further: the frame F) is determined by their coordinates ξ_i, η_i, ζ_i, but we shall immediately introduce, supposing that our system of particles is scleronomous with respect to F, the independent Lagrangian coordinates q^α as the coordinates which we shall use further to determine its position. Throughout the paper the indices α, β, γ take the values $1, 2, ..., n$, with summation over this range of values in the case of repeated index. The index i, which has not tensorial meaning, runs from 1 to N. The motion of the frame F is prescribed by its angular velocity $\vec{\omega} = \vec{\omega}(t)$ and the velocity $\vec{v}_A = \vec{v}_A(t)$ of its origin A.

The relative velocities of M_i are

$$\overrightarrow{v}_i = \frac{d_r \overrightarrow{\rho_i}}{dt} = \frac{\partial \overrightarrow{\rho_i}}{\partial q^\alpha} \dot{q}^\alpha, \tag{1}$$

where $\frac{d_r}{dt}$ denotes the relative differentiation with respect to time and where

$$\overrightarrow{\rho}_i(q^\alpha, t) = \overrightarrow{AM_i} = \xi_i(q^\alpha)\overrightarrow{\lambda} + \eta_i(q^\alpha)\overrightarrow{\mu} + \zeta_i(q^\alpha)\overrightarrow{\nu}, \tag{2}$$

with $\overrightarrow{\lambda} = \overrightarrow{\lambda}(t)$, $\overrightarrow{\mu} = \overrightarrow{\mu}(t)$ and $\overrightarrow{\nu} = \overrightarrow{\nu}(t)$ as the unit vectors of the moving frame F. We notice that ξ_i, η_i and ζ_i do not depend on time explicitly, since the constraints to which the system of particles is subject are scleronomous with respect to F. So, t appears in $\overrightarrow{\rho}_i$ as a consequence of the rotation of the frame F only.

Having in mind (1), the apparent kinetic energy of the system, given by

$$\tau = \frac{1}{2} \sum_{i=1}^{N} m_i v_i^2,$$

can be written in the form

$$\tau = \frac{1}{2} A_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta, \tag{3}$$

where

$$A_{\alpha\beta}(q^1, q^2, ..., q^n) = \sum_{i=1}^{N} m_i \frac{\partial \overrightarrow{\rho}_i}{\partial q^\alpha} \cdot \frac{\partial \overrightarrow{\rho}_i}{\partial q^\beta}. \tag{4}$$

The potential energy function of our system we denote by

$$V = V(q^\alpha, t). \tag{5}$$

Let us now start from the Hamilton's principle for relative motion, expressed by ([2])

$$\delta \int_{t_0}^{t_1} \Lambda dt = 0, \tag{6}$$

where Λ is the apparent Lagrangian function of the system, and where t_0 and t_1 are fixed instants corresponding to the fixed terminal points in q-space. The function Λ has the form (see, e.g.,[4], p.171)

$$\Lambda = \tau - \pi + \overrightarrow{\omega} \cdot \overrightarrow{L_A}, \tag{7}$$

where

$$\overrightarrow{L_A} = \sum_{i=1}^{N} \overrightarrow{\rho_i} \times m_i \overrightarrow{v_i} \tag{8}$$

and

$$\begin{gathered} \pi = V + m\overrightarrow{a_A} \cdot \overrightarrow{\rho_c} - \frac{1}{2} I_A \omega^2, \\ m\overrightarrow{\rho_c} = \sum_{i=1}^{N} m_i \overrightarrow{\rho_i}, \end{gathered} \tag{9}$$

$\overrightarrow{a_A}$ being the acceleration of the origin A of the moving frame and I_A the moment of inertia of our system at the instant considered about a line through A in the direction of $\overrightarrow{\omega}$. We note that (9) can be accepted as the apparent potential energy function of the system ([3], [4]).

Having in mind (2) and (8), we find

$$\overrightarrow{\omega} \cdot \overrightarrow{L_A} = A_\alpha \dot{q}^\alpha, \tag{10}$$

where

$$A_\alpha(q^\beta, t) = \overrightarrow{\omega} \cdot \sum_{i=1}^{N} m_i \overrightarrow{\rho_i} \times \frac{\partial \overrightarrow{\rho_i}}{\partial q^\alpha},$$

while from (9) we conclude that

$$\pi = \pi(q^\alpha, t). \tag{11}$$

Thus, using (3), (10) and (11), we can write (7) in the form

$$\Lambda(q^\alpha, \dot{q}^\alpha, t) = \frac{1}{2} A_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta + A_\alpha \dot{q}^\alpha - \pi(q^\alpha, t). \tag{12}$$

Let us define, further, the relative generalized momenta components by

$$P_\alpha = \frac{\partial \Lambda}{\partial \dot{q}^\alpha} = A_{\alpha\beta} \dot{q}^\beta + A_\alpha, \tag{13}$$

from where, denoting by $A^{\alpha\beta}$ the conjugate tensor of $A_{\alpha\beta}$, we obtain

$$\dot{q}^\beta = A^{\alpha\beta}(P_\alpha - A_\alpha). \tag{14}$$

In virtue of (14), from (12) we have

$$\tilde{\Lambda} = \tilde{\tau} + A_\alpha A^{\alpha\beta} (P_\beta - A_\beta) - \pi, \tag{15}$$

where

$$\tilde{\tau} = \frac{1}{2} A^{\alpha\beta} P_\alpha P_\beta - A^{\alpha\beta} P_\alpha A_\beta + \frac{1}{2} A^{\alpha\beta} A_\alpha A_\beta \tag{16}$$

and where the symbol " $\backsim$ " above the letter denotes that the generalized velocities are eliminated by means of the equations (14).

Introducing now the generalized momenta components (13) into the integrand in (6) and keeping in mind relations (14), we obtain the constrained variational problem given by

$$\delta \int_{t_0}^{t_1} \left[\tilde{\Lambda} \left(q^\alpha, P_\alpha, t \right) + \lambda_\alpha \left(\dot{q}^\alpha - A^{\alpha\beta} P_\beta - A^{\alpha\beta} A_\beta \right) \right] dt = 0, \tag{17}$$

where the multipliers λ_α are functions of t to be determined. We note that the coordinates q^α have fixed values at the end-points in (17), while the values of P_α are free.

Writing the Euler's equations for q^α and P_α, which are

$$\frac{d}{dt} \frac{\partial f}{\partial \dot{q}^\alpha} - \frac{\partial f}{\partial q^\alpha} = 0, \qquad \frac{d}{dt} \frac{\partial f}{\partial \dot{P}_\alpha} - \frac{\partial f}{\partial P_\alpha} = 0,$$

where

$$f = \tilde{\Lambda} \left(q^\alpha, P_\alpha, t \right) + \lambda_\alpha \left(\dot{q}^\alpha - A^{\alpha\beta} P_\beta - A^{\alpha\beta} A_\beta \right), \tag{18}$$

we obtain

$$\dot{\lambda}_\alpha - \frac{\partial}{\partial q^\alpha} \left(\tilde{\Lambda} - \lambda_\gamma A^{\gamma\beta} P_\beta + \lambda_\gamma A^{\gamma\beta} A_\beta \right) = 0 \tag{19}$$

and

$$\lambda_\beta A^{\beta\alpha} - \frac{\partial \tilde{\Lambda}}{\partial P_\alpha} = 0. \tag{20}$$

The end-condition

$$\left(\frac{\partial f}{\partial \dot{q}^\alpha} \delta q^\alpha + \frac{\partial f}{\partial \dot{P}_\alpha} \delta P_\alpha \right) \Bigg|_{t=t_0}^{t=t_1} = 0$$

is fulfilled identically, since $\delta q^\alpha(t_0) = \delta q^\alpha(t_1) = 0$, and f does not contain $\dot{P}^\alpha$.

Finding, further,

$$\frac{\partial \tilde{\Lambda}}{\partial P_\beta} = \frac{\partial \tilde{\tau}}{\partial P_\beta} + A^{\alpha\beta} A_\alpha = A^{\alpha\beta} P_\alpha, \tag{21}$$

we get from (20)

$$\lambda_\alpha = P_\alpha, \tag{22}$$

after what (19) reads

$$\dot{P}^\alpha + \frac{\partial}{\partial q^\alpha}\left(A^{\gamma\beta}P_\gamma P_\beta - A^{\gamma\beta}P_\gamma A_\beta - \tilde{\Lambda}\right) = 0. \tag{23}$$

The function

$$K(q^\alpha, P_\alpha, t) = A^{\alpha\beta}P_\alpha P_\beta - A^{\alpha\beta}P_\alpha A_\beta - \tilde{\Lambda}, \tag{24}$$

appearing in (23), which can be written in the form

$$K = \frac{\partial\Lambda}{\partial\dot{q}^\alpha}\dot{q}^\alpha - \Lambda = \tau + \pi, \tag{25}$$

we recognize as the apparent Hamiltonian function of our system. Using this function, we can write (23) in the form

$$\dot{P}^\alpha = -\frac{\partial K}{\partial q^\alpha}. \tag{26}$$

Further, as

$$\frac{\partial K}{\partial P_\beta} = 2A^{\beta\gamma}P_\gamma - A^{\beta\gamma}A_\gamma - \frac{\partial\tilde{\Lambda}}{\partial P_\beta},$$

we get, keeping in mind (2l),

$$\frac{\partial K}{\partial P_\beta} = A^{\alpha\beta}\left(P_\alpha - A_\alpha\right),$$

which, comparing with (14), leads to

$$\dot{q}^\alpha = \frac{\partial K}{\partial P_\beta}. \tag{27}$$

Equations (26) and (27) exhibits the Hamilton's equations for the relative motion of our dynamical system.

3. The use of the canonical transformations

We next establish the relation between the two Hamiltonian functions of our system - the function H corresponding to its absolute motion and the apparent Hamiltonian function K, given by (25). We can derive this starting from the kinetic energy T of the system expressed in the form ([3],[4])

$$T = \tau + \frac{1}{2}mv_A^2 + \frac{1}{2}I_A\omega^2 + \overrightarrow{\omega}\cdot\overrightarrow{L_A} + \frac{d}{dt}\left(m\overrightarrow{u_A}\cdot\overrightarrow{\rho_c}\right) - m\overrightarrow{a_A}\cdot\overrightarrow{\rho_c},$$

what allows, using (7) and (9), to write the Lagrangian function of the system, given by

$$L = T - V,$$

in the form

$$L = \Lambda + \frac{d}{dt}(m\overrightarrow{v_A}\cdot\overrightarrow{\rho_c}) + \frac{1}{2}mv_A^2. \tag{28}$$

Now, since

$$H = \frac{\partial L}{\partial \dot{q}^\alpha}\dot{q}^\alpha - L,$$

from (28) we easily obtain

$$H = K - \frac{\partial}{\partial t}(m\overrightarrow{v_A}\cdot\overrightarrow{\rho_c}) - \frac{1}{2}mv_A^2, \tag{29}$$

where (25) was taken into account. We notify that the term $\frac{1}{2}mv_A^2$ may be ommited in (25), as well as in (28), since it is a function of t only. Thus, we may write

$$H = K - \frac{\partial}{\partial t}(m\overrightarrow{v_A}\cdot\overrightarrow{\rho_c}). \tag{29'}$$

Using (28) we can also find the relations between the generalized momenta components corresponding to absolute motion, defined by

$$p_\alpha = \frac{\partial T}{\partial \dot{q}^\alpha} = \frac{\partial L}{\partial \dot{q}^\alpha}, \tag{30}$$

and the relative generalized momenta components (13). These relations are

$$P_\alpha = p_\alpha - m\overrightarrow{v_A}\cdot\frac{\partial\overrightarrow{\rho_c}}{\partial q^\alpha}. \tag{31}$$

Denoting further by Q^α the generalized coordinates q^α when they describe the relative motion, and keeping the denotations q^α for the coordinates used in the case of absolute motion, we may write, since there is no difference between them,

$$Q^\alpha = q^\alpha. \tag{32}$$

Let us return now to the equations (26) and (27). It is interesting to notice that they have been obtained in [3] in a way which differs from the one exposed here. Namely, the quantities P_α were defined by the relations (31) as quantities without any physical meaning, then it was proved that (31) and (32) are the canonical transformations from the variables q_α, p_α to the variables Q_α, P_α, and the generating function

$$W(q^\alpha, P_\alpha, t) = P_\alpha q^\alpha + m\overrightarrow{v_A} \cdot \overrightarrow{\rho_c} \tag{33}$$

was introduced, what made it possible to construct the Hamiltonian function $K(q^\alpha, P_\alpha, t)$ in the form (24), and then to write the equations (26), (27). Introducing the quantities P_α by the relations (31), instead by (13), however, does not give the possibility to recognize immediately P_α as the relative generalized momenta components. On the contrary, from (13), which is analogous to (30), valid for the absolute motion, it is natural to accept P_α as the momenta components for the relative motion.

At the end, we shall demonstrate that the generating function (33) can be obtained starting from (29'), (31) and (32) and having in mind the formulae wellknown from the theory of canonical transformations ([3]):

$$p_\alpha = \frac{\partial W}{\partial q^\alpha} \tag{34}$$

$$Q^\alpha = \frac{\partial W}{\partial P_\alpha} \tag{35}$$

and

$$K = \left(H + \frac{\partial W}{\partial t}\right)_{p_\alpha, q^\alpha \Rightarrow P_\alpha, Q^\alpha}. \tag{36}$$

Namely, comparing (36) and (29'), we obtain

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial t}\left(m\overrightarrow{v_A} \cdot \overrightarrow{\rho_c}\right)$$

which leads to

$$W(q^\alpha, P_\alpha, t) = m\overrightarrow{v_A} \cdot \overrightarrow{\rho_c} + W_1\left(q^\alpha, P_\alpha\right). \tag{37}$$

Further, finding from (37)

$$\frac{\partial W}{\partial q^\alpha} = m\overrightarrow{v_A} \cdot \frac{\partial \overrightarrow{\rho_c}}{\partial q^\alpha} + \frac{\partial W_1}{\partial q^\alpha}$$

and using (34) and (31), we have

$$\frac{\partial W_1}{\partial q^\alpha} = P_\alpha,$$

from where

$$W_1\left(q^\alpha, P_\alpha\right) = P_\alpha q^\alpha + W_2\left(P_\alpha\right).$$

Then (37) reads

$$W\left(q^{\alpha}, P_{\alpha}, t\right) = m\overrightarrow{v_A} \cdot \overrightarrow{\rho_c} + P_{\alpha} q^{\alpha} + W_2\left(P_{\alpha}\right). \tag{38}$$

Finally, as from (38)

$$\frac{\partial W}{\partial P_{\alpha}} = q^{\alpha} + \frac{\partial W_2}{\partial P_{\alpha}},$$

(35) and (32) lead to

$$\frac{\partial W_2}{\partial P_{\alpha}} = 0,$$

i.e. to

$$W_2 = C, \quad C = \text{const.}$$

and (38) takes the form

$$W\left(q^{\alpha}, P_{\alpha}, t\right) = P_{\alpha} q^{\alpha} + m\overrightarrow{v_A} \cdot \overrightarrow{\rho_c} + C, \tag{39}$$

which coincides with (33), since the constant term may be omitted.

4. Conclusion

Although somewhat extensive, the method of deducing the Hamilton's equations for relative motion presented here brings, by our opinion, an advantage in comparison with the other well-known methods. Namely, an expression which can be recognized as the apparent Hamiltonian function of the system arises in the very process of deducing (see equation (23)), and there is no need to start from such a function as a given one.

Further, the relative generalized momenta components, P_{α}, we introduce by the relations (13). It must be born in mind that these quantities, which include the effects of the inexorable motion of the frame F, differ from the quantities given by $P_{\alpha}^{*} = \frac{\partial \tau}{\partial \dot{q}^{\alpha}}$, which are considered in [3] as the relative generalized momenta components. Having in mind that P_{α} are introduced in [3] by (31), and not by (13), it is easy to understand why an adventage is ascribed to P_{α}^{*} as the generalized momenta components, in comparison with P_{α}. But the considerations exposed in this paper offer, we suppose, an argument in favor of the opposite standpoint, too.

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Mirjana Lukačević
Mašinski fakultet
27. marta 80
11000 Beograd

Olivera Jeremić
Mašinski fakultet
27. marta 80
11000 Beograd