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ENERGY EXCHANGE THEOREMS IN SYSTEMS WITH TIME-DEPENDENT CONSTRAINTS[1,2]

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Introduction

It has been proven that the mechanical energy change theorem of a system, with potential energy Π and time-dependent constraints, can be written in a general form $dE/dt = \mathcal{R}_0$, where $\mathcal{R}_0$ is the power of time dependent constraints, and $E = T + \Pi$ is the mechanical energy of the system. Introducing the term "generalized rheonomic coordinate q^0", as the known faction of time and parametre γ, which is appears in the equations of constraints, the above relation of energy is generalized on the form $dE/dq^0 = \mathcal{R}_0$ where $\mathcal{R}_0$ is called "generalized force of rheonomic constraint". This unknown function $\mathcal{R}_0$ has the dimension of force (ML^2T^{-2}), if the rheonomic coordinate has the dimension of length (L), or the dimension of torque (MLT^{-2}), if q^0 is a nondimensional coordinate (angle) and, as already shown, $\mathcal{R}^0$ has the dimension of power (ML^2T^{-3}), if q^0 has the dimension of time (T). It is shown that the equivalent energy change theorem cannot be proven for system with time-dependent constraints by n standard Lagrange's equations of second kind, $2n$ Hamilton's equations of motion, or with so-called "Homogeneous Formalism".

To clarify the above mentioned statement, we shall, in Section 1., derive the energy change theorem by Lagrange's equations of the first kind; then, in Section 2., show that, using standard Lagrange's equations of second kind, it is not possible to derive the previous equations of energy for systems of point masses constrained by time-dependent (rheonomic) constraints. In Section 3., an enlarged equivalent system of differential equations of motion is derived and the invariant enrgy change theorem is proven. Later, in Section 4. an enlarged suitable system of canonical differential equations of motion is written, and

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using this system, the validity of the derived energy theorem for systems with rheonomic constraints in an enlarged phase space, is proven. Finally, in Section 5., the relations incorrectly called Energy change theorem, will be shown.

1. The law of energy change

We consider the motion of a system of N point masses, with masses m_r, and position vectors $\mathbf{r}_\nu \in R^3$, where R^3 is the linear space. The vector $\mathbf{F}_\nu$ is the force acting on the ν-th point. The kinetic energy of a system is $T = \frac{1}{2}\sum m_\nu \mathbf{v}_\nu^2$ where $\mathbf{v}_\nu = \dot{\mathbf{r}}_\nu$ is the velocity of the ν-th point. The flow of kinetic energy holds

$$\dot{T} = \sum_{\nu=1}^{N} \mathbf{F}_\nu \cdot \mathbf{v}_\nu \tag{1.1}$$

If the point masses are constrained by holonomic constraints dependent on time, i.e., by

$$f_\mu(\mathbf{r}_1, ..., \mathbf{r}_N, t) \equiv f_\mu(\mathbf{r}, t) = 0, \quad \mathbf{r} = (\mathbf{r}_1, ..., \mathbf{r}_N), \quad \mu = 1, ..., k \leq 3N \tag{1.2}$$

then we have

Theorem 1. *The flow of kinetic energy change of a system of point masses, constrained by holonomic time-dependent constraints, is equal to the power of all forces, including the forces of the constraints, acting on the points of the systems, i.e.,*

$$\dot{T} = \sum_{\nu=1}^{N} \mathbf{F}_\nu \cdot \mathbf{v}_\nu - \sum_{\mu=1}^{N} \lambda_\mu \frac{\partial f_\mu}{\partial t} \tag{1.3}$$

Proof. The differential equation of motion of the system can be written in the form of Lagrange's equation the first kind [1], [8],

$$m_\nu \frac{d\mathbf{v}_\nu}{dt} = \mathbf{F}_\nu + \sum_{\mu=1} k\lambda_\mu \frac{\partial f_\mu}{\partial \mathbf{r}_\nu}, \qquad (\mu = 1, ..., N) \tag{1.4}$$

where $\lambda_1, ..., \lambda_k$ are Lagrange's undetermined multipliers.

The velocities of point masses satisfy equations of constraints

$$\sum_{\nu=1}^{N} \frac{\partial f_\mu}{\partial \mathbf{r}_\nu} \frac{\partial \mathbf{r}_\nu}{dt} + \frac{\partial f_\nu}{dt} = 0, \qquad (\mu = 1, ..., k)$$

Therefore,

$$\begin{aligned} \dot{T} &= \frac{1}{2} \frac{d}{dt} \sum_{\nu=1}^{N} m_\nu \mathbf{v}_\nu^2 = \sum_{\nu=1}^{N} m_\nu \frac{d\mathbf{v}_\nu}{dt} \cdot \mathbf{v} = \\ &= \sum_{\nu=1}^{N} \left(\mathbf{F} + \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial \mathbf{r}_\nu} \right) \mathbf{v}_\nu = \sum_{\nu=1}^{N} \mathbf{F} \cdot \mathbf{v}_\nu - \sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial t} \end{aligned} \tag{1.5}$$

This is the base theorem which must hold under all mathematical transformations.

Corollary 1. *If the forces F_ν acting on points masses, constrained by holonomic rheonomic constraints, are potential with potential energy $\Pi(\mathbf{r})$ then the flow of mechanical energy $E = T + \Pi$ is equal to the power of the constraining "force" λ_0,* [13]

$$\dot{E} = -\sum_{\mu=1}^{k} \lambda_\mu \frac{\partial f_\mu}{\partial t} = \lambda_0 \tag{1.6}$$

Proof. A set of forces is potential if and only if there exists a function $\Pi(\mathbf{r})$ such that $\mathbf{F} = -\partial\Pi/\partial\mathbf{r}$. Therefore,

$$\sum_{\nu=1}^{N} \mathbf{F} \cdot \mathbf{v}_\nu = -\sum_{\nu=1}^{N} \left(\frac{\partial \Pi}{\partial \mathbf{r}_\nu}\right) \cdot \mathbf{v}_\nu = -d\Pi$$

and Eq. (1.6) follows from Eq. (1.5).

2. The standard Lagrangian formulation

The motion of system of N point masses, constrained by holonomic constraints, expressed in k equations (1.2), is described in standard analytical mechanics by $n = 3N - k$ independent generalized variables $q = (q^1, ..., q^n)$, $n = 3N - k$ generalized velocities $\dot{q} = (\dot{q}^1, ..., \dot{q}^n)^T$ and well known n Lagrange's differential equations of second kind

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^i} - \frac{\partial T}{\partial q^i} = Q_i, \qquad (i = 1, ..., n = 3N - k), \tag{2.1}$$

where Q_i are generalized forces. For potential energy $\Pi(q)$ we have

$$Q_i = -\partial\Pi/\partial q^i$$

The kinetic energy T this is described by non-homogeneous quadratic form in the generalized velocities

$$T = \underbrace{\frac{1}{2}\sum_{i,j=1}^{n} a_{ij}\dot{q}^i\dot{q}^j}_{T_2} + \underbrace{\sum_{i}^{n} a_i\dot{q}^i}_{T_1} + \underbrace{\frac{1}{2}a}_{T_0} \tag{2.2}$$

Multiplying equations of motion (2.1) by velocities $\dot{q}^i$ and summing on index i, we shoul obtain the equation of energy (1.6). But, it is not the case, as we obtain:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^i}\dot{q}^i - T + \Pi\right) \equiv \frac{d}{dt}(T_2 - T_0 + \Pi) = \frac{\partial \Pi}{\partial t} - \frac{\partial T}{\partial t} \tag{2.3}$$

Some authors formally add or subtract various additions to this equations, or write it in the different form, but that does not change the contents of Eqs (2.1), because the n Eqs. (2.1) are not equivalent to the equation of motion (1.4). This will be proven in the following section, and here it will suffice to show following: If the motion of only one point masses is limited by 3 constraints $f_\mu(x, y, z, t) = 0$, $(\mu = 1, 2, 3)$ and if in the general case, they are moving, their intersection $f_1 \cap f_2 \cap f_3$ is also moving. But, according to the previous theory $n = 3N - K = 3 - 3 = 0$. Therefore, neither theh coordinate q, the velocity q nor Eq. (2.1) do exist. And that does not correspond to the situation described by Eqs (1.4). This fact was also shown by H.Goldstein in his book ([5], p.54), quote: "Actually the two (energy) theorems are not talking about quite the same energy. In the previous statement, the energy change of the system included the work done by all forces, including the forces of constraint. Here, in the Lagrangian formulation, (energy) V contains only the work of the external or applied forces, excluding the forces of constraint... . However, if there is a moving constraint the force of constraint need not be perpendicular to the actual displacement and the work done by such forces will not be zero". That problem is obvious in the modern book of V.T. Arnold ([2], p. 86). Among a number of energy theorems in the part "Lagrangian mechanics on manifolds" it was written: "A system of n mass points, constrained by holonomic constraints dependent on time, is defined with the help of a time-dependent submanifold of the configuration space of a free system. Such a manifold is given by a mapping $i : M \times R \to E^{3n}$, $i(q, t) = x$, which for any fixed $t \in R$, defines an embedding $M \to E^{3n}$." The question "why" is arises.

In the standard Lagrangian formulation for a system of N point masses , constrained by k holonomic constraints, expressed in Eqs. (1.2), well known that $\mathbf{r}_\nu = \mathbf{r}_\nu\left(q^1, ..., q^n, t\right)$ and

$$\mathbf{v}_\nu = \frac{\partial \mathbf{r}_\nu}{\partial q^1}\dot{q}^1 + ... + \frac{\partial \mathbf{r}_\nu}{\partial q^n}\dot{q}^n + \frac{\partial \mathbf{r}_\nu}{\partial t}\dot{q}^n$$

where $\dot{q}^1, ..., \dot{q}^n$ are generalized velocities. The reader should notices this important statement, that standard Lagrange's mechanics treats **n-generalized velocities** through the fact that velocity vectors

$$\mathbf{v}_\nu = \sum_{i=1}^{n} \frac{\partial \mathbf{r}_\nu}{\partial q^i}\dot{q}^i + \frac{\partial \mathbf{r}_\nu}{\partial t} \tag{2.4}$$

have $\mathrm{n} + 1$ component.

Only if constrains do not depend on time, the velocity vectors have n-components. Such as

$$\mathbf{v}_\nu = \sum_{i=1}^{n} \frac{\partial \mathbf{r}_\nu}{\partial q^i}\dot{q}^i = \frac{\partial \mathbf{r}_\nu}{\partial q^i}\dot{q}^i.$$

Dot-multiplying the equation of motion (10) by the corresponding coordinate vectors $\frac{\partial \mathbf{r}}{\partial q}$ and summing on index ν, Eqs. (2.1) are obtained. where

$$Q_j = \sum_{\nu=1}^{n} \mathbf{F}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^j}$$

are generalized forces, and kinetic energy T stays homogenous square positive form $T = \frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j$.

Differential Eqs. (2.1) are used in standard mechanics and for systems with time-dependent constrains, even if Eqs. (1.4) are not projected in the direction of $\partial \mathbf{r}_\nu / \partial t$.

3. Introducing "rheonomous coordinate" and consequences

In time-dependent constraints a known time function $\tau(\gamma, t)$ is present, it contains a certain physical parameter, such that equations of constraints are dimensionally homogenous.

For example, $f_1 = x^2 + y^2 + z^2 - 4t^2 \equiv x^2 + y^2 + z^2 - (vt)^2 = 0$, $f_2 = y - tx \equiv y - \omega t x = 0$, where, x, y, z are Decart's rectrangular coordinates, having the dimension of length [dim x]=L. In the first equation, parameter $\gamma = v = 2[\mathrm{LT}^{-1}]$ has the dimension of velocity, and in the second, the dimension of angular velocity $\gamma = \omega = 1[\mathrm{T}]^{-1}$.

Definition. *The generalized variable q^0, added to the Lagrange's independent coordinates q^1, ..., q^n and which is equal to a choosen known function $\tau(t)$, is called "rheonomic coordinate".*

For these examples, it is suitable to take for "rheonomic coordinate" $q^0 = vt$, [dim q^0]=L or $q^0 = \omega t$, [dim q^0] = [$\emptyset$].

In the special case it can be taken $q^0 = t$, [dim q^0] = T, but that equality (the constraint) should be distinguished from identity $q^0 \equiv t$.

Then the constraints can be written in the form:

$$f_\mu(\mathbf{r}, t) = f_\mu\left(\mathbf{r}, q^0\right) = 0, \quad q^0 = \tau(t) \tag{3.1}$$

The conditions for velocities are, therefore,

$$\dot{f}_\mu = \sum_{\nu=1}^{N} \frac{\partial f_\mu}{\partial \mathbf{r}_\nu} \cdot \mathbf{v}_\nu + \frac{\partial f_\mu}{\partial q^0} \dot{q}^0 = 0, \quad \dot{q}^0 = \dot{\tau}(t)$$

Because of the independence of k-equations (3.1) and implicite functions theorem, equations of constraints can be written in parametric form:

$$\mathbf{r}_\nu = \mathbf{r}_\nu\left(q^0, q^1, ..., q^n\right)$$

and

$$\mathbf{v}_\nu = \frac{\partial \mathbf{r}_\nu}{\partial q^0} \dot{q}^0 + \sum_{i=1}^{N} \frac{\partial \mathbf{r}_\nu}{\partial q^i} \dot{q}^i = \sum_{\alpha=0}^{N} \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha} \dot{q}^\alpha, \tag{3.2}$$

where now we have $n+1$ independent velocity vectors $\frac{\partial \mathbf{r}_\nu}{\partial q^0}\dot q^0$, ..., $\frac{\partial \mathbf{r}_\nu}{\partial q^n}\dot q^n$ and corresponding number of $n+1$ generalized velocities $\dot q^0$, $\dot q^1$,..., $\dot q^n$. That way, the observed discrepancy (see 2.4) in standard Lagrange's mechanics is eliminated. The coordinate q^0 is not formally introduced as $q^0 \equiv t$, but it is function of time (contained in rheonomic constraints).

A "generalized force of the rheonomic constraint" as an unknown function $\mathcal{R}_0(q^0)$ is corresponded to that coordinate q^0. That rheonomic coordinate yields an additional differential equations of motion [6, 11]

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q^0} - \frac{\partial T}{\partial q^0} = Q_0 = Q_0^* + \mathcal{R}_0 \tag{3.3}$$

where, in contrast to Section (2.2), kinetic energy is homogenous quadrate form of $n+1$ generalized velocities $\dot q^0$, $\dot q^1$, ..., $\dot q^n$, i.e.,

$$T = \frac{1}{2}a_{\alpha\beta}\dot q^\alpha \dot q^\beta = \frac{1}{2}a_{ij}\dot q^i \dot q^j + a_{0i}\dot q^0 \dot q^i + \frac{1}{2}a_{00}\dot q^0 \dot q^0,$$
$$(\alpha,\beta = 0,1,...,n; \qquad i,j = 1,2,...,n). \tag{3.4}$$

Indeed, if one multiplies differential equaitions (1.4) by the basic vectors $\partial \mathbf{r}_\nu/\partial q^1$, ..., $\partial \mathbf{r}_\nu/\partial q^n$; the known Lagrange's equations of the second kind (2.1) are obtained. Thus, also Eq. (3.3) is obtained if Eqs. (1.4) are dot-multiplied by corresponding vectors $\frac{\partial \mathbf{r}_\nu}{\partial t}$ or $\frac{\partial \mathbf{r}_\nu}{\partial q^0}$ as follows

$$\left(m_\nu \dot{\mathbf{v}}_\nu = \mathbf{F}_\nu + \sum_\mu^k \lambda_\nu \nabla_\nu f_\mu\right)\cdot \frac{\partial \mathbf{r}_\nu}{\partial q^0}, \qquad (\nabla_\nu = grad_\nu)$$

$$\rightarrow \sum_{i=1}^N m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^0}\frac{d}{dt}\left(\frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}\dot q^\alpha\right) = \sum_{\nu=1}^N \mathbf{F}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^0} + \sum_{\nu=1}^N\sum_{\mu=1}^k \lambda_\nu \nabla_\mu f_\mu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^0},$$

$$\sum_{\nu=1}^N m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^0}\frac{d}{dt}\left(\frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}\dot q^\alpha\right) = \frac{d}{dt}\sum_{\nu=1}^N m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^0}\frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}\dot q^\alpha - \sum_{\nu=1}^N m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}\dot q^\alpha \frac{d}{dt}\frac{\partial \mathbf{r}_\nu}{\partial q^0} =$$
$$= \frac{d}{dt}\frac{\partial T}{\partial \dot q^0} - \sum_{\nu=1}^N m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}\cdot\frac{\partial^2 \mathbf{r}_\nu}{\partial q^\beta \partial q^0}\dot q^\alpha \dot q^\beta = \frac{d}{dt}\frac{\partial T}{\partial \dot q^0} - \frac{\partial T}{\partial q^0},$$

$$T = \frac{1}{2}\sum_\nu m_\nu \mathbf{v}_\nu \cdot \mathbf{v}_\nu = \frac{1}{2}\sum_{\nu=1}^N m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}\cdot\frac{\partial \mathbf{r}_\nu}{\partial q^\beta}\dot q^\alpha \dot q^\beta = \frac{1}{2}a_{\alpha\beta}\dot q^\alpha \dot q^\beta,$$

$$Q_0^* = \sum_\nu \mathbf{F}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^0}, \quad (\alpha,\beta = 0,1,...,n = 3N-k),$$

$$\sum_{\nu}^{N}\sum_{\mu}^{k}\lambda_{\mu}\nabla_{\nu}f_{\mu}\cdot\frac{\partial\mathbf{r}_{\nu}}{\partial q^{0}}=\sum_{\mu}^{k}\lambda_{\mu}\sum_{\nu}^{N}\frac{\partial f_{\mu}}{\partial\mathbf{r}_{\nu}}\frac{\partial\mathbf{r}_{\nu}}{\partial q^{0}}=-\sum_{\mu}^{k}\lambda_{\mu}\frac{\partial f_{\mu}}{\partial q^{0}}=\mathcal{R}_{0}\left(q^{0}\right).$$

what was be proved. Therefore, can to write

Theorem 2. *The system of Lagrange's equations of the first kind (1.4) is equivalent to the system of equations*

$$\begin{aligned}&\frac{d}{dt}\frac{\partial T}{\partial\dot{q}^{i}}-\frac{\partial T}{\partial q^{i}}=Q_{i},\quad(i=1,...,n=3N-k)\\&\frac{d}{dt}\frac{\partial T}{\partial\dot{q}^{0}}-\frac{\partial T}{\partial q^{0}}=Q_{0},\end{aligned}\qquad(3.5)$$

where q^0 is the "rheonomic coordinate".

Corollary 2. *If generalized forces $Q=(Q_0^*,Q_1,...,Q_n)$ and the generalized forces of rheonomic constraints act on the system of point-masses, then the total differential of the kinetic energy is equal to the work all generalized forces including, the generalized force of the rheonomic constraints $\mathcal{R}_0$ i.e.* ,

$$dT=Qdq^{i}+Q_{0}dq^{0}\equiv Qdq+\left(Q_{0}^{*}+\mathcal{R}_{0}\right)dq^{0}=Q_{\alpha}dq^{\alpha},\quad Q_{0}=Q_{0}^{*}+\mathcal{R}_{0},\qquad(3.6)$$

Proof I. Multiplying Eqs. (3.5) with $dq^i=\dot{q}^i dt$, and Eq.(3.3) with $dq^0=\dot{q}^0 dt$ and adding on index i it is obtained Eq. (3.6).

ProofII. Substituting the expression (3.2) in Eq. (1.1) i.e.

$$\dot{T}=\sum_{\nu=1}^{N}\mathbf{F}_{\nu}\cdot\frac{\partial\mathbf{r}_{\nu}}{\partial q^{\alpha}}\dot{q}^{\alpha}=Q_{\alpha}\dot{q}^{\alpha}$$

Eq. (3.6) follows.

Corollary 3. *If the generalized forces act on a system of point masses with time-dependent constraints have a potential $\Pi\left(q^0,q^1,...,q^n\right)$, $Q=-\partial\Pi/\partial q$, $Q_0^*=-\partial\Pi/\partial q^0$, then the total differential of the mechanical energy $E=T+\Pi$ is equal to the work of the generalized constraint force $\mathcal{R}_0$, i.e.,*

$$dE=\mathcal{R}_{0}dq^{0}\qquad(3.7)$$

Eq. (3.7) is equivalent to Eqs. (1.6). For $q^0=t$ Eqs. (3.7) and (1.6) are identical and for $q^0=\tau(t)$ only the dimension of the generalized force $\mathcal{R}_0$ changes, because of $\mathcal{R}\left(q^0\right)dq^0=\left(\mathcal{R}_0\partial q^0/\partial t\right)dt=\bar{\mathcal{R}}_0(t)\,dt$, $\bar{\mathcal{R}}_0(t)=\mathcal{R}_0\partial q^0/\partial t=\lambda_0$.,

Introducing the rheonomic coordinate q^0 as a known function of time, from the holonomic constraints and the corresponding force of constraint $\mathcal{R}_0$, the law of the kinetic energy flow (1.3) and (3.6) and the theorem of the mechanical energy change (3.7) derived from the differential equations of motion, can be brought into agreement with the invariant and equivalent cases for nonholonomic systems with time-dependent constraints.

4. The Hamiltonian formulation

The theorem of the energy change, equivalent to relations (3.7) or (1.6), for a system with the holonomic time dependent constraints, can be obtained from $2n+2$ cannonic differential equations of the form

$$\dot{q}^\alpha = \frac{\partial \mathcal{E}}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial \mathcal{E}}{\partial q^\alpha}, \quad (\alpha = 0, 1, ..., n) \tag{4.1}$$

where,

$$\mathcal{E} = T + \Pi + P = \frac{1}{2} a^{\alpha\beta} p_\alpha p_\beta + \Pi + \int \mathcal{R}_0 \left(q^0\right) dq^0$$

is the "total mechanical energy", $P = -\int \mathcal{R}_0 \left(q^0\right) dq^0$ is the "rheonomic potential" [6,10,11] and p_0, p_1, ..., p_n are generalized impulses. These $2n+2$ equations are equivalent to the system of equations

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{q}^0 = -\frac{\partial H}{\partial p_0}, \tag{4.2}$$

$$\dot{p}_i = \frac{\partial H}{\partial q^i}, \quad \dot{p}_0 = -\frac{\partial H}{\partial q^0} + \mathcal{R}_0, \tag{4.3}$$

where

$$H = T + \Pi = \frac{1}{2} a^{ij} p_i p_j + a^{0j} p_0 p_j + \frac{1}{2} a^{00} p_0 p_0 + \Pi = E \tag{4.4}$$

Eqs. (4.1) are equivalent to Eqs. (3.3), (3.5), as well as Eqs. (1.4) for a system of potential forces. Indeed, q^α are generalized coordinates, as it was shown in Section 3. The corresponding generalized impulses are sums of projection of momentum on the point masses direction of the coordinate q, i.e.

$$p_i = \sum_{\nu=1}^{N} m_\nu \mathbf{v}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^i} = \sum_{\nu=1}^{N} m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^i} \dot{q}^\alpha = a_{i\alpha} \dot{q}^\alpha = a_{ij} \dot{q}^j + a_{i0} \dot{q}^0. \tag{4.5}$$

In the same way, we obtain the additional impulses

$$p_0 = \sum_{\nu=1}^{N} m_\nu \mathbf{v}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^0} = a_{0j} \dot{q}^0 = \frac{\partial T}{\partial \dot{q}^0}. \tag{4.6}$$

Since there are linear constraints connecting the generalized impulses p_0, p_1, ..., p_n with generalized velocities $\dot{q}^0$, $\dot{q}^1$, ... , $\dot{q}^n$ then,

$$\dot{q}^{\beta} = a^{\alpha\beta} p_{\alpha} =: \sum_{\alpha=0}^{n} a^{\beta\alpha} p_{\alpha} = a^{\beta 0} p_0 + \sum_{i=1}^{n} a^{\beta i} p_i \tag{4.7}$$

Substituting into Eq. (3.4) the kinetic energy is obtained, as well as the homogeneous quadratic form of the generalized impulses

$$T = \frac{1}{2} a_{\alpha\beta} \dot{q}^{\alpha} \dot{q}^{\beta} = \frac{1}{2} a_{\alpha\beta} a^{\beta\gamma} p_{\gamma} a^{\beta\rho} p_{\rho} = \frac{1}{2} a^{\gamma\rho} p_{\gamma} p_{\rho}. \tag{4.8}$$

As the Hamiltonian function H is equal to the mechanical energy, $E = T + \Pi$, considering that

$$\frac{\partial T(q,\dot{q})}{\partial \dot{q}} = -\frac{\partial T(q,p)}{\partial p}$$

Eqs. (4.2) and (4.3) follow from Eqs. (3.3) and (3.5) and vice versa

If we multiply Eqs. (4.2) by impulses p_{α} and Eq. (4.3) by corresponding generalized velocities, and by adding, the following relation is obtained

$$\dot{H} \equiv \dot{E} = \mathcal{R}_0 \dot{q}^0 \tag{4.9}$$

This is identical to the relations of energy (3.7) and (1.6).

In case that the constraints are time-independent, neither the rheonomic coordinate q^0, nor the corresponding impulses p_0 ($q^0 \equiv 0$, $p_0 \equiv 0$, $\rightarrow \mathcal{R}_0 = 0$) do exist, so all the results of the classical Lagrange's, Jacoby's and Hamiltonian analytical mechanics are obtained

In the standard analytical dynamics, Eq. (4.9) is not obtained, but it is known that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}, \tag{4.10}$$

where $H = T_2 - T_0 + \Pi$. This is equivalent to Eq. (2.3) because of

$$\frac{\partial T(q,\dot{q},t)}{\partial t} = -\frac{\partial T(q,p,t)}{\partial t},$$

but it is not the flow of energy of rheonomic systems

Eq. (4.10) can also be obtained by the so-called homogeneous formalism (see for example [8],[6] or [3]), which is different from the proven energy theorem for rheonomous systems (4.9). Only the denotations q^0 and p_0 are similar in "the Homogeneous formalism". Everything else is different. This is not the question of formalism but of the description of the real mechanical motion. The coordinate q^0 is not introduced formally, but by the time-dependent constraints. The impulses are not introduced by the definition $p = -H$ as in the homogenous formalism but are obtained from Newton's definition of "Quantitas motus", using the expression (4.5) Those are such big differences that we can't talk about the same mechanics. In the homogeneous formalism $n+1$ equation of the form (3.5) exist, but it doesn't contain the force of rheonomic constraints $\mathcal{R}_0$. Also, two relations

$$\dot{q}^0 = \frac{\partial H}{\partial p_0}, \quad \dot{p}_0 = \frac{\partial H}{\partial q^0}$$

are similar to our Eqs. (4.1). But, it is known that in the homogeneous formalism those equations are identities, which solve nothing in the classical mechanics, while here, using one equation, the unknown generalized force $\mathcal{R}_0$ is determined, and using the other equation, the relations between velocities $\dot{q}_0, \dot{q}^1, \ldots, \dot{q}^n$ and $p_0, p_1, \cdots, p_n$ are determined.

For the case of the Decart's rectangular coordinate y_i is easy to proove that is

$$a^{ij} = \delta^{ij} = \begin{cases} 0, & i \neq j \\ \dfrac{1}{m_i}, & i = j \end{cases}.$$

For a system of point masses which are not connected by rheonomic constraints, for example, N celestial bodies, the "rheonomic coordinate" q^0 does not exist, as well as the corresponding impulse p_0. In that case the Newtonian equations of motion are

$$m_\nu \ddot{\mathbf{r}}_\nu = -\frac{\partial \Pi}{\partial \mathbf{r}_\nu} + \mathbf{P}_\nu \ldots,$$

where Π is potential of forces, $\mathbf{P}_\nu$ are non-potential forces, or

$$m_i \ddot{y}_i = -\frac{\partial \Pi}{\partial y_i} + P_i, \qquad (m_{3i-2} = m_{2i-1} = m_{3i}) \tag{4.12}$$

where, here, y_i Descart's rectangular coordinate.

From Eqs. (4.12) it follows that the flow of energy holds, as in Eq. (1.1), i.e.

$$\dot{E} = \sum_{i=1}^{3} P_i \dot{y}^i$$

In the conjugate variables $y_1, \ldots, y_{3N}; p_1, \ldots, p_{3N}$ the motion is described by the set of differential equations

$$\dot{y}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial y_i} + P_i,$$

where

$$E = H = \sum_{i,j=1}^{3N} \frac{1}{2} a^{ij} p_i p_j + \Pi(y)$$

So, for such systems, the kinetic energy is a homogeneous quadratic form, therefore there is no need for additional introduction of conjugate variables, y_0 and p_0.

5. The misconception of the energy theorem

In many textbooks and references on Analytical Mechanics of systems with time-dependent constraints, the flow of energy theorem is either not precisely well described, or reduced to the Jacobi's form (see for example [1]),

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L\right) = -\frac{\partial L}{dt} + \bar{Q}_i\dot{q}^i, \quad L = T - \Pi \tag{5.1}$$

or in the form ([3] p. 58, 88)

$$\frac{dE}{dt} = \bar{Q}_i\dot{q}^i + \frac{d}{dt}(T_1 + 2T_0) - \frac{\partial T}{\partial t} + \frac{\partial \Pi}{\partial t} \tag{5.2}$$

where: $i = 1, ..., n$ is the summation index, $\Pi(q,t)$ is the natural potential energy, $E = T + \Pi$ is the mechanical energy, Q_i are non-potential generalized forces and T is the kinetic energy in the non-homogeneous quadratic form (2.2).

Similarly, in the Hamiltonian formulation Eq. (1) is described in the form

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \bar{Q}_i\dot{q}^i \tag{5.3}$$

where the Hamilton's function H is equal to the energy E only in the case of a system with time-independent constraints.

Eqs. (5.1), (5.2) and (5.3) are describing the same energy, because one equation is following from the other. Indeed, if we add the differential $d(T_1 + 2T_0)$ either to the left and the right side of Eq. (5.1), we shall obtain Eq. (5.2).

In fact, Eq. (5.2) neither contains the total energy E, nor its total time-derivative, although it looks like it. It is an identity to Eq. (5.1) for the case of potential forces. In fact, if the mechanical energy is written in the form $E = T + \Pi = T_2 + T_1 + T_0 + \Pi$ it follows that

$$\frac{dE}{dt} \equiv \frac{d}{dt}(T_2 + T_1 + T_0 + \Pi) = \frac{d}{dt}(T_1 + 2T_0) - \frac{\partial T}{\partial t} + \frac{\partial \Pi}{\partial t}$$

and this is reduced to

$$\frac{d}{dt}(T_2 - T_0 + \Pi) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} - L\right) = -\frac{\partial L}{\partial t},$$

because of

$$\frac{\partial L}{\partial \dot{q}^i}\dot{q}^i = 2T_2 - T_1, \quad L = T - \Pi$$

Therefore, Eq. (5.2) and other equivalent Eqs. (5.1) and (5.3) describe the part of the energy of a system with time dependent constraints, but not the total energy.

Example*: To illustrate Sections 1, 2 and 3 of this paper, let us choose the system shown in Figure a (**case a**) and Figure b (**case b**).

We choose two kinds of constraints

(case a) $f_1 = y_1 - a\cos\Omega t = 0$,

(case b), $f_2 = y_2 - y_1 - a\cos\Omega t = 0$; $a\&\Omega = const.$

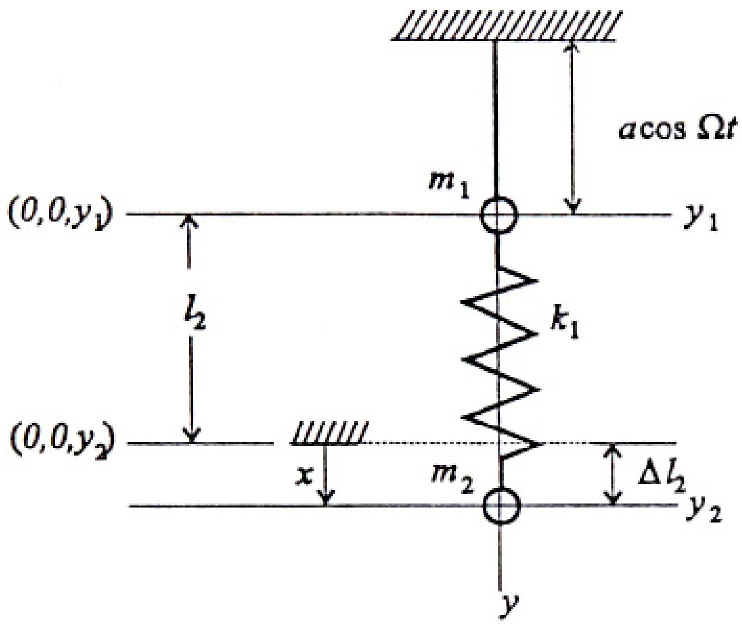

Fig.a.

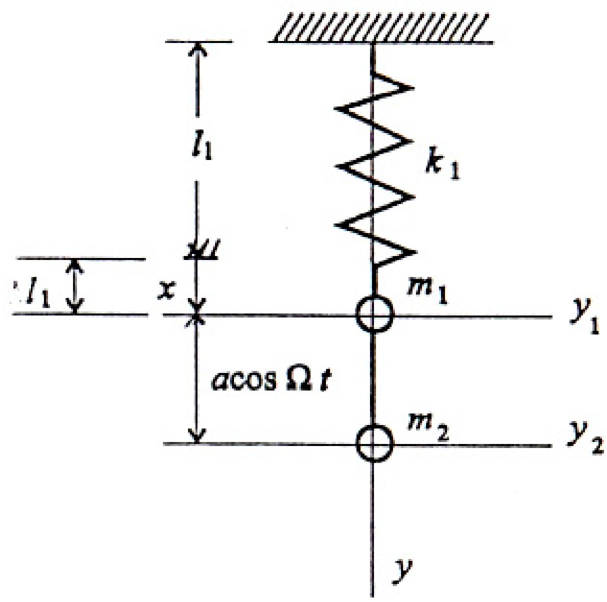

Fig.b.

Case a.

Section 1. The equations of motion (1.4) in this case are

$$\begin{aligned} m_1\ddot{y}_1 &= k_2\Delta l_2 + \lambda_1 = k_2\left(y_2 - y_1 - l_2\right) + \lambda_1 \\ m_2\ddot{y}_2 &= -k_2\Delta l_2 = -k_2\left(y_2 - y_1\right); \end{aligned} \tag{1.1,a}$$

$$\begin{aligned} &f_1 = y_1 - a\cos\Omega t = 0, \\ &\ddot{f}_1 = 0 \quad \rightarrow \quad \lambda_1 = k_2\left(y_2 - y_1 - l_2\right) - m_1 a\Omega^2\cos\Omega t \end{aligned} \tag{1.2,a}$$

*This example was proposed to the author by anonymous referee

Multiplying the Eq. (1.1,a) by dy_i and the Eq. (1,2.a) by dy_2 we will get Eq. (1.6) in the form

$$d\left[\left(\frac{1}{2}\right)\left(m_1\dot{y}_1^2+m_2\dot{y}_2^2\right)\right]=d\left[\left(\frac{k_2}{2}\right)(y_2-y_1-l_2)^2\right]+\lambda_1 dy_1, \tag{1.3,a}$$

or

$$\dot{E}=\dot{T}+\dot{\Pi}=\lambda_1 a\Omega\sin\Omega t=R_0 \tag{1.4,a}$$

or

$$E=T+\Pi=\int \mathcal{R}_0 dt+C. \tag{1.5,a}$$

Section 2. The degree of freedom $n=1$; Lagrange' s independent coordinate $q\equiv x$. Then constraint f_1 in the parametric form: $y_1=a\cos\Omega t$, $y_2=a\cos\Omega t+x$.

The equation of motion (2.1) is

$$m_2(\dot{x}-a\Omega\sin\Omega t)=-k_2 x \tag{2.1,a}$$

because the kinetic energy gets the form

$$T=\underbrace{\frac{1}{2}\dot{x}^2}_{T_2}-\underbrace{am_2\Omega\dot{x}\sin\Omega t}_{T_1}+\underbrace{\frac{1}{2}(m_1+m_2)a^2\sin^2\Omega t}_{T_0} \tag{2.2,a}$$

and potential

$$\Pi=\frac{k_2}{2}x^2 \tag{2.3,a}$$

Multiplying the Eq. (2.1,a) by dx we will get

$$\left(\frac{1}{2}\right)\left(m_2\dot{x}^2+k_2\dot{x}^2\right)=m_2 a\Omega^2\int\cos\Omega t dx+\bar{C} \tag{2.4,a}$$

or

$$E\neq T_2+\Pi=m_2 a\Omega^2\int\cos\Omega t dx+\bar{C} \tag{2.5,a}$$

It is follows that Eq.(1.5,a) $\neq$ Eq.(2.5,a).

Section 3. In this case a the rheonomic coordinate $q^0\equiv x_0$ could be taken as functions: $x_0=\Omega t$, $x_0=\cos\Omega t$, $x_0=a\cos\Omega t$ and $x_0=t$. Let as

$$x_0=\Omega t \tag{3.1,a}$$

Then we have: $y_1=a\cos x_0$, $y_2=a\cos x_0+l_2+x$; $T=(m_2/2)x^2$, $\Pi=(k_2/2)x^2$. The Eqs. (3.4) & (3.5) or more concrete

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} - \frac{\partial T}{\partial x} = -\frac{\partial \Pi}{\partial x}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_0} - \frac{\partial T}{\partial x_0} = -\frac{\partial \Pi}{\partial x^0} + \mathcal{R}_0$$

in this case are

$$m_2\left[\ddot{x} + \frac{d}{dt}(a\dot{x}_0 \sin x_0)\right] = -k_2 x... \tag{3.2,a}$$

$$\frac{d}{dt}\left[(m_1+m_2)\dot{x}_0 a^2 \sin^2 x_0 + m_2 a\dot{x}\sin x_0\right] -$$
$$(m_1+m_2)\dot{x}_0^2 a^2 \sin x_0 \cos x_0 + am_2\dot{x}_0\dot{x}\cos x_0 = \mathcal{R}_0. \tag{3.3,a}$$

Multiplying the Eq. (3.2,a) by dx and the Eq. (3.3,a) by dx_0 and then adding, we will get

$$\frac{m_2}{2}\dot{x}^2 + am_2\dot{x}_0\dot{x}\sin x_0 + \frac{m_1+m_2}{2}a^2\dot{x}_0^2\sin^2 x + \frac{k_2}{2}x^2 = \int \mathcal{R}_0(x_0)\,dx_0 + C, \tag{3.4,a}$$

or

$$E = T + \Pi = \int \mathcal{R}_0 dx_0 + C. \tag{3.5,a}$$

It is follows that (1.5,a) = (3.5,a) $\neq$ (2.5,a) what was to be proved.

Case b.

Section 1. In similar way we will consider and this case

$$m_1\ddot{y}_1 = -k_1(y_1 - l_1) - \lambda_2, \tag{1.1,b}$$

$$m_2\ddot{y}_2 = \lambda_2; \tag{1.2,b}$$

$$f_2 = y_2 - y_1 + a\cos\Omega t = 0, \tag{1.3,b}$$

$$\ddot{f}_2 = 0 \quad \rightarrow \quad \lambda_2 = -\frac{km_2}{m_1+m_2}(y_1 - l_1),$$

$$m_1\dot{y}_1^2 + m_2\dot{y}_2^2 = -k_1(y_1-l_1)^2 + 2\int \lambda_2 a\Omega\sin\Omega t dt + C \tag{1.4,b}$$

$$E = T + \Pi = \int \lambda_2 a\Omega \sin\Omega t dt + C \tag{1.5,b}$$

Section 2.

$$y_1 = l_1 + x, \qquad y_2 = l_1 + x + a\cos\Omega t$$

$$T = \underbrace{\frac{m_1+m_2}{2}\dot{x}^2}_{T_2} - \underbrace{m_2 a\Omega\dot{x}\sin\Omega t}_{T_1} + \underbrace{\frac{m_2 a\Omega^2}{2}\sin^2\Omega t}_{T_0} \tag{2.1,b}$$

$$\Pi = \frac{k_1}{2}x^2, \tag{2.2,b}$$

$$(m_1 + m_2)\,\ddot{x} - m_2 a\Omega^2 \cos \Omega t = -k_1 x, \tag{2.3,b}$$

$$\frac{m_1 + m_2}{2}\dot{x}^2 + \frac{k_1}{2}x^2 = m_2 a\Omega^2 \int \cos \Omega t dc + C^*, \tag{2.4,b}$$

$$E \neq T_2 + \Pi = m_2 a\Omega 2 \int \cos \Omega t dx + C^* \tag{2.5,b}$$

Section 3.

$$y_1 = l_1 + x, \qquad y_2 = l_1 + x + x_0; \qquad x_0 = a \cos \omega t, \tag{3.1,b}$$

$$T = \frac{m_1 + m_2}{2}\dot{x}^2 + m_2 \dot{x}_0 \dot{x} + \frac{m_2}{2}\dot{x}_0^2, \quad \Pi = \frac{k_1}{2}x^2 \tag{3.2,b}$$

$$\begin{aligned} &(m_1 + m_2)\,\ddot{x} + m_2 \ddot{x} = -k_1 x, \\ &m_2\,(\ddot{x} + \ddot{x}_0) = \mathcal{R}_0 \end{aligned} \tag{3.3,b}$$

$$\frac{m_1 + m_2}{2}\dot{x}^2 + m_2 \dot{x}_0 \dot{x} + \frac{m_2}{2}\dot{x}_0^2 + \frac{k_1}{2}x^2 = \int \mathcal{R}_0 dx_0 + C, \tag{3.4,b}$$

$$E = T + \Pi = \int \mathcal{R}_0 dx_0 + C. \tag{3.5,b}$$

It is follows that

$$(1.5,b) = (3.5,b) \neq (2.5,b)$$

what was to be proved.

This simple example is give a clear explanation our assertion that

1. Lagrange's equations first kind and standard Lagrange's equations second kind, neither the same nor the equivalent Energy exchange theorems in systems with time - dependent constraints are obtained,
2. Using standard Lagrange's equations second kind a correct energy exchange theorem could not be obtained, and
3. That it is necessary that Lagrange's standard systems of equations second kind be supplemented by one more appropriate equation which corresponds to "Rheonomic coordinate".

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ТЕОРЕМЫ ОБ ИЗМЕНЕНИИ ЭНЕРГИИ СИСТЕМ С РЕОНОМНЫМИ СВЯЗЯМИ

Для механических систем с реономными связями доказано:

1. что n стандартных дифференциальных уравнений движения Лагранжа второго рода (2.1) не являются эквивалентом системы уравнений Лагранжа первого рода (1.4); для эквивалентности необходимо уравнениям (2.1) добавить уравнение (3.3).
2. На основании n дифференциальных уравнений Лагранжа (2.1) или им соотвественных $2n$ дифференциальных уравнений Гамильтона нельзя доказать теорему об изменении энергии (1.1) или (1.6). Чтобы доказать эту теорему на основании уранений (2.1), надо им добавить уравнение (3.3), т.е. доказательство обосновать на расширеной системы уравнений (4.1).
3. Соотношение "теоремы" или "закона" об изменении энергии (5.2) приведено формальным способом из уравнения (5.1) и не выражает теорему об изменении энергии, соответствующую теореме (1.6). Инвариантнное соотношение об изменении механической энергии, эквивалентное соотношению (1.3) и (1.6) в обобщенным независимым координатам Лагража или относительно каноническим координатам Гамильтона имеет вид (3.6) или соответственно (3.7).

O TEOREMAMA PROMENE ENERGIJE REONOMNIH SISTEMA

Za mehaničke sisteme sa reonomnim vezama dokazano je:

1. da sistem od n standardnih Lagranžovih diferencijalnih jednačina kretanja druge vrste (2.1) nije ekvivalentan sistemu Lagranževih jednačina prve

vrste (1.4); za njihovu ekvivalentnost potrebno je jednačinama (2.1) dodati jednačinu (3.3).

2. Na osnovu n standardnih Lagranžovih diferencijalnih jednačina (2.1), ili njima odgovarajućih $2n$ Hamiltonovih diferencijalnih jednačina nije moguće dokazati teoremu promene energije (1.1) ili (1.6). Da bi se dokazala ta teorema na osnovu jednačina (2.1), potrebno je sistemu jednačina (2.1) dodati jednačinu (3.3), tj. dokaz teoreme bazirati na proširenom sistemu jednačina (4.1).

3. Relacija "teoreme" ili "zakona" o promeni energije (5.2) izvedena je na formalan način iz jednačine (5.1) i ne izražava teoremu o promeni energije, koja odgovara relaciji (5.1). Invariantna relacija o izmeni mehaničke energije, koja je ekvivalentna relaciji (1.3) ili (1.6) u generalisanim nezavisnim Lagranžovim koordinatama ili u odnosu na Hamiltonove kanonske promenljive ima oblik (3.6) ili (3.7).

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