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THE STRAIN STATE OF THE ELLIPTICAL - ANNULAR PLATE BY THE COMPLEX VARIABLE FUNCTION AND CONFORMAL MAPPING METHOD

K. Hedrih(Stevanović), D.B. Jovanović

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Introduction

An inclusive survey of the research on the problem of the plain stress state and plain strain state can be find in the paper [25] by P. P. Tedorescu. The application of the complex variable function first time appears in solving the problems of the plain elasticity theory in the papers of G. V. Kolosov (1909). N. I. Mushelishvili has given fundamental contribution in the field of the systematic development and application of the complex variable function to the theory of elasticity problems. These contributions are summarized in the well-known monograph [4] of the mathematical theory of elasticity. This monograph republished in 1966. gives a short survey of the papers and authors who have given further contributions to the application of the complex variable function method to the theory of elasticity problems. Among the listed authors we would like to emphasize the papers of D. I. Sherman in the period from 1949. to 1959. He dealt with the boundary condition problems for double-connected areas and gave an important contribution to the complex variable function and to the study of stress state of the multi-holed plates.

The paper of Hedrih, Jecić and Jovanović [15] and [17] give an analysis of the principal stresses state at the points of the elliptical-annular plate contour stressed by one and two pairs of the concentrated forces. In this analysis the photoelastic experimental method is used by which the isiochromatic and isoclinic families are obtained for three cases of stresses included either by a pair or by pairs of concentrated forces. By these isochromatic and isoclinic families the principal stress distribution is determinated at the points of the external and internal contours of the elliptical-annular plate and respective graphic displays are made.

In our expose [16] at the congress of the Yugioslav Society of Mechanics held in 1990. we gave our contribution to the application of the complex variable function and of the conformal mapping to the study of the stress state of the plain stressed plates whose contours can be expressed by means of the confocal ellipses and arches of the hyperbolas from the respective families of the orthogonal curves. The strain tensor components are derived in the system of the hyperbolic-elliptical coordinates with analytical functions of the complex variable z in the conformally mapped plane ζ. By means of these expressions in the paper we derived the expressins for the strain tensor components and displacement vector in the system of hypelrbolic-elliptical coordinates at the points of the elliptical-annular plate segmentally stressed by the stress distributed along the external and internal contour.

1. Definition of the problems of the plain stressed elliptical-annular plate

The subject of our analysis here is the strain state of the elliptical-annular plate segmentally stressed along the external and internal contour by the stress distributed in the middle plane in the form of pressure perpendicular to the contours of the plate as it is shown in the Fig.1. Let $p_s(\varphi)$ and $p_u(\varphi)$ denote the pressures on the external and internal contour and α and β parameters by which in the hyperbolic-elliptical coordinates we give the contour segments along which these pressures are normally distributed. Let's use the hyperbolic-elliptical coordinate system with the coordinates p and φ, where their relations with the Descartes coordinates are given in the form:

$$
\begin{aligned}
x &= R(\rho + \frac{m}{\rho})\cos\varphi \\
y &= R(\rho - \frac{m}{\rho})\sin\varphi \\
&(\frac{x}{R(\rho + \frac{m}{\rho})})^2 + (\frac{y}{R(\rho - \frac{m}{\rho})})^2 = 1 \\
&(\frac{x}{R\cos\varphi})^2 + (\frac{y}{R\sin\varphi})^2 = 1
\end{aligned}
\tag{1}
$$

therefore for $\rho = const.$ we obtain ellipses and for $\varphi = const.$ we obtain a hyperbole from the orthogonal curves family. Along the hyperbole the parameter ρ changes, whereas along the ellipse the parameter φ changes. For $\rho = \rho_2$ the internal ellipse contour is defined. By using thus adopted coordinates of hyperbolic-elliptical coordinate system and according to the Fig.1. the boundary conditions can be written in the form:

a) for the points on the internal contour

$$
\sigma_\rho(\rho_1,\varphi) = \begin{cases} -p_u, & \text{for } (\frac{\pi}{2}-\alpha) \le \varphi \le (\frac{\pi}{2}+\alpha) \wedge (\frac{3\pi}{2}-\alpha) \le \varphi \le (\frac{3\pi}{2}+\alpha) \\ 0, & \text{for } \varphi \in (0,\frac{\pi}{2}-\alpha) \cup (\frac{\pi}{2}+\alpha, \frac{3\pi}{2}-\alpha) \cup (\frac{3\pi}{2}+\alpha, 2\pi) \end{cases}
$$

$$
\tau_{\rho\varphi}(\rho_1,\varphi) = 0 \text{ for } \varphi \in (0,2\pi) \tag{2}
$$

b) for the points on the external contour

$$\sigma_\rho(\rho_2,\varphi) = \begin{cases} -p_s, & \text{for } (2\pi-\beta) \le \varphi \le (+\beta) \wedge (\pi-\beta) \le \varphi \le (\pi+\beta) \\ 0, & \text{for } \varphi \in (\beta, \frac{\pi}{2}+\beta) \cup (\pi+\beta, 2\pi-\beta) \end{cases}$$

$$\tau_{\rho\varphi}(\rho_2,\varphi) = 0 \text{ for } \varphi \in (0, 2\pi) \tag{3}$$

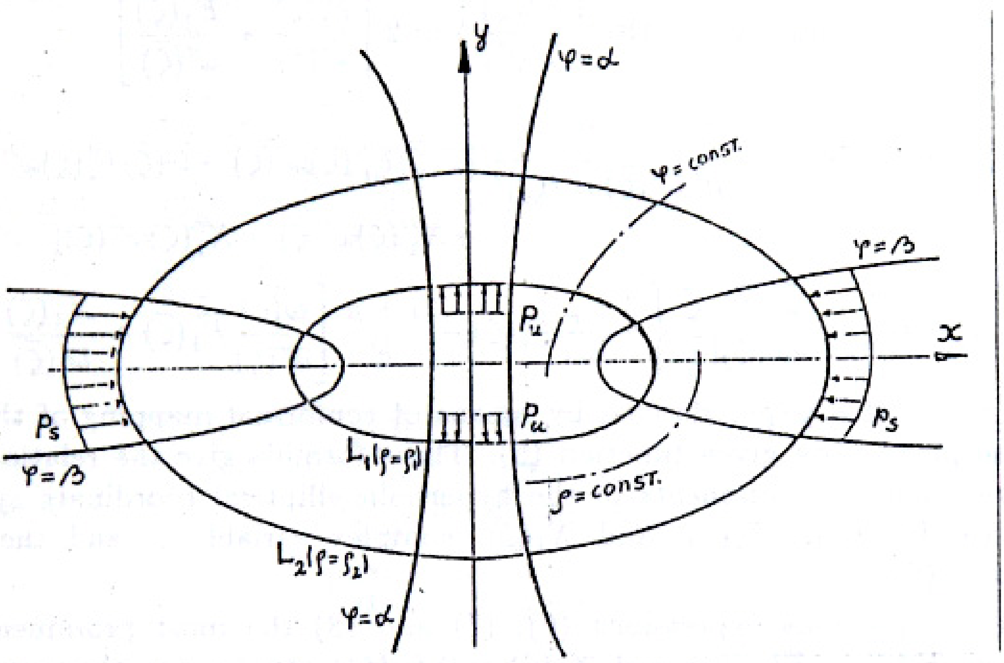

Fig.1.

2. Essentials of the complex variable function method with interpretation within hyperbolic-elliptical coordinate system

For determining the stress tensor components in the hyperbolic-elliptical system, that is the normal stresses σ_h, and σ_e at the points of plate for the sections with the normals in the direction of the tangent lines to the orthogonal family of hyperbolas, that is, the ellipses and shear stresses τ_{he}, that is, τ_{eh} respective to these planes, we use the complex variable function method requiring that the stress biharmonic function $\varphi(x,y)$, should be expressed by means of the complex variable analytical function [7], [4], [2], [12] in the form:

$$\varphi = \mathrm{Re}\{\overline{z}F(z) + X(z)\} \tag{4}$$

where z is a complex argument. Since the elliptical-annular plate contours in the plane z can be mapped in two concentric circles in ζ plane by means of mapping function:

$$z = \omega(\zeta) = R(\zeta + \frac{m}{\zeta}), \qquad R = \frac{a+b}{2} \text{ and } m = \frac{a-b}{a+b} \tag{5}$$

in which then by means of the same mapping function the hyperbolic-elliptical coordinate system, coordinate lines - confocal ellipses and hyperbolas orthogonal to them, map in the concentric circles family and beam of straight lines.

We use following transformation formulas (15), (16) and (17) given in the paper of Hedrih and Jovanović [16]:

$$\sigma_\rho + \sigma_\varphi = 4\mathrm{Re}\left\{\frac{F_1'(\zeta)}{\omega'(\zeta)}\right\} = 2\left[\frac{F_1'(\zeta)}{\omega'(z)} + \frac{\overline{F}_1'(\overline{\zeta})}{\overline{\omega}'(\overline{\zeta})}\right] \tag{6}$$

$$\sigma_\varphi - \sigma_\rho + 2i\tau_{\rho\varphi} = \frac{\zeta^2}{\rho^2}\frac{2}{\overline{\omega}'(\overline{\zeta})[\omega'(\zeta)]^2}\left[\overline{\omega}(\overline{\zeta})F_1''(\zeta)\omega'(\zeta) - \overline{\omega}(\overline{\zeta})F_1'(\zeta)\omega''(\zeta) + \right.$$
$$\left. + X_1''(\zeta)\omega'(\zeta) - X_1'(\zeta)\omega''(\zeta)\right] \tag{7}$$

$$u_\rho + iu_\varphi = \frac{\overline{\omega'(\zeta)}}{|\omega'(\zeta)|}\frac{\overline{\zeta}}{\rho}\left\{\frac{3-\mu}{E}F_1(\zeta) - \frac{1+\mu}{E}\left[\frac{\omega(\zeta)}{\overline{\omega}'(\overline{\zeta})}\overline{F}_1'(\overline{\zeta}) + \frac{\overline{X}_1'(\overline{\zeta})}{\overline{\omega}'(\overline{\zeta})}\right]\right\} \tag{8}$$

which are in this paper derived by means of conformal mapping of the z plane into the plane ζ by given function (5). These formula give the relation between the stress tensor components in the hyperbolic-elliptical coordinate system and analytical functions $F_1(\zeta)$, and $X_1(\zeta)$, complex variable ζ, and the mapping function $\omega(\zeta)$.

In the previous expressions (6), (7) and (8) the most prominent are the functions $F_1(\zeta) = F(\omega(\zeta))$ and $X_1(\zeta) = X(\omega(\zeta))$ can be directly represented in the mapping plane ζ in the form of the Laurent series along the complex variable $\zeta = \rho e^{i\varphi}$ with the unknown coefficients A_n and B_n. The unknown coefficients of these series are determined from the boundary conditinns (2) and (3), along the elliptical-annular plate whose stress state is being studied. By the reason of an infinite number of coefficients the reduction of the number of the order members to the finite number in concrete calculations is defined with a desired approximate accuracy of the boundary conditions.

3. Determination of the boundary conditions development coefficients

Using the boundary conditions (2) and (3) for the sake of the comfortable application of the expressions (6), (7) and (8) let's write the boundary conditions by introducing the series:

$$[\sigma_\rho - i\tau_{\rho\varphi}]_{\rho=\rho_1} = \sum_{-\infty}^{\infty} C_n^{(1)} e^{in\varphi} = -p_u(\varphi) \tag{9}$$

$$[\sigma_\rho - i\tau_{\rho\varphi}]_{\rho=\rho_2} = \sum_{-\infty}^{\infty} C_n^{(2)} e^{in\varphi} = -p_s(\varphi) \tag{10}$$

with the development coefficients $C_n^{(1)}$ and $C_n^{(2)}$ in the form:

$$C_n^{(1)} = -\frac{1}{2\pi}\int_0^{2\pi} p_u(\varphi)d\varphi \tag{11}$$

$$C_n^{(2)} = -\frac{1}{2\pi}\int_0^{2\pi} p_s(\varphi)d\varphi \tag{12}$$

After calculating the integral in the expressions (11) and (12) according to the boundary conditions (2) for these development coefficients we obtain:

$$\begin{aligned} C_0^{(1)} &= -\frac{2p_u\alpha}{\pi} \\ C_{2k}^{(1)} &= -\frac{(-1)^k p_u}{\pi k}\sin 2k\alpha = -\frac{p_u}{\pi k}e^{ik\pi}\sin 2k\alpha \\ C_{2k+1}^{(1)} &= 0 \end{aligned} \tag{11'}$$

$$\begin{aligned} C_0^{(2)} &= -\frac{2p_s\beta}{\pi} \\ C_{2k}^{(2)} &= -\frac{p_s}{\pi k}\sin 2k\beta \\ C_{2k+1}^{(2)} &= 0 \end{aligned} \tag{12'}$$

By analyzing we conclude that all coefficients $C_{2n+1}^{(1)}$ and $C_{2n+1}^{(2)}$ with odd indexes are equal to zero, except $C_{2n}^{(1)}$ and $C_{2n}^{(2)}$ with even indexes which are different then zero.

4. Determination of the coefficients A_n and B_n of the development of the analytical functions $F_1(\zeta)$ and $X_1(\zeta)$ in the Laurent series

Let's present the functions $F_1(\zeta)$ and $X_1(\zeta)$ in the form of their derivations along the complex variable ζ as the Laurent series:

$$F_1'(\zeta) = \sum_{-\infty}^{\infty} A_n\zeta^n$$

$$\overline{F}_1'(\overline{\zeta}) = \sum_{-\infty}^{\infty} \overline{A}_n\overline{\zeta}^n$$

$$F_1''(\zeta) = \sum_{-\infty}^{\infty} nA_n\zeta^{n-1}$$

$$X_1''(\zeta) = \sum_{-\infty}^{\infty} B_n \zeta^n$$

$$F_1(\zeta) = A_0\zeta + A_{-1}\ln\zeta + \sum_{n=-\infty, n\neq -1,0}^{\infty} \frac{A_n \zeta^{n+1}}{n+1} + c_1$$

$$X_1'(\zeta) = B_0\zeta + B_{-1}\ln\zeta + \sum_{n=-\infty, n\neq -1,0}^{\infty} \frac{B_n \zeta^{n+1}}{n+1} + c_2 \tag{13}$$

with unknown coefficients A_n and B_n which would be determined from the boundary conditions by means of the relation between the stress at the points on the stressed elliptical-annular plate contour and assumed analytical functions of the complex variable. Therefore the expressions (6), (7) or (8) are written for the points on the external and internal contours by means of the series (13) and made equal with the expressions on the right side of the relations (9) and (10) in which the development coefficients $C_n^{(1)}$ and $C_n^{(2)}$ are known by using the expressions (12). By transforming the expressions - sums according to the indexes n and by making equal the coeficients on the left and on the right sides with the equal degrees of the complex unit $e^{in\varphi}$ we obtain the desired relations between the coefficients A_n, $\overline{A_n}$, B_n, $\overline{B_n}$ with the coefficients $C_n^{(1)}$ and $C_n^{(2)}$ as well as ρ_1, and ρ_2 defining contours. In order to simplify these relations according to the concrete defined stress problem of the elliptical-annular plate we carry the following a· 'ysis.

Let's now consider the fact that some of coefficients are equal to zero on the basis of the defined boundary conditions, the characteristics of the symmetry of the elliptical-annular plate and form the symmetry of the given stress, as well as from the limited value of the plate points displacements.

Since the displacements area in the hyperbolic-elliptical coordinates is given in the form:

$$u_\rho + iu_\varphi = \frac{1}{E}\left\{(3-\mu)\sum_{-\infty}^{\infty}\frac{A_{2k}\zeta^{2k+1}}{2k+1} - \right.$$
$$\left. -(1+\mu)\left[\frac{\zeta + \frac{m}{\zeta}}{1-\frac{m}{\bar{\zeta}^2}}\sum_{-\infty}^{\infty}\overline{A}_{2k}\bar{\zeta}^{-2k} + \frac{\sum_{-\infty}^{\infty}\frac{B_{2k}}{2k+1}\zeta^{2k+1}}{R(1-\frac{m}{\bar{\zeta}^2})}\right]\right\}\frac{\bar{\zeta}}{\rho}\frac{(1-\frac{m}{\bar{\zeta}^2})}{|1-\frac{m}{\zeta^2}|} \tag{14}$$

and in the order to define unanimously the displacement vector from the previous expression (14) it follows that the coefficient next to the denoted member would be equal to zero, hence we conclude that $A_{-1} = 0$ and $B_{-1} = 0$.

From the condition that the elliptical-annular plate as well as the given external stress are with two symmetry axes, we conclude that $A_{-n} = A_n$ and $B_{-n} = B_n$.

If we use the previous conclusions, and since the coefficients $C_n^{(1)}$ and $C_n^{(2)}$ are real, we can assume that the coeficients $A_n = \overline{A_n}$ are real, then from the boundary conditions we obtain the following relation:

$$\left.\frac{\sum_{-\infty}^{\infty} A_n\zeta^n}{R(1-\frac{m}{\zeta^2})}\right|_{\zeta=\zeta_s} + \left.\frac{\sum_{-\infty}^{\infty} \overline{A}_n\overline{\zeta}^n}{R(1-\frac{m}{\zeta^2})}\right|_{\zeta=\zeta_s} -$$

$$\left\{ R(\overline{\zeta}+\frac{m}{\overline{\zeta}^2})\frac{\left[\sum_{-\infty}^{\infty} nA_n\zeta^{n-1}\right] R\left(1-\frac{m}{\zeta^2}\right) - \left[\sum_{-\infty}^{\infty} A_n\zeta^n\right]\frac{2mR}{\zeta^3}}{R^3\left(1-\frac{m}{\zeta^2}\right)^3} + \right.$$

$$\left. \frac{\left[\sum_{-\infty}^{\infty} B_n\zeta^n\right] R\left(1-\frac{m}{\zeta^2}\right) - \left[B_{-1}\ln\zeta + \sum_{-\infty}^{\infty} B_n\frac{\zeta^{n+1}}{n+1}\right]\frac{2mR}{\zeta^3}}{R^3\left(1-\frac{m}{\zeta^2}\right)^3}\right\}$$

$$\left.\frac{\zeta^2 R(1-\frac{m}{\zeta^2})}{\rho^2 R(1-\frac{m}{\zeta^2})}\right|_{\zeta=\zeta_s} = \sum_{-\infty}^{\infty} C_n^{(s)} e^{in\varphi} \tag{15}$$

If we use the relation (15) then the relations between unknown and known coefficients can only be written in the function of six unknown and eight known coefficients A_n, A_{n+2}, A_{n-2}, A_{n+4}, B_{n-2}, B_n and $C_n^{(1)}$, $C_n^{(2)}$, $C_{n+2}^{(1)}$, $C_{n+2}^{(2)}$, $C_{n-2}^{(1)}$, $C_{n-2}^{(2)}$, $C_{n+4}^{(1)}$, $C_{n+4}^{(2)}$:

$$\left\{A_n[(1-n)\rho^{n+2} + (3+n)m^2\rho^{n-2} + \rho^{-(n-2)}] + \right.$$

$$mA_{n+2}\left[(n+3)\rho^{n+2} - 2\rho^{-(n+2)}\right] -$$

$$\left. m(n-1)\rho^{n-2}A_{n-2} + m^2\rho^{-(n+6)}A_{n+4} - \frac{1}{R}\rho^n B_{n-2} + \frac{m}{R}\frac{n+3}{n+1}\rho^n B_n\right\}\Big|_{\rho=\rho_s} =$$

$$R\left[\rho^2\left(1+2\frac{m^2}{\rho^4}\right)C_n^{(s)} + m\left(2+\frac{m^2}{\rho^4}\right)C_{n+2}^{(s)} + \frac{m^2}{\rho^2}C_{n+4}^{(s)} - mC_{n-2}^{(s)}\right]\Big|_{\rho=\rho_s}$$

$$s = 1,2 \tag{16}$$

If we introduce the following notations in the previous equations:

$$L_n^{(n)}(\rho) = R\left[\rho^{-(n-2)}\left(1+2\frac{m^2}{\rho^4}\right)C_n^{(s)} + \right.$$

$$\left. m\rho^{-n}\left(2+\frac{m^2}{\rho^4}\right)C_{n+2}^{(s)} + \frac{m^2}{\rho^2}\rho^{-n}C_{n+4}^{(s)} - m\rho^{-n}C_{n+2}^{(s)}\right]\Big|_{\rho=\rho_s} \tag{17}$$

and if the expressions (16) and (17) are divided by ρ_1^n and ρ_2^n respectively then we write them in the following form:

$$\left\{A_n\left[(1-n)\rho^2+(3+n)\frac{m^2}{\rho^2}+\rho^{-2(n-1)}\right]+\right.$$
$$mA_{n+2}\left[(n+3)\rho^2-2\rho^{-2(n+1)}\right]-m(n-1)\rho^{-2}A_{n-2}+$$
$$\left.m^2\rho^{-2(n+3)}A_{n+4}-\frac{1}{R}B_{n-2}+\frac{m}{R}\frac{n+3}{n+1}B_n\right\}\Bigg|_{\rho=\rho_s}=L_n^{(s)}(\rho)\Bigg|_{\rho=\rho_s} \tag{18}$$

If we introduce the following notation:

$$E_n(\rho_1,\rho_2)=L_n^{(2)}(\rho_2)-L_n^{(1)}(\rho_1) \tag{19}$$

and the we make subtract of the equations (18) we obtain the following equations along A_{n-2}, A_n, A_{n+2}, A_{n+4} in the following form:

$$\begin{aligned}
&m(1-n)(\rho_1^{-2}-\rho_2^{-2})A_{n-2}+\\
&A_n\left[(1-n)(\rho_2^{-2}-\rho_1^{-2})+(n+3)m^2(\rho_2^{-2}-\rho_1^{-2})+\rho_2^{-2(n-1)}-\rho_1^{-2(n-1)}\right]+\\
&mA_{n+2}\left[(n+3)(\rho_2^{-2}-\rho_1^{-2})-2(\rho_2^{-2(n+1)}-\rho_1^{-2(n+1)})\right]+\\
&A_{n+4}m^2(\rho_2^{-2(n+3)}-\rho_1^{-2(n+3)})=E_n(\rho_1,\rho_2)\\
&n=-\infty,\dots,-2,-1,0,1,2,\dots,\infty
\end{aligned} \tag{20}$$

If we introduce $n=-4,-2,0$ and 2 and if take into consideration that $A_n=A_{-n}$, so that we can obtain non-homogeneous algebraic equations with only unknowns A_0, A_2, A_4, A_6 in the form:

$$D\{A\}=E \tag{21}$$

where

$$\{A\}=\begin{Bmatrix}A_0\\A_2\\A_4\\A_6\end{Bmatrix},\qquad E=\begin{Bmatrix}E_{-4}(\rho_1,\rho_2)\\E_{-2}(\rho_1,\rho_2)\\E_0(\rho_1,\rho_2)\\E_2(\rho_1,\rho_2)\end{Bmatrix}=\begin{Bmatrix}l_1\\l_2\\l_3\\l_4\end{Bmatrix} \tag{22}$$

$$D=(a_{ik})_{\overrightarrow{1k}}^{\,i}=\begin{bmatrix}a_{11}&a_{21}&a_{31}&a_{41}\\a_{12}&a_{22}&a_{32}&a_{42}\\a_{13}&a_{23}&a_{33}&a_{43}\\a_{14}&a_{24}&a_{34}&a_{44}\end{bmatrix} \tag{23}$$

$$\begin{aligned}
a_{11}&=m^2\left(\rho_2^2-\rho_1^2\right)\\
a_{21}&=-m\left[\left(\rho_2^2-\rho_1^2\right)+2\left(\rho_2^6-\rho_1^6\right)\right]\\
a_{31}&=5\left(\rho_2^2-\rho_1^2\right)-m^2\left(\frac{1}{\rho_2^2}-\frac{1}{\rho_1^2}\right)+\left(\rho_2^{10}-\rho_1^{10}\right)\\
a_{41}&=5m\left(\frac{1}{\rho_2^2}-\frac{1}{\rho_1^2}\right)
\end{aligned}$$

$$a_{12} = -m\left(\rho_2^2 - \rho_1^2\right)$$
$$a_{22} = 3\left(\rho_2^2 - \rho_1^2\right) + 2m^2\left(\rho_2^{-2} - \rho_1^{-2}\right) + \left(\rho_2^6 - \rho_1^6\right)$$
$$a_{32} = 3m\left(\rho_2^{-2} - \rho_1^{-2}\right)$$
$$a_{42} = 0$$

$$a_{13} = 2\left(\rho_2^2 - \rho_1^2\right) + 3m^2\left(\rho_2^{-2} - \rho_1^{-2}\right)$$
$$a_{23} = m\left[3\left(\rho_2^2 - \rho_1^2\right) - \left(\rho_2^{-2} - \rho_1^{-2}\right)\right]$$
$$a_{33} = m^2\left(\rho_2^{-6} - \rho_1^{-6}\right)$$
$$a_{43} = 0$$

$$a_{14} = -m\left(\rho_2^{-2} - \rho_1^{-2}\right)$$
$$a_{24} = -\left(\rho_2^2 - \rho_1^2\right) + \left(5m^2 + 1\right)\left(\rho_2^{-2} - \rho_1^{-2}\right)$$
$$a_{34} = m\left[5\left(\rho_2^2 - \rho_1^2\right) - 2\left(\rho_2^{-6} - \rho_1^{-6}\right)\right]$$
$$a_{44} = m^2\left(\rho_2^{-10} - \rho_1^{-10}\right)$$

(23')

$$E_n(\rho_1, \rho_2) = R\Bigg[C_n^{(2)}\rho_2^{2-n}\left(1 - \frac{2m^2}{\rho_2^4}\right) - C_n^{(1)}\rho_1^{2-n}\left(1 + \frac{2m^2}{\rho_1^4}\right) - m\rho_2^{-n}\left(2 + \frac{m^2}{\rho_2^4}\right)C_{n+2}^{(2)} + m\rho_1^{-n}\left(2 + \frac{m^2}{\rho_1^4}\right)C_{n+2}^{(1)} + \frac{m^2}{\rho_2^2}\rho_2^{-n}C_{n+4}^{(2)} - \frac{m^2}{\rho_1^2}\rho_1^{-n}C_{n+4}^{(1)} - m\rho_2^{-n}C_{n-2}^{(2)} + m\rho_1^{-n}C_{n-2}^{(1)}\Bigg] \tag{24}$$

$$l_1 = E_n(\rho_1, \rho_2)\big|_{n=-4} = R\Bigg\{m^2(\rho_2^2C_0^{(2)} - \rho_1^2C_0^{(1)}) - C_4^{(1)}\rho_1^6\left(1 + \frac{2m^2}{\rho_1^4}\right) + C_4^{(2)}\rho_2^6\left(1 - \frac{2m^2}{\rho_2^4}\right) - m\left[\rho_2^4\left(2 + \frac{m^2}{\rho_2^4}\right)C_2^{(2)} + \rho_1^4\left(2 + \frac{m^2}{\rho_1^4}\right)C_2^{(1)}\right] - m\left(\rho_2^4C_6^{(2)} + \rho_1^4C_6^{(1)}\right)\Bigg\}$$

$$l_2 = E_n(\rho_1, \rho_2)\big|_{n=-2} = R\Bigg\{C_2^{(2)}\rho_2^4\left(1 + \frac{3m^2}{\rho_2^4}\right) - C_2^{(1)}\rho_1^4\left(1 + \frac{3m^2}{\rho_1^4}\right) - m\left[\rho_2^2\left(2 + \frac{m^2}{\rho_2^4}\right)C_0^{(2)} + \rho_1^2\left(2 + \frac{m^2}{\rho_1^4}\right)C_0^{(1)}\right] -$$

$$m\left(\rho_2^2 C_4^{(2)} + \rho_1^2 C_4^{(1)}\right)\Big\}$$

$$l_3 = E_n(\rho_1,\rho_2)|_{n=0} = R\Big\{C_0^{(2)}\rho_2^2\left(1+\frac{2m^2}{\rho_2^4}\right) - C_0^{(1)}\rho_1^2\left(1+\frac{2m^2}{\rho_1^4}\right) - m\left[\left(3+\frac{m^2}{\rho_2^4}\right)C_2^{(2)} + \left(3+\frac{m^2}{\rho_1^4}\right)C_2^{(1)}\right] + m^2\left(\frac{1}{\rho_2^2}C_4^{(2)} - \frac{1}{\rho_1^2}C_4^{(1)}\right)\Big\}$$

$$l_4 = E_n(\rho_1,\rho_2)|_{n=2} = R\Big\{C_2^{(2)}\left(1+\frac{2m^2}{\rho_2^4}\right) - C_2^{(1)}\left(1+\frac{2m^2}{\rho_1^4}\right) - m\left[\left(2+\frac{m^2}{\rho_2^4}\right)\frac{1}{\rho_2^2}C_4^{(2)} + \left(2+\frac{m^2}{\rho_1^4}\right)\frac{1}{\rho_1^2}C_4^{(1)}\right] + m^2\left(\frac{1}{\rho_2^4}C_6^{(2)} - \frac{1}{\rho_1^4}C_6^{(1)}\right) - m\left(\frac{1}{\rho_2^2}C_0^{(2)} - \frac{1}{\rho_1^2}C_0^{(1)}\right)\Big\} \quad (24')$$

Then the solution of the system of equations (21) according to the unknown coefficients A_0, A_2, A_4, and A_6 are obtained in the form

$$A_0 = \frac{D_1}{D}, \quad A_2 = \frac{D_2}{D}, \quad A_4 = \frac{D_3}{D}, \quad A_6 = \frac{D_4}{D}, \tag{25}$$

in which for the sake of simplification the following notation is introduced

$$D = |a_{ik}| = \sum_{j=1}^{4} a_{ik}K_{ij}$$

$$D_i = = \sum_{j=1}^{4} l_j K_{ij} \tag{26}$$

of the system determinant (21) and of the determinant cofactors.

By means of these constants expressed in the function $C_0^{(1)}$, $C_0^{(2)}$, $C_2^{(1)}$, $C_2^{(2)}$, $C_4^{(1)}$, $C_4^{(2)}$, $C_6^{(1)}$, $C_6^{(2)}$ and **m** and **R**, ρ_1 and ρ_2, we express all the other coefficients A_n and B_n from the equations (20), (16) and (18) by writing them further for **n** $=$ **2k**, **k** $= 2,3,4,5,6,\ldots$ so that we always obtain another new equation with another new unknown A_{2k+4} or B_{2k} determined by the four or five previously determined coefficients. The coefficients determined by the following recursion formulas:

$$A_{n+4} = \frac{1}{m^2\left(\rho_2^{-2(n+3)} - \rho_1^{-2(n+3)}\right)}\Big\{\left[L_n^{(2)}(\rho_2) - L_n^{(1)}(\rho_1)\right] + m(n-1)\left(\rho_2^{-2} - \rho_1^{-2}\right)A_{n-2} -$$

$$A_n\left[-(n-1)\left(\rho_2^2-\rho_1^2\right)+(n+3)m^2\left(\rho_2^{-2}-\rho_1^{-2}\right)+\rho_2^{-2(n-1)}-\rho_1^{-2(n-1)}\right]-$$
$$mA_{n+2}\left[(n+3)\left(\rho_2^2-\rho_1^2\right)-2\left(\rho_2^{-2(n+1)}-\rho_1^{-2(n+1)}\right)\right]\Big\} \tag{27}$$

$$B_0=\frac{R}{15m^2-3}\left\{5mL_0^{(s)}(\rho)+3L_2^{(s)}(\rho)-mA_0\left[10\rho^2+3(5m^2-1)\rho^{-2}\right]-\right.$$
$$A_2\left[3\left(5m^2-1\right)\rho^2+(10m^2+3)\rho^{-2}\right]-mA_4\left[(5m^2-6)\rho^{-6}+15\rho^2\right]-$$
$$\left.3m^2\rho^{-10}A_6\right\} \tag{28}$$

$$B_2=\frac{R}{5m^2-1}\left\{L_0^{(s)}(\rho)+3mL_2^{(s)}(\rho)-2A_0\rho^2-\right.$$
$$\left.-mA_2(2+15m^2)\rho^{-2}-m^2(15\rho^2-5\rho^{-6})A_4-3m^3\rho^{-10}A_6\right\} \tag{29}$$

$$B_n=\frac{R(n+1)}{m(n+3)}\left\{L_n^{(s)}(\rho)-A_n\left[(1-n)\rho^{-2}+(3+n)\frac{m^2}{\rho^2}+\rho^{-2(n-1)}\right]-\right.$$
$$mA_{n+2}\left[(n+3)\rho^2-2\rho^{-2(n+1)}\right]+m(n-1)\rho^2A_{n-2}-$$
$$\left.m^2A_{n+4}\rho^{-2(n+3)}+\frac{1}{R}B_{n-2}\right\} \tag{30}$$

Thus we have formally solved the problem since we use coefficients to determine the Laurent series by which we form the stress biharmonic function, hence, the stress tensor components, the small strain tensor and displacement vector.

5. Relative deformation tensor components in the hyperbolic-elliptical coordinates

By applying the Hooke's law by means of the expressions (33), (34) and (35) given in the paper [20] we determine the relative deformation tensor components in the hyperbolic-elliptical coordinate system so that we obtain the following:

a) The expression for the dilatation $\epsilon_\rho(\rho,\varphi)$ of the line element drawn from the point $N(\rho,\varphi)$ of the plate in the unit direction $\overrightarrow{\mathbf{h}_0}$ which is tangent to the hyperbolic coordinate line:

$$\epsilon_\rho(\rho,\varphi)=\epsilon_h(\rho,\varphi)=$$
$$\frac{1}{ERg}\sum_{-\infty}^{\infty}A_{2k}\rho^{2k}\left\{(1-\mu)\left[\cos 2k\varphi-\frac{m}{\rho^2}\cos 2(k+1)\varphi\right]+\right.$$
$$(1+\mu)\left[-\frac{2k}{g}\left[\cos 2k\varphi-\frac{m}{\rho^2}\cos 2(k-1)\varphi\right]+\right.$$

$$\frac{m^3}{\rho^6}\cos 2(k+1)\varphi - \frac{m^2}{\rho^4}\cos 2(k+2)\varphi\Big] +$$
$$\frac{2m}{g^2\rho^2}\left[\left(1+3\frac{m^2}{\rho^4}\right)\cos 2(k-1)\varphi -\right.$$
$$\frac{m}{\rho^2}\cos 2(k-2)\varphi - 3\frac{m}{\rho^2}\left(1+\frac{m^2}{\rho^4}\right)\cos 2k\varphi +$$
$$\left.\left.\left.\frac{m^2}{\rho^4}\left(3+\frac{m^2}{\rho^4}\right)\cos 2(k+1)\varphi - \frac{m^3}{\rho^6}\cos 2(k+2)\varphi\right]\right]\right\} +$$
$$\frac{1+\mu}{Eg^2R^2}\sum_{-\infty}^{\infty} B_{2k}\rho^{2k}\left\{\left[\frac{m}{\rho^2}\cos 2k\varphi + \frac{m}{\rho^2}\cos 2(k+2)\varphi -\right.\right.$$
$$\left.\left(1+\frac{m^2}{\rho^4}\right)\cos 2(k+1)\varphi\right] +$$
$$\frac{2m}{\rho^2 g(2k+1)}\left[\left(1+2\frac{m^2}{\rho^4}\right)\cos 2k\varphi - \frac{m}{\rho^2}\cos 2(k-1)\varphi -\right.$$
$$\left.\left.\frac{m}{\rho^2}\left(2+\frac{m^2}{\rho^4}\right)\cos 2(k+1)\varphi + \frac{m^2}{\rho^4}\cos 2(k+2)\varphi\right]\right\} \tag{31}$$

b) The expressions for the dilatation $\epsilon_\varphi(\rho,\varphi)$ of the line element drawn from the point $N(\rho,\varphi)$ of the plate in the unit direction $\vec{e}_0$ which is tangent to the elliptical coordinate line:

$$\epsilon_\varphi(\rho,\varphi) = \epsilon_e(\rho,\varphi) =$$
$$\frac{1}{EgR}\sum_{-\infty}^{\infty} A_{2k}\rho^{2k}\left\{-2\mu\left[\cos 2k\varphi - \frac{m}{\rho^2}\cos 2(k+1)\varphi\right] +\right.$$
$$(1+\mu)\left[\frac{2k}{g}\left[\cos 2k\varphi - \frac{m}{\rho^2}\cos 2(k-1)\varphi\right] +\right.$$
$$\frac{m^3}{\rho^6}\cos 2(k+1)\varphi - \frac{m^2}{\rho^4}\cos 2(k+2)\varphi\Big] -$$
$$\frac{2m}{g^2\rho^2}\left[(1+3\frac{m^2}{\rho^4})\cos 2(k-1)\varphi -\right.$$
$$\frac{m}{\rho^2}\cos 2(k-2)\varphi - 3\frac{m}{\rho^2}\left(1+\frac{m^2}{\rho^4}\right)\cos 2k\varphi +$$
$$\left.\left.\left.\frac{m^2}{\rho^4}(3+\frac{m^2}{\rho^4})\cos 2(k+1)\varphi - \frac{m^3}{\rho^6}\cos 2(k+2)\varphi\right]\right]\right\} +$$
$$\frac{1+\mu}{Eg^2R^2}\sum_{-\infty}^{\infty} B_{2k}\rho^{2k}\left\{\left[\frac{m}{\rho^2}\cos 2k\varphi + \frac{m}{\rho^2}\cos 2(k+2)\varphi -\right.\right.$$
$$\left.\left(1+\frac{m^2}{\rho^4}\right)\cos 2(k+1)\varphi\right] +$$

$$\frac{2m}{\rho^2 g(2k+1)}\left[\left(1+2\frac{m^2}{\rho^4}\right)\cos 2k\varphi-\frac{m}{\rho^2}\cos 2(k-1)\varphi-\right.$$
$$\left.\left.\frac{m}{\rho^2}(2+\frac{m^2}{\rho^4})\cos 2(k+1)\varphi+\frac{m^2}{\rho^4}\cos 2(k+2)\varphi\right]\right\} \qquad (32)$$

c) The expressions for shear strain $\gamma_{\rho\varphi}(\rho,\varphi)$, that is $\gamma_{\varphi\rho}(\rho,\varphi)$, the right angle change between the line elements drawn in the point $N(\rho,\varphi)$ of the plate in the unit directions $\overrightarrow{h_0}$ and $\overrightarrow{e_0}$ to the elliptical and hyperbolic coordinate line:

$$\gamma_{\rho\varphi}(\rho,\varphi)=\gamma_{he}(\rho,\varphi)=$$
$$\frac{2(1+\mu)}{EgR}\sum_{-\infty}^{\infty}A_{2k}\rho^{2k}\left\{\frac{2k}{g}\left[\sin 2k\varphi-\frac{m}{\rho^2}\sin 2(k-1)\varphi+\right.\right.$$
$$\left.\frac{m^3}{\rho^6}\sin 2(k+1)\varphi-\frac{m^2}{\rho^4}\sin 2(k+2)\varphi\right]+$$
$$\frac{2m}{g^2\rho^2}\left[\frac{m}{\rho^2}\sin 2(k-2)\varphi-\left(1+3\frac{m^2}{\rho^4}\right)\sin 2(k-1)\varphi+\right.$$
$$3\frac{m}{\rho^2}\left(1+\frac{m^2}{\rho^4}\right)\sin 2k\varphi-\frac{m^2}{\rho^4}\left(3+\frac{m^2}{\rho^4}\right)\sin 2(k+1)\varphi+$$
$$\left.\left.\frac{m^3}{\rho^6}\sin 2(k+2)\varphi\right]\right\}+$$
$$\frac{2(1+\mu)}{Eg^2R^2}\sum_{-\infty}^{\infty}B_{2k}\rho^{2k}\left\{\left[\left(1+\frac{m^2}{\rho^4}\right)\sin 2(k+1)\varphi-\right.\right.$$
$$\left.\frac{m}{\rho^2}\sin 2k\varphi-\frac{m}{\rho^2}\sin 2(k+2)\varphi\right]+$$
$$\frac{2m}{\rho^2 g(2k+1)}\left[\frac{m}{\rho^2}\sin 2(k-1)\varphi-\left(1+2\frac{m^2}{\rho^4}\right)\sin 2k\varphi+\right.$$
$$\left.\left.\frac{m}{\rho^2}\left(2+\frac{m^2}{\rho^4}\right)\sin 2(k+1)\varphi-\frac{m^2}{\rho^4}\sin 2(k+2)\varphi\right]\right\} \qquad (33)$$

6. Displacement vector components in the hyperbolic-elliptical coordinate system

By splitting up the real and imaginary parts in the expression (8) as well as:

$$u_\rho+iu_\varphi=\frac{1}{E}\left\{(3-\mu)b_1-(1+\mu)\left[b_2+b_3\right]\right\}$$
$$|1-\frac{m}{\zeta^2}|=\sqrt{g}$$

$$b_1 = \sum_{-\infty}^{\infty} \frac{A_{2k}\rho^{2k+1}}{(2k+1)\sqrt{g}} \left(e^{i2k\varphi} - \frac{m}{\rho^2} e^{i2(k+1)\varphi}\right)$$

$$b_2 = \frac{1}{\sqrt{g}} \sum_{-\infty}^{\infty} A_{2k}\rho^{2k+1} \left(e^{-i2k\varphi} - \frac{m}{\rho^2} e^{-i2(k+2)\varphi}\right)$$

$$b_3 = \frac{1}{Rg\sqrt{g}} \sum_{-\infty}^{\infty} \frac{B_{2k}\rho^{2k+1}}{(2k+1)} \left[e^{-i2(k+1)\varphi}\left(1 + \frac{m^2}{\rho^4}\right) - \frac{m}{\rho^2} e^{-i2k\varphi} - \frac{m}{\rho^2} e^{-i2(k+2)\varphi}\right] \tag{34}$$

or in the next expression:

$$u_\rho + iu_\varphi = u_h + iu_e = \frac{1}{\sqrt{g}E}\left\{\sum_{-\infty}^{\infty} \frac{\rho^{2k+1}}{2k+1}\left[A_{2k}\left\{(3-\mu)\left(e^{i2k\varphi} - \frac{m}{\rho^2}e^{i2(k+1)\varphi}\right) - (1+\mu)(2k+1)\left(e^{-i2k\varphi} + \frac{m}{\rho^2}e^{-i2(k+1)\varphi}\right)\right\} - \frac{1+\mu}{Rg}B_{2k}\left[\left(1+\frac{m^2}{\rho^4}\right)e^{-i2(k+1)\varphi} - \frac{m}{\rho^2}e^{-i2k\varphi} - \frac{m}{\rho^2}e^{-i2(k+2)\varphi}\right]\right]\right\} \tag{35}$$

and by taking into consideration our conclusions about the coefficients A_n and B_n we can finally write the expressions for the displacement vector components in the hyperbolic-elliptical coordinates:

a) For the displacement $u_\rho(\rho,\varphi)$ in the direction $\vec{\mathbf{h}}_0$ tangent to the hyperbolic coordinate line:

$$u_\rho(\rho,\varphi) = \frac{1}{\sqrt{g}E}\left\{\sum_{-\infty}^{\infty} \frac{\rho^{2k+1}}{2k+1}\left[A_{2k}\left\{(3-\mu)\left(\cos 2k\varphi - \frac{m}{\rho^2}\cos 2(k+1)\varphi\right) - (1+\mu)(2k+1)(\cos 2k\varphi + \frac{m}{\rho^2}\cos 2(k+1)\varphi)\right\} - \frac{1+\mu}{Rg}B_{2k}\left[(1+\frac{m^2}{\rho^4})\cos 2(k+1)\varphi - \frac{m}{\rho^2}\cos 2k\varphi - \frac{m}{\rho^2}\cos 2(k+2)\varphi\right]\right]\right\} \tag{36}$$

(b) For the displacement $u_\varphi(\rho,\varphi)$ in the direction $\vec{\mathbf{e}}_0$ tangent to the elliptical coordinate line:

$$u_\varphi(\rho,\varphi) = \frac{1}{\sqrt{g}E}\left\{\sum_{-\infty}^{\infty} \frac{\rho^{2k+1}}{2k+1}\left[A_{2k}\left\{(3-\mu)\sin 2k\varphi - \frac{m}{\rho^2}\sin 2(k+1)\varphi) + \right.\right.\right.$$

$$(1+\mu)(2k+1)(\sin 2k\varphi + \frac{m}{\rho^2}\sin 2(k+1)\varphi)\Big\} +$$

$$\frac{1+\mu}{Rg}B_{2k}\left[(1+\frac{m^2}{\rho^4})\sin 2(k+1)\varphi - \frac{m}{\rho^2}\sin 2k\varphi - \frac{m}{\rho^2}\sin 2(k+2)\varphi\right]\Big]\Big\} \quad (37)$$

The displacement components $u_\rho(\rho,\varphi)$ and $u_\varphi(\rho,\varphi)$ are along the coordinate lines of the hyperbole and of the ellipse at each plate point.

7. Conclusion

Our contribution in this paper is the analysis of the strain state of the elliptical-annular plate strained by pairs or two opposing segmentally distributed forces along the external and internal contours by means of complex variable function and conformal mapping method. In these cases, the expressions for the strain tensor components at an arbitrary point of the elliptical-annular plate, have been derived.

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ПРИМЕНЕНИЕ ФУНКЦИИ КОМПЛЕКСНОЙ ПЕРЕМЕННОЙ И КОНФОРМНОГО ОТОБРАЖЕНИЯ ДЛЯ ОПРЕДЕЛЕНИЯ ДЕФОРМАЦИИ В ЕЛЛИПТИЧЕСКО-КОЛЬЦЕВОЙ ПЛАСТИНИ

В настоящей работе сделано исследование состояния деформации в эллиптическо-кольцевой пластини, плоско, сегментно нагруженной по внешней и внутренной контурах постоянным распределеним давлением. Для этого случая нагружения, посроенни выражения для компоненти тензора деформации и вектора перемещения точек пластини. Использован метод аналитических функций комплекснной переменной в эллиптическо-гиперболических координат и метод конформного отоброжения.

PRIMENA FUNKCIJE KOMPLEKSNE PROMENLJIVE I KONFORMNOG PRESLIKAVANJA ZA ODREDJIVANJE STANJA DEFORMACIJE U ELIPTIČKO-PRSTENASTOJ PLOČI

U ovom radu su izvedeni analitički izrazi za komponente tenzora deformacije i komponente vektora pomeranja u sistemu eliptičko-hiperboličkih koordinata u tačkama eliptično-prstenaste ploče segmentno opterećeno kontinualnim opterećenjem na spoljašnjoj i unutrašnjoj konturi. Funkcija kompleksne promenljive, metoda konformnog preslikavanja i hiperbolično-eliptične koordinate su korišćene za rešavanje ovog problema.

Prof. dr Katica (Stevanović) Hedrih, Matematički Institut SANU,
Mašinski fakultet

Beogradska 14
18000 Niš, Jugoslavija

Mr Dragan B. Jovanović, asistent
Mašinski fakultet
Beogradska 14
18000 Niš, Jugoslavija